A SPIRAL OF TRIANGLES RELATED TO THE GREAT PYRAMID

Revised

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1. Introduction

In this article we will study an interesting spiral built from a single remarkable triangle. The method we use to generate the spiral is shown in Figure 1. We start by drawing any right triangle *OAB* with *O* at the origin of the *x*-*y* plane and *OA* placed along the *x*-axis. The side lengths are *a*, *b*, and *c* as shown. From point *B* we construct a line of length *b* perpendicular to *OB* terminating at *C*. *OBC* is now our second triangle. We continue in this way to construct the other triangles shown. Our spiral is the rectilinear arc *ABCDE*… .

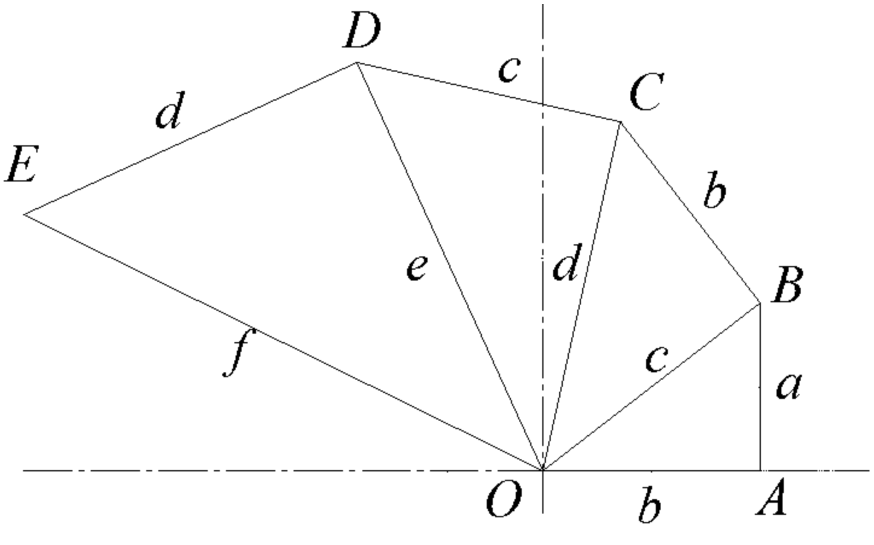

Figure 1: Spiral of Triangles

We can start with any right triangle *OAB*, and a rectilinear spiral with interesting mathematical properties will be generated. However in this article we will concentrate our attention on a very special triangle whose sides are related to the golden ratio [2], $\phi = \frac{1+\sqrt{5}}{2}$. We will select the following lengths for the sides of our first triangle: $a = \sqrt{\phi}$, $b = 1$, and $c = \phi$. The reader will see at once that these values satisfy the Pythagorean theorem $c^2 = a^2 + b^2$ which becomes the well-known equation (see [2])

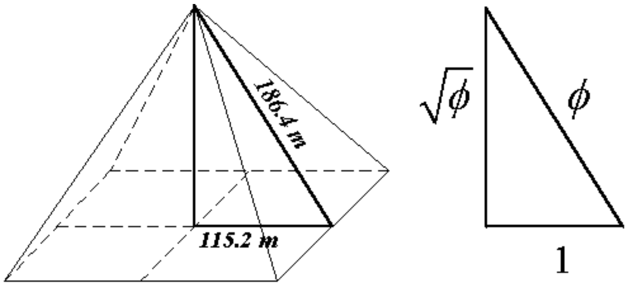

Figure 2: The Great Pyramid of Giza and a Related Triangle

$\phi^2 = \phi + 1$. Figure 2 also shows some dimensions of the great pyramid of Giza in Egypt. Our triangle is nearly similar to the triangle shown in the pyramid since $\frac{186.4}{115.2} = 1.618055555...$ while $\phi = 1.618033989...$. No one knows if this is a happy accident, or if the architect of the pyramid had this mathematical fact in mind. For this reason we will call the triangle with sides $a = \sqrt{\phi}$, $b = 1$ and $c = \phi$ the *pyramid triangle*.

In Figure 3 we see this new spiral starting with our pyramid triangle. In the next section we will examine some properties of these triangles and the rectilinear spiral. Later we will investigate a smooth spiral that contains the points *A*, *B*, *C*, … . Finally we will evaluate the length of these spirals and the areas generated by the radius vector as it traces the spirals.

2. Similar triangles

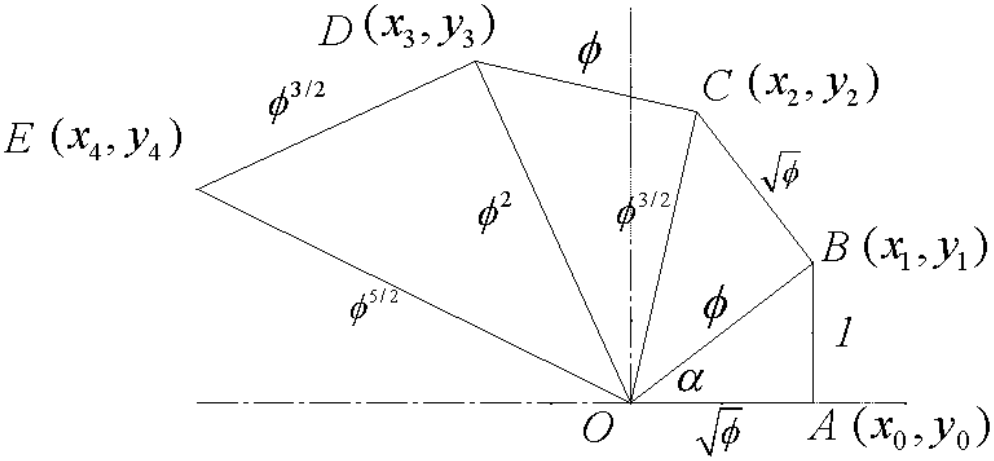

Figure 3: Spiral of Pyramid Triangles

In Figure 3 we have denoted the coordinates of the points *A*, *B*, *C*, … as $(x_0, y_0), (x_1, y_1), (x_2, y_2), \cdots$ respectively. We have also denoted by α the angle *AOB*. (We note that $\alpha = 0.666239432...$ radians $= 38.17270762\ldots$ degrees.) It is easy to see that all the triangles are similar and that the sides of successive triangles differ by the factor $\sqrt{\phi}$.

Thus all of the angles of each triangle at the origin equal α . The polar coordinates of the vertex (x_n, y_n) are

(1) $$\theta = n\alpha \text{ and } r = \phi^{(n+1)/2}.$$

It follows that the rectangular coordinates of the same vertex are

(2) $$x_n = \phi^{(n+1)/2} \cos n\alpha \text{ and } y_n = \phi^{(n+1)/2} \sin n\alpha .$$

In the following theorem we show that the only way to have all the triangles similar is to start with this pyramid triangle, or one similar to it.

Theorem: If the triangles that generate the spiral shown in Figure 1 are all similar, then the starting triangle has sides $a = 1$, $b = \sqrt{\phi}$ and $c = \phi$, or is a triangle similar to it.

Proof: If the triangles are similar, then from Figure 1 we see by examining the first two triangles that

$$\frac{a}{b} = \frac{b}{c} = \frac{b}{\sqrt{a^2 + b^2}} .$$

Thus $a\sqrt{a^2 + b^2} = b^2$, and squaring we get $a^4 + a^2b^2 = b^4$ which we rewrite as

$$\left(\frac{b^2}{a^2}\right)^2 = \frac{b^2}{a^2} + 1 .$$

This last equation is quadratic in $\frac{b^2}{a^2}$ and has the solution $\frac{b^2}{a^2} = \frac{1+\sqrt{5}}{2}$. We thus have $b = \sqrt{\phi}\, a$ and it follows that $c = \phi\, a$. Thus the theorem is proved.

3. The smooth spiral

In the parametric equations (1) and (2) with the discrete parameter $n = 0, 1, 2, \ldots$, we can easily replace *n* by a continuous parameter *t* to obtain a smooth spiral that passes through all the points A, B, C, … of our previous rectilinear spiral. The polar equations now become

(3) $$\theta = \alpha t \text{ and } r = \phi^{(t+1)/2},$$

and the Cartesian equations become

(4) $$x(t) = \phi^{(t+1)/2} \cos \alpha t \text{ and } y(t) = \phi^{(t+1)/2} \sin \alpha t.$$

It will be convenient to rewrite these equations in terms of the exponential function. Let

(5) $$\beta = \frac{\log \phi}{2}.$$

(Note that $\beta = 0.240605912\ldots$.) Then $\phi^{t/2} = e^{\beta t}$ and the parametric equations in polar form become

(6) $$\theta = \alpha t \text{ and } r = \sqrt{\phi}\, e^{\beta t},$$

and in Cartesian form

(7) $$x(t) = \sqrt{\phi}\, e^{\beta t} \cos \alpha t \text{ and } y(t) = \sqrt{\phi}\, e^{\beta t} \sin \alpha t.$$

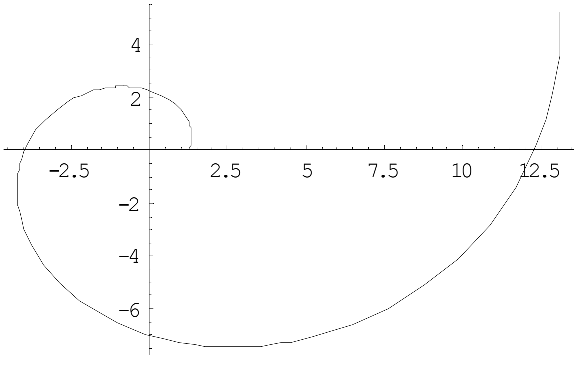

Figure 4: The Smooth Spiral for 0 < t < 10

In Figure 4 we see a plot generated by the software program Mathematica of our smooth spiral.

4. Arc length of the spirals

We begin by finding the length of a segment of our rectilinear spiral.

Theorem: The length of the rectilinear spiral described by equations (1) and (2) from the point (x_0, y_0) to the point (x_n, y_n) is

$$(8) \qquad L(n) = \frac{\phi^{n/2} - 1}{\phi^{1/2} - 1}.$$

Proof: From Figure 3 we see that the desired arc length is the sum

$$L(n) = 1 + \left(\phi^{1/2}\right) + \left(\phi^{1/2}\right)^2 + \left(\phi^{1/2}\right)^3 + \cdots + \left(\phi^{1/2}\right)^{n-1}.$$

This is a geometric series, and it is well known that $1 + x + x^2 + \cdots + x^{n-1} = \dfrac{x^n - 1}{x - 1}$, when $n \neq 1$; therefore we have (8) and the theorem is proved.

It is also easy to find the corresponding length of our smooth spiral.

Theorem: The length of the segment of the smooth spiral described by equations (6) and (7) from the point $(x(0), y(0))$ to the point $(x(t), y(t))$ is

$$(9) \qquad S(t) = \frac{\sqrt{\phi}}{\log \phi} \sqrt{4\alpha^2 + (\log \phi)^2} \left(\phi^{t/2} - 1\right).$$

Proof: We know that the arc length can be calculated in polar coordinates by

$$(10) \qquad S(t) = \int_0^t \sqrt{\left(\frac{dr}{dt}\right)^2 + r^2 \left(\frac{d\theta}{dt}\right)^2}\, dt.$$

From (6) we see that

$$\frac{dr}{dt} = \beta \sqrt{\phi}\, e^{\beta t} \text{ and } \frac{d\theta}{dt} = \alpha.$$

Substituting these derivatives into (10) and simplifying we get

$$S(t) = \sqrt{\phi} \sqrt{\alpha^2 + \beta^2} \int_0^t e^{\beta t} dt$$

$$=\frac{\sqrt{\phi}\sqrt{\alpha^2+\beta^2}}{\beta}\left(e^{\beta t}-1\right).$$

Since $e^{\beta t}=\phi^{t/2}$, and since $\beta=\dfrac{\log\phi}{2}$, this last expression can be written as (9) and our theorem is proved.

5. Area under the spirals

We begin by finding the area of the first *n* triangles that generate our rectilinear spiral.

Theorem: The area of the first *n* triangles that generate our rectilinear spiral given by equations (1) and (2) is

$$A(n)=\frac{\sqrt{\phi}\left(\phi^n-1\right)}{2(\phi-1)}. \tag{11}$$

Proof: From Figure 3 it is easy to add the areas of the first *n* triangles. We get

$$\frac{\sqrt{\phi}}{2}+\frac{\sqrt{\phi}}{2}\phi+\frac{\sqrt{\phi}}{2}\phi^2+\frac{\sqrt{\phi}}{2}\phi^3+\cdots+\frac{\sqrt{\phi}}{2}\phi^{n-1}=\frac{\sqrt{\phi}\left(\phi^n-1\right)}{2(\phi-1)},$$

and the theorem is proved.

We end by finding the area generated by the radius vector as it traces the smooth spiral.

Theorem: The area generated by the radius vector as it traces the smooth spiral given by equations (6) while the parameter ranges from 0 to *t* is

$$a(t)=\frac{\alpha\phi}{2\log\phi}\left(\phi^t-1\right). \tag{12}$$

Proof: The area is given by the integral

$$a(t) = \frac{1}{2}\int_0^t r^2 \frac{d\theta}{dt} dt .$$

Using the parametric equations (6) this integral becomes after a little computation

$$a(t) = \frac{\alpha\phi}{2\log\phi}\left(e^{2\beta t} - 1\right).$$

Since $e^{\beta t} = \phi^{t/2}$ we get (12) at once and the theorem is proved.

6. Final remarks

We can investigate the spirals with other initial triangles. Starting with the lengths $a = 1$, $b = 1$ and $c = \sqrt{2}$ the Fibonacci numbers appear in the spiral. If we start with the triangle $a = 1$, $b = \sqrt{3}$ and $c = 2$ our spiral involves the Lucas numbers. Many other spirals of interest are found in [1].

References

[1] Davis, Philip J., *Spirals from Theodorus to Chaos,* A. K. Peters, Wellesley, Massachusetts, 1993.

[2] Graham, Ronald L., Knuth, Donald L., Patashnik, Oren, *Concrete Mathematics*, Addison-Wesley Publishing, Massachusetts, 1989.