Table of Contents

Table of Contents for Problems in Real Analysis

T J Osler

PAGE 1
HW- REAL
OSLER /
[redacted]

REAL ANALYSIS
HOMEWORK PROBLEMS
DR T J OSLER

CHAPTER 1 REAL AND COMPLEX NUMBERS

1.1 Infinite Decimal Expansions

---

1. What is the last digit in each of the following decimal expansions?
   (a) 0.9999999...
   (b) 0.12121212...
   (c) The square root of 2: 1.41421356...

---

2. Give all possible decimal expansions of the following numbers:
   (a) 1/3
   (b) 1/2
   (c) 1/9
   (d) 5/4

---

3. Is the ternary expansion of 1/3 infinite? (Ternary means base 3. Use only the digits 0, 1 and 2).

---

4. What is the largest real number less than 2 ?

---

5. What is the largest rational number less than 2 ?

---

6. What is the largest integer less than 2 ?

---

7. What is the largest irrational number less than 2 ?

---

8. Prove that there is no smallest positive real number.

---

9. Prove that there is no largest negative number.

---

10. Give two definitions of a rational number, one in terms of the integers, and the other in terms of decimal expansions.

1.2 Rational and Irrational Numbers

1. Find fractions whose decimal expansions are:
   (a) 0.11111...
   (b) 4.99999...
   (c) 3.12121212...
   (d) -123.45123123123123...

---

2. Using long division, (not your calculator), divide out the fraction 15/7 until you can determine its repeating decimal expansion. Think about what you saw in this calculation and also about other fractions like 1/2, 1/4, 1/3, 1/6 etc. Now give a proof that the decimal representation of any fraction either terminates or repeats. If it repeats, what can you say about the length of the repeating part?

---

3. Find an irrational number by exhibiting its decimal expansion.

1.21 Closure of the Rationals under +, - , x , /

1. If x and y are rational numbers, is z = x + y always a rational number? Why?

---

2. If x and y are rational numbers, is z = x - y always a rational number? Why?

---

3. If x and y are rational numbers, is z = x y always a rational number? Why?

---

4. If x and y are rational numbers, and y is not zero, is x/y always a rational number? Why?

---

5. Are the rational numbers closed under addition, subtraction, multiplication and division?

---

6. State the axioms of a group.

---

7. State the axioms of a field.

---

8. Do the rational numbers form a field?

---

9. If x and y are irrational numbers, is z = x + y always an irrational number? Why?

---

10. If x and y are irrational numbers, is z = x - y always an irrational number? Why?

---

11. If x and y are irrational numbers, is z = x y always an irrational number? Why?

12. If x and y are irrational numbers, and y is not zero, is $z = x/y$ always an irrational number? Why?

---

13. Are the irrational numbers closed under addition, subtraction, multiplication and division?

---

14. Do the irrational numbers form a field?

---

15. If x and y are real numbers with $x < y$, give a quick proof that there is always a real number z between them. In other words, $x < z < y$.

---

16. If x and y are rational numbers with $x < y$, give a quick proof that there is always a rational number z between them.

---

17. If x and y are rational numbers with $x < y$, give a proof that there is always an irrational number z between them.

---

18. If r is a rational number and i is an irrational number, what can you say about

(a) $r + i$
(b) $r\, i$

Give reasons for your answers. (Should be short)

1.3 Prime numbers

---

1. What is a prime number? What is a composite number?

---

2. Prove that there are infinitely many prime numbers.

---

3. Let prime(n) be the the n th prime number. Thus prime(1) = 2, prime(2) = 3, prime(3) = 5, etc. Let

$$p(n) = 2 * 3 * 5 * 7 * 11 * \dots * \text{prime}(n) + 1.$$

Is p(n) always a prime number?

---

4. Find the prime factorization of the following numbers:
(a) 12, (b) 132, (c) 3072, (d) 101

---

5. Find a natural number with two different prime factorizations.

---

6. State the prime number theorem.

---

7. Suppose you are given a large number N and asked to test it for primality. You could try dividing N by all numbers 2, 3, 4, 5, ... up to N-1. If none of these numbers divide N then you would know that N is prime. Describe how you could shorten this test.

8. Write a computer program to test the primality of a given number.

---

9. Write a computer program to print out the first 1000 prime numbers.

1.4 Proofs of irrationality

---

1. Give two proofs that the square root of 2 is irrational.

---

2. Prove that the square root of each of the following numbers is irrational:
   (a) 3, (b) 5, (c) 24, (d) 6

---

3. Prove that the cube root of each of the following numbers if irrational:
   (a) 2, (b) 6, (c) 24

---

4. What can you say about the irrationality of the n th root of a natural number.

---

5. Prove that e = 2.71828... (the base of the natural logarithm) is
irrational.

1.5 Constructable numbers

---

1. What two instruments are allowed in the construction of numbers according to classical Euclidean geometry?

---

2. Show how to construct to sum of two quantities given as line segments.

---

3. Show how to construct the difference of two quantities given as line segments.

---

4. Show how to construct the product of two given line segments if a unit length is also given.

---

5. Show how to construct the ratio of two given line segments if a unit length is also given.

---

6. Show how to construct the square root of a given line segment if a unit length is also given.

---

7. Show how to construct the cube root of a given line segment if a unit length is also given.

---

8. Show how to construct the bisection of a given angle.

---

9. Show how to construct the trisection of a given angle.

---

10. Show how to construct the square of a circle. That is, given a circle, construct a square having the same area.

---

11. Show how to duplicate the cube. Given a cube, find a cube having twice the volume.

1.6 Algebraic vs Transcendental Numbers

---

1. (T/F) A number is said to be "algebraic" if it is the root of a polynomial with rational coefficients.

---

2. Show that the word "rational" can be replaced by the word "integer" in the above definition.

---

3. Prove that all rationals are algebraic.

---

4. Prove that the square root of any rational number if algebraic.

---

5. Let n be a natural number. Prove that the n th root of any rational number is algebraic.

---

6. Show that the following numbers are algebraic:

(a) $2\sqrt{3}$

(b) $2+3\sqrt{7}$

(c) $\sqrt[3]{5-\sqrt{3}}$

(d) $\sqrt{2}+\sqrt{3}$

---

7. Who first proved that e is transcendental and when?

---

8. Is "pi" algebraic?

---

9. (T/F) Every rational number is algebraic.

---

10. (T/F) Every irrational number is transcendental.

---

11. (T/F) Some irrational numbers are algebraic.

---

12. (T/F) All constructable numbers are algebraic.

1.7 Complex Numbers

1. Let $a = 3 + 4i$, $b = 5 - 2i$. Find

   (i) $a + b$, (ii) $a - b$, (iii) $a b$, (iv) a/b

   (v) $|a|$, (vi) $\arg(a)$

2. Convert the following complex numbers to polar form $r \exp(i\theta)$

   (i) $1 + i$, (ii) $3i$, (iii) -5, (iv) $2 + i2\sqrt{3}$,

   (v) $3 + 4i$

3. Convert the following polar form numbers to rectangular $x + i y$ form:

   (i) $\exp(i\pi)$, (ii) $2\exp(i\pi/3)$,

   (iii) $4\exp(-i\pi/6)$

4. Prove Euler's formula:

   $$\exp(i t) = \cos(t) + i \sin(t)$$

5. Using the above formula show that

   (i) $\cos(2t) = \cos^2 t - \sin^2 t$

   (ii) $\sin(2t) = 2 \sin t \cos t$

6. Find formulas in terms of $\cos t$ and $\sin t$ for:

   (i) $\cos(3t)$, (ii) $\sin(3t)$,
   (iii) $\cos(4t)$, (iv) $\sin(4t)$

7. Solve $z^3 - 8 = 0$

8. Solve $z^3 + 27 = 0$

1.8 Inequalities, Open and Closed Sets

1. Decide if each of the following statements is true or false. If it is false, give a counterexample using explicit numbers. Assume all numbers are real.

(a) If $a < b$ then $a + c < b + c$
(b) If $a < b$ then $a - c < b - c$
(c) If $a < b$ then $ac < bc$
(d) If $a < b$, and c is not zero, then $a/c < b/c$
(e) If $a < b$ and $c < d$, then $a + c < b + d$
(f) If $a < b$ and $c < d$, then $a - c < b - d$

---

2. Rewrite each inequality using absolute values. Show the set on the real line.

(a) $-3 < x < 3$.
(b) $0 < x < 2$
(c) $-3 \leq x \leq -2$
(d) $1/2 < x < 5/2$
(e) $a < x < b$

---

3. Rewrite each inequality without using absolute values. Show the set on the real line.

(a) $|x| < 5$
(b) $|x - 4| \leq 8$
(c) $|x + 4| > 8$
(d) $|x + 1/2| < 3$
(e) $|2x - 8| < 4$
(f) $|x - 3| \geq 1/2$
(g) $|x - a| < b$

---

4. Decide which of the following terms applies to each set of real numbers. (More than one answer is possible.)

(A) bounded, (B) unbounded, (C) open, (D) closed, (E) compact

(a) $(2, 5)$
(b) $(3, \infty)$
(c) $(2, 10]$
(d) $\{ x \mid x \text{ is rational} \}$
(e) $(2, 5) \cup (7, 10)$
(f) $(-\infty, 3]$
(g) $\{ x \mid x \text{ is irrational} \}$
(h) $\{ x \mid x \text{ is a natural number} \}$
(i) $[4, 5] \cup [7,12]$

---

5. (a) (T/F) The union of open sets is open
(b) (T/F) A finite intersection of open sets is open.
(c) (T/F) The union of closed sets is closed.
(d) (T/F) The intersection of closed sets is closed.
(e) (T/F) A SET CAN BE BOTH OPEN AND CLOSED,

HW - PAGE 8
REAL
OSLER

1.9 Additional Problems for Chapter 1

In each of the following theorems, check the proofs carefully. If a proof is correct, say so. If a proof is not correct, identify the statements in the proof that are not true.

---

1. **THEOREM:** Let x and y be irrational numbers such that $x < y$. There exists an irrational number z such that $x < z < y$.

PROOF: Let $z = (x + y)/2$. Since z is irrational and also since z is the average of x and y, z must be an irrational between x and y.

---

2. **THEOREM:** Let r be rational and i be irrational. Let $r < i$. There exists an irrational number z such that $r < z < i$.

PROOF: Let $z = (r + i)/2$. Since z is irrational and the average of r and i, z must be an irrational between r and i.

---

3. **THEOREM:** The square root of 6 is irrational.

PROOF: By contradiction assume that the square root of 6 is a rational number, say N/D, where N and D are whole numbers and the fraction N/D is completely reduced. A little algebra gives us

(A) $$6D^2 = N^2$$

Suppose the prime factorizations of D and N are

(B) $$D = d_1 d_2 d_3 \ldots d_m \quad \text{and}$$

(C) $$N = n_1 n_2 n_3 \ldots n_k$$

Substituting relations (B) and (C) into (A) we see that there are $2m + 1$ prime factors on the left and $2k$ prime factors on the right. Since the prime factorization is unique, we have a contradiction because the number of factors on the left is odd while the number of factors on the right is even.

---

4. **THEOREM:** There exists a smallest positive number.
PROOF: Consider the number $x = 0.00000\ldots1$, where there are infinitely many zeroes between the decimal point and the last digit which is a one. This number is positive, and no number can be smaller and positive.

---

5. **THEOREM:** 3.9999... is the largest real number less than 4.
PROOF: 3.999... is less than 4 since it begins with a 3. As you add digits to the right of the "3." in an attempt to increase the number, the largest digit you can use is a "9". Thus using all nines after the "3." produces the largest number less than 4.

HW-9
REAL
OSLER

6. Decide which of the following terms applies to each number. (More than one answer is possible.)

(A) natural number, (B) integer, (C) rational, (D) irrational, (E) constructable (Euclidean), (F) algebraic, (G) transcendental, (H) prime, (I) composite

(a) 5, (b) e = 2.71828..., (c) π = 3.14159... (d) -8,

(e) square root of 2, (f) 234/137, (g) cube root of two,

(h) square root of 16, (i) seventh root of 11

(j) $\sqrt{2}+\sqrt{7}$ (k) $\sqrt[3]{\frac{2+\sqrt{3}}{2}}$

---

7. Is the sum of an infinite number of positive numbers always infinite?

---

8. (a) Consider the set of 1,000,000 numbers:

C = { x+2, x+3, x+4, ..., x + 1,000,001 } . Suppose x = 1,000,001 !.

Show that every number in this set C is composite.

(b) Show that given any number N, however large, there exists a set of N consecutive whole numbers all of which are composite.

---

9 (a) (T/F) The complement of an open set is closed.
(b) (T/F) The complement of a closed set is open.

---

10. Find a collection of open sets O_1, O_2, ...

such that $\bigcap_{n=1}^{\infty} O_n$ is closed.

---

11. Find a collection of closed sets C_1, C_2, ...

such that $\bigcup_{n=1}^{\infty} C_n$ is open.
