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Chapter 2: SEQUENCES

2.1 Definition of a Sequence

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1. Write out the first five terms of the sequence $\{x_n\}$, $(n = 1, 2, \ldots)$

(a) $x_n = (-1)^n$

(b) $x_n = 2n - 1$

(c) $x_n = \frac{1}{n!}$

(d) $x_n = (-2)^n$

(e) $x_n = \left(1 + \frac{1}{n}\right)^n$

(f) $x_n = 1 + 1/2 + 1/3 + 1/4 + \ldots + 1/n - \log n$

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2. Find a simple expression for the general term of each sequence.

(a) 1, -1, 1, -1, ...

(b) 1, -1/2, 1/3, -1/4, 1/5, ...

(c) 1, 1/4, 1/9, 1/16, 1/25, ...

(d) 2, 3/2, 4/3, 5/4, 6/5, ...

(e) 1 ,- 1/2, 1/6, -1/24, 1/120, -1/720, ...

(f) 1, i, -1, -i, 1, i, -1, -i, 1, ...

(g) 1, i/2, -1/4, -i/8, 1/16, i/32, -i/64, 1/128, ...

(h) 1, 1/3, 1/5, 1/7, 1/9, 1/11, ...

(i) 2/3, -4/5, 6/7, -8/9, ...

(j) 2, 4/2, 8/6, 16/24, 32/120, 64/720, ...

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3. The following terms frequently occur in expressions for the general term of a sequence. Arrange them in order according to the speed with which they grow as n approaches infinity. (Slowest to fastest)

(a) exp(n), (b) n, (c) n^2, (d) n^3, (e) n!, (f) log n, (g) n^n, (h) c = constant, (i) 2^n, (j) 3^n

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4. Determine the limit of each sequence as n approaches infinity. In many cases you can use the list from problem 3 above to see the answer quickly.

(a) $x_n = \dfrac{2n + 1}{n + 3}$

(b) $x_n = \dfrac{n^2 + 2n - 1}{3n^2 + 1}$

(c) $x_n = \dfrac{2n^2 + 1}{3n^2 + \log n}$

(d) $x_n = \dfrac{n^5 + 5}{e^n + 2}$

(e) $x_n = \dfrac{e^n}{n!}$

(f) $x_n = \dfrac{e^n + n}{n^n + 2}$

(g) $x_n = (1 + 1/n)^n$

(h) $x_n = (1 + 2/n)^n$

(i) $x_n = \dfrac{n^2 + 1}{n^2 \log n}$

(j) $x_n = \dfrac{n!\, e^n}{n^n \sqrt{n}}$

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5. Write a computer program to study the convergence of each of the following sequences. Determine how large "n" must be before the terms of the sequence demonstrate one, two, three, and four decimal place accuracy when compared to their limiting value.

(a) $x_n = (1 + 1/n)^n \qquad \lim x_n = e$

(b) $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n} - \log n$

$\lim x_n = \gamma = 0.57721\,56649 \dots$ = *Euler's constant*

(c) $$x_n = \frac{\sqrt{2}}{2} \; \frac{\sqrt{2+\sqrt{2}}}{2} \; \cdots \; \underbrace{\frac{\sqrt{2+\sqrt{2+\dots+\sqrt{2}}}}{2}}_{n \text{ radicals}}$$

$\lim x_n = 2/\pi$ *F. Vieta* (1540 - 1603)

(d) $$x_n = \frac{2}{1}\frac{2}{3} \; \frac{4}{3}\frac{4}{5} \; \frac{6}{5}\frac{6}{7} \; \cdots \; \frac{(2n)(2n)}{(2n-1)(2n+1)}$$

$\lim x_n = \pi/2$ *J. Wallis* (1616 - 1703)

2.2 Recursion Formulas

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1. Solve the following recursion formulas. (Find x_n in general.) Assume $x_1 = 1$.

(a) $x_{n+1} = x_n + 3$

(b) $x_{n+1} = 1 - x_n$

(c) $x_{n+1} = 2\,x_n$

(d) $x_{n+1} = \frac{n}{n+1}\,x_n$

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2. The Fibonacci sequence is generated by the recursion relation

$$x_{n+2} = x_{n+1} + x_n \quad \text{and} \quad x_1 = x_2 = 1.$$

(a) Determine the first ten terms of the Fibonacci sequence.

(b) Find $\lim \left(\frac{x_{n+1}}{x_n}\right)$

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3. Consider the recursion formula:

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) \quad \text{with} \quad x_1 = 1$$

(a) Find $\lim x_n$

(b) With $a = 2$, study the convergence of this sequence using your calculator. How many terms of the sequence are needed to get eight decimal place accuracy?

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4. Find $\lim \left(\frac{x_{n+1}}{x_n}\right)$ for each of the following sequences defined by recursion relations. Assume $x_1 = x_2 = 1$

(a) $x_{n+2} = 3\,x_{n+1} + x_n$

(b) $x_{n+2} = x_{n+1} - 2\,x_n$

(c) $x_{n+2} = 2\,x_{n+1} + 3\,x_n$

5. Find $\lim x_n$ for each sequence

(a) $x_n = \sqrt{2 + \sqrt{2 + \sqrt{2 + \ldots \sqrt{2}}}}$

$\underbrace{\qquad\qquad\qquad}_{n \text{ two's}}$

(b) $x_n = \sqrt{3 + \sqrt{3 + \sqrt{3 + \ldots + \sqrt{3}}}}$

$\underbrace{\qquad\qquad\qquad}_{n \text{ three's}}$

2.3 The Limit of a Sequence

1. Memorize the following definition:

Let (x_n) be a given sequence of real numbers, and let l be a given real number. Then

$$\lim_{n \to \infty} x_n = l$$

means that for any given $\varepsilon > 0$, there is a corresponding integer N (which may depend on ε) such that

$$|x_n - l| < \varepsilon \quad \text{for every } n \geq N.$$

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2. (T/F) A sequence can have two distinct limits.

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3. (T/F) Suppose ε is given as .01 in the above definition and the corresponding value of N is found to be 42. Any value larger than 42 will do as well for N.

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4. (T/F) Suppose $\varepsilon = .001$ in the above definition and the corresponding N is found to be 101. This value of N will do as well for any larger ε.

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5. (T/F) We can replace the absolute value in the above definition by

$$1 - \varepsilon < x_n < 1 + \varepsilon$$

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6. (T/F) Given a sequence (x_n), changing a finite number of terms of this sequence will not change the limit of the sequence.

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7. If for a given sequence, the value of $N(\varepsilon)$ is found to be $N(\varepsilon) = 2/\varepsilon$, which of the following values of $N(\varepsilon)$ also are suitable for this sequence.

(a) $N(\varepsilon) = 1/\varepsilon$

(b) $N(\varepsilon) = 3/\varepsilon$

(c) $N(\varepsilon) = 4 + 2/\varepsilon$

(d) $N(\varepsilon) = 2\,\varepsilon$

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8. Draw a graph illustrating how the above definition "works". Plot n as the horizontal axis and x_n as the vertical. Be sure to show the limiting value 1, the tollerance ε and the value $N(\varepsilon)$ on the graph.

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9. (T/F) As ε decreases, we usually expect $N(\varepsilon)$ to increase.

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10. Describe a sequence for which any integer value can be used for N regardless of the value of ε.

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11. (T/F) Doubling the value of ε usually doubles the value of N.

2.3 Using the Definition of Limit with Specific Sequences

EXAMPLE PROBLEM: Prove that $\lim \frac{n+1}{n} = 1$.

Preliminary Analysis:

We must find N such that

$\left\| \frac{n+1}{n} - 1 \right\| < \varepsilon$	*From the basic definition of limit*
$\left\| 1 + \frac{1}{n} - 1 \right\| < \varepsilon$	*Expanding the fraction*
$\left\| \frac{1}{n} \right\| < \varepsilon$	$1 - 1 = 0$
$\frac{1}{n} < \varepsilon$	*Since n is a natural number Absolute values can be dropped.*
$\frac{1}{\varepsilon} < n$	*Laws of inequalities*

The above analysis shows that a value of N could be the next integer greater than

$$N(\epsilon) = \frac{1}{\epsilon}$$

Proof:

Given any number $\varepsilon > 0$, (however small), we select N to be

(1) $N = \frac{1}{\epsilon}$

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(2) $N > 1/\varepsilon$

For all $n \geq N$, we have from (1) that

$n > 1/\varepsilon$ and from the laws of inequalities that

(3) $\varepsilon > 1/n$

Since n is positive adding absolute values makes no change:

(4) $| 1/n | < \varepsilon$

Adding and subtracting one inside the absolute value we get

(5) $| 1 + 1/n - 1 | < \varepsilon$

(6) $\left| \frac{n+1}{n} - 1 \right| < \varepsilon$

Since this last inequality is true for all $n \geq N$, the conditions imposed by the basic definition of limit are seen to be true. Thus we have proved that the limit of the given sequence is 1.

---------- *end of sample problem* ----------------------

1. Why was the preliminary analysis in the above example performed?

~~2. What are the following:~~

~~(a) $\lceil 123.4 \rceil$, (b) $\lceil 100 \rceil$, (c) $\lceil -1.5 \rceil$~~

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3. Using (1) above, determine the numeric value of N for each value of ε given below:
(a) $\varepsilon = .1$, (b) $\varepsilon = .02$, (c) $\varepsilon = .001$
(d) As ε decreases, we expect N to ______________

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4. Is not the preliminary analysis itself a proof? Do we really need the formal proof that follows it?

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5. What other values could be used for N in (1) in the above example?

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6. Two important inequalities that ~~of~~ ARE use in proving limits are:

(a) $|x + y| \leq |x| + |y|$ (*triangle inequality*)

(b) $| \, |x| - |y| \, | \leq |x - y|$

Both these inequalities are true for all real (or complex) values

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7. Using the basic definition of the limit of a sequence, prove that the following limits are true:

(a) $\lim_{n \to \infty} 1/n = 0$

(b) $\lim \dfrac{5n + 2}{n} = 5$

(c) $\lim \dfrac{5n + 2}{3n + 1} = \dfrac{5}{3}$

(d) $\lim \dfrac{3n^2 + n + 2}{4n^2 + 2n + 1} = \dfrac{3}{4}$

(e) $\lim \dfrac{3n^2 - 2n + 1}{5n^2 - 2n - 3} = \dfrac{3}{5}$

(f) $\lim \dfrac{n^3 + 1}{n^3 - n + 3} = 1$

(g) $\lim \dfrac{4\ n}{3n^5 + 1} = 0$

(h) $\lim \dfrac{n^2}{2^n} = 0$

(i) $\lim \dfrac{2^n}{3^n} = 0$

8. Prove that the following limits do not exist:

(a) $\lim (-1)^n$, (b) $\lim n$, (c) $\lim \sin(n\pi/3)$