Real Analysis
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Homework on Limits, Continuity and Differentiability

1. Memorize the following definition.

   We say that $\lim_{x \to a} f(x) = l$ if given any $\varepsilon > 0$ we can find $\delta(\varepsilon)$ such that

   $|f(x) - l| < \varepsilon$ for all x satisfying $0 < |x - a| < \delta$.

2. Using the above definitions, prove the following
   (a) $\lim_{x \to 2} 3x - 2 = 4$, (b) $\lim_{x \to 3} x^2 = 9$, (c) $\lim_{x \to 2} x^3 = 8$, (d) $\lim_{x \to a} x^2 = a^2$.

3. For each function listed below find (a) $\lim_{x \to a} f(x)$, (b) decide if $f(x)$ is continuous at $x = a$, and (c) decide if $f(x)$ is differentiable at $x = a$. For each function, draw an appropriate graph.

3.1 $f(x) = \begin{cases} 1 & for \quad 0 < x \\ 0 & for \quad x < 0 \end{cases}$, $a = 0$

3.2 $f(x) = |x - 1|$, $a = 1$.

3.3 $f(x) = x^{-2}$, $a = 0$.

3.4 $f(x) = \sin(1/x)$, $a = 0$

3.5 $f(x) = x \sin(1/x)$, $a = 0$

3.6 $f(x) = x^2 \sin(1/x)$, $a = 0$

(In the following problems, take $a = 0$.)

3.7 $f(x) = \begin{cases} x \sin(1/x) & for \ x \neq 0 \\ 1 & for \ x = 0 \end{cases}$

3.8 $f(x) = \begin{cases} x^2 \sin(1/x) & for \ x \neq 0 \\ 0 & for \ x = 0 \end{cases}$

3.9 $f(x) = \begin{cases} x \sin(1/x) & for \ x \neq 0 \\ 0 & for \ x = 0 \end{cases}$

3.10 $f(x) = \begin{cases} \sin(1/x) & for \ x \neq 0 \\ 0 & for \ x = 0 \end{cases}$

3.11 $f(x) = \begin{cases} 1 & if \ x \ is \ rational \\ 0 & if \ x \ is \ irrational \end{cases}$

3.12 $f(x) = \begin{cases} x & if \ x \ is \ rational \\ 0 & if \ x \ is \ irrational \end{cases}$

3.13 $f(x) = \begin{cases} x^2 & if \ x \ is \ rational \\ 0 & if \ x \ is \ irrational \end{cases}$

4. Memorize the following: We say that the function $f(x)$ is continuous at the point $x = a$ if the following three conditions are true: (1) $f(a)$ exists, (2) $\lim_{x \to a} f(x)$ exists, and (3) $\lim_{x \to a} f(x) = f(a)$. We say that $f(x)$ is continuous on the interval $c < x < d$ if $f(x)$ is continuous at each point of that interval.

5. Find all points at which the following functions are discontinuous

5.1 $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$

5.2 $f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$

5.3 $f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$

5.4 $f(x) = 1/x$

5.5 $f(x) = (\sin x)/x$,

5.6 $f(x) = \begin{cases} (\sin x)/x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

5.7 $f(x) = \begin{cases} \dfrac{x}{e^x - 1} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

6 Memorize the following definition: We say that the function $f(x)$ is differentiable at the point where $x = a$ if $\lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ exists. We call the value of this limit $f'(a)$. We say that $f(x)$ is differentiable on the interval $c < x < d$ if $f(x)$ is differentiable at each point of this interval.

7. Find all points at which the following functions are differentiable

7.1 $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$

7.2 $f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$

7.3 $f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$

7.4 $f(x) = 1/x$

7.5 $f(x) = (\sin x)/x$,

7.6 $f(x) = \begin{cases} (\sin x)/x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

True or false:

8.1 If $f(x)$ is continuous at $x = a$, then it is differentiable at $x = a$.

8.2 If $f(x)$ is differentiable at $x = a$, then it is continuous at $x = a$.

8.3 In the definition of limit of a function in problem 1, we can always replace the expression $0 < |x - a| < \delta(\varepsilon)$ by the expression $|x - a| < \delta(\varepsilon)$

8.4 The function $\dfrac{\sin x}{x}$ is defined at $x = 0$.

8.5 In the definition of limit of a function in problem 1, we can always replace $\delta(\varepsilon)$ by a smaller value.

9. Memorize the following definition: We say that the function $f(x)$ is uniformly continuous on the interval $c \le x \le d$ if given any $\varepsilon > 0$ we can find $\delta(\varepsilon)$ (but not depending on a } such that for each number a in the interval we have $|f(x) - f(a)| < \varepsilon$ for all x satisfying $|x - a| < \delta(\varepsilon)$.

10. Prove that the function $f(x) = x^2$ is uniformly continuous on the interval $0 \le x \le 1$.