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REARRANGING TERMS OF A HARMONIC-LIKE SERIES

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1. Introduction

The commutative law of addition is familiar to all our students. However, if the sum involves an infinite number of terms, the commutative law might not be true. This is indeed a surprise! In this paper we will study one example of a series which illustrates the failure of the commutative law. Our series is

(1) $$S = \frac{1}{1} - \frac{1}{1} + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \frac{1}{4} - \frac{1}{4} + \cdots .$$

Notice that we have taken the terms of the harmonic series, and subtracted them from themselves. The study of this one example should help motivate the remarkable general theory behind all such series.

If we call S_n the n th partial sum of our series (1), then we see at once that

$$S_1 = \frac{1}{1}$$

$$S_2 = \frac{1}{1} - \frac{1}{1}$$

$$S_3 = \frac{1}{1} - \frac{1}{1} + \frac{1}{2}$$

... .

Now we find that $S_n = 0$ if *n* is even, and $S_n = \dfrac{2}{n+1}$ if *n* is odd. Therefore

$S = \lim_{n\to\infty} S_n = 0$.

Now suppose we *rearrange* the terms in this series in the following way:

$$S(2,1) = \left(\frac{1}{1}+\frac{1}{2}-\frac{1}{1}\right)+\left(\frac{1}{3}+\frac{1}{4}-\frac{1}{2}\right)+\left(\frac{1}{5}+\frac{1}{6}-\frac{1}{3}\right)+\left(\frac{1}{7}+\frac{1}{8}-\frac{1}{4}\right)+\cdots \qquad (2)$$
$$= \quad m_1 \quad + \quad m_3 \quad + \quad m_3 \quad + \quad m_4 \quad +\cdots.$$

The notation $S(2,1)$ was selected to show that each *major term* m_n contains 2 + 1 = 3 *minor terms.* We used 2 successive positive minor terms followed by 1 negative minor term to construct each major term. Combining the second and third minor terms in each major term of (2) we get

$$S(2,1) = \left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+\left(\frac{1}{7}-\frac{1}{8}\right)+\cdots. \qquad (3)$$

Because $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots$, we can let $x = 1$ and determine that the series in (3) sums to $S(2,1) = \log 2$.

The fundamental theory that applies to our observations is stated below in theorems (a) and (b). (For proofs see [1, pages 318-319], and [2, page 401].)

> **Theorem (a):** Let $s = a_1 + a_2 + a_3 + \cdots$ be an *absolutely convergent* series. Then the terms of this series may be rearranged in any way and the series remains convergent to the sum *s*.
>
> **Theorem (b):** Let $a_1 + a_2 + a_3 + \cdots$ be a *conditionally convergent* series. Then for *any* number *s* there is a rearrangement of the terms of this series for which the sum of the series is *s*.

In short these theorems say that the commutative law for addition only works for absolutely convergent series. Our series (1) is conditionally convergent, and we have seen that the arrangement of the terms as given yields the sum 0, while the rearrangement (2) has the sum $\log 2$. But theorem (b) promises more surprises. It says that (1) can be made to converge to any number whatsoever by suitably rearranging its terms! At first glance, this seems impossible! It is the purpose of this paper to use the series (1) to provide motivation for this remarkable fact.

These ideas can be used in the traditional calculus course when the notions of absolute and conditional convergence are introduced. At this level, simply stating theorems (a) and (b) without proof and showing the previous examples would be appropriate. The remaining material would be suitable for a first course in real analysis. In addition, the different rearrangements of series shown here could be used as exercises in an introductory course in computer programming.

2. Rearrangements of the series

We will now prove our main theorem concerning the conditionally convergent series $\frac{1}{1}-\frac{1}{1}+\frac{1}{2}-\frac{1}{2}+\frac{1}{3}-\frac{1}{3}+\frac{1}{4}-\frac{1}{4}+\cdots$ and the result of rearranging its terms in a special way. Let M and N be positive integers. Define the infinite series $S(M,N)$ by

(4) $$S(M,N)=\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{M}-\frac{1}{1}-\frac{1}{2}-\frac{1}{3}-\cdots-\frac{1}{N}\right)+$$

$$\left(\frac{1}{M+1}+\frac{1}{M+2}+\cdots+\frac{1}{2M}-\frac{1}{N+1}-\frac{1}{N+2}-\cdots-\frac{1}{2N}\right)+$$

$$\left(\frac{1}{2M+1}+\frac{1}{2M+2}+\cdots+\frac{1}{3M}-\frac{1}{2N+1}-\frac{1}{2N+2}-\cdots-\frac{1}{3N}\right)+$$

$$\cdots$$

Notice that each major term (...) contains $M + N$ minor terms. First M positive terms of the harmonic series are used, followed by N negative terms. To simplify the notation we introduce $h(m,d) = \frac{1}{m+1} + \frac{1}{m+2} + \frac{1}{m+3} + \cdots + \frac{1}{m+d}$. Then our series (4) becomes

$$S(M,N) = \sum_{k=0}^{\infty} \left(h(kM, M) - h(kN, N) \right). \tag{5}$$

The Main Theorem:

$$S(M,N) = \log(M/N). \tag{6}$$

Proof:

Consider the partial sum of the series defining $S(M,N)$ involving n terms:

$$S_n(M,N) = \sum_{k=0}^{n-1} \left(h(kM, M) - h(kN, N) \right).$$

Adding and subtracting $\log(nM)$ and $\log(nN)$ we have

$$S_n(M,N) = \left\{ \sum_{k=0}^{n-1} h(kM, M) - \log(nM) \right\} - \left\{ \sum_{k=0}^{n-1} h(kN, N) - \log(nN) \right\} + \log(nM) - \log(nN)$$

This simplifies to

$$S_n(M,N) = \left\{ \sum_{k=1}^{nM} \frac{1}{k} - \log(nM) \right\} - \left\{ \sum_{k=1}^{nN} \frac{1}{k} - \log(nN) \right\} + \log \frac{M}{N}.$$

Using the fact that $\lim_{n\to\infty} \left\{ \sum_{k=1}^{n} \frac{1}{k} - \log n \right\} = \gamma = 0.5772156649\cdots$, (see [1, page 225]), where γ is Euler's constant we have

$$S(M,N) = \lim_{n\to\infty} S_n(M,N) = \gamma - \gamma + \log \frac{M}{N} = \log \frac{M}{N}.$$

This completes the proof of the main theorem.

3. Final remarks

Theorem (b) from section 1 promised that our conditionally convergent series (1) could be rearranged so that it converged to *any* number whatsoever. The theorem just proved in section 2 showed how to rearrange the terms so that the sum of the series is $\log(M/N)$, where M and N are any positive integers. Now suppose we are given any number s and we ask how to rearrange the terms of (1) so that it converges to s. We can solve the equation $s = \log x$ by using $x = \exp(s)$. Now if x happens to be a rational number $x = M/N$, then our theorem shows us how to rearrange (1) so that it sums to the number s. If s is irrational, we know that there are rational numbers M/N arbitrarily close to s. In this case our theorem does not show us how to rearrange terms to sum exactly to the number s, but does show us how to approximate s to any degree of accuracy. After seeing this example, theorem (b) no longer seems so strange. This example also makes it easier to follow the proof of theorem (b) as given in [1] and [2]. This completes our exploration of the rearrangements of the series (1).

References:

[1] Knopp, K., *Theory and Application of Infinite Series*, Dover Pub., New York, 1951.

[2] Spivak, M. *Calculus*, W. A. Benjamin, Inc., New York, 1967.