GENERALIZING INTEGRALS INVOLVING x^x AND SERIES INVOLVING n^n

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1. Introduction

In a series of three interesting papers [3], [4], and [5], that appeared recently in this journal, P. Glaister established the following power series evaluation of the integral

(1) $$G(c;x)=\int_0^1 t^{-xt+c}\,dt=\sum_{n=0}^{\infty}\frac{x^n}{(n+c+1)^{n+1}}.$$

The integral and the series are convergent for all x and $c>-1$. Special cases of this series include

$$xG(0,x)=\sum_{n=1}^{\infty}\frac{x^n}{n^n}\text{, and } xG(0,-x)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^n}{n^n}.$$

Glaister also considers other series that he can derive from these such as

$$\frac{xG(0,x)-xG(0,-x)}{2}=\sum_{n=1}^{\infty}\frac{x^{2n}}{(2n)^{2n}}\quad\text{and}\quad\frac{xG(0,1)+xG(0,-1)}{2}=\sum_{n=1}^{\infty}\frac{x^{2n-1}}{(2n-1)^{2n-1}}.$$

These functions seem intriguing and invite investigation.

In the papers cited above, Glaister is interested in obtaining lower and upper bounds on the sum of his series. For this purpose he calculates upper and lower Riemann sums of the integral representation in (1) using rectangles. For example in [3], page 205, he uses 10,000 rectangles to obtain the bounds $1.291241\le\sum_{n=1}^{\infty}\frac{1}{n^n}\le 1.291331$.

In this paper we study the more general function

(2) $$F(P,Q,c;x)=\int_0^1 t^{-xt^P+c}(\log t)^Q dt\,,$$

where the integral converges if $0<P$, $Q=0, 1, 2, \cdots$, $-1<c$, and x is any real number. In his previous papers, Glaister was interested in the special cases of (2) in which $P=1$ and $Q=0$. We will see that this function has the Maclauren series expansion

(3) $$F(P,Q,c;x)=\sum_{n=0}^{\infty}\frac{(-1)^Q(n+Q)!}{n!(Pn+c+1)^{n+Q+1}}\,x^n.$$

(The series in (1) is the special case of (3) in which $P=1$ and $Q=0$.)

The unusual feature of the series (3) is that it contains an expression of the form $(an+b)^{n+d}$ in the coefficients. In the extensive collection of over 1100 series given in Jolley [7], we could only find one such series. Jolley gives us

$$e^x=1+\sum_{n=1}^{\infty}\frac{(n+1)^{n-1}}{n!}\left(\frac{x}{e^x}\right)^n.$$

Bromwich [1] also lists one such series which he derives from Lagrange's expansion. A few such series are found in Hanson's collection [6] on pages 60, 62, 64, and 65. Edwards [2] gives the series for $G(0,-1)$ as a problem.

We will derive the series (3) from the integral (2) in two ways. The first derivation uses the technique employed by Glaister in [3] and [4] and is found in section 2. The second derivation uses a change in variable in the integral (2) and is found in section 3. In section 4 we derive the (possibly new) functional equation

$$F(ab,Q,c;x)=\frac{1}{a^{Q+1}}F\left(b,Q,\frac{c-a+1}{a};\frac{x}{a}\right).$$

A special case of this equation was used by Glaister in his paper [5]. In section 5 we examine an interesting special case of (2) and (3).

The material found in this paper could be used in calculus courses by students studying integration and series. It could also be used in courses in real analysis and numerical analysis.

2. First derivation of the series expansion

We will derive the series (3) starting with the integral (2) using the technique used by Glaister in [3] and [4]. First we expand the integrand of

$$F(P,Q,c;x)=\int_0^1 t^{-xt^P+c}(\log t)^Q dt$$

using the series for the exponential function $e^z=\sum_{n=0}^{\infty}\frac{z^n}{n!}$

$$t^{-xt^P+c}(\log t)^Q = t^c(\log t)^Q e^{-xt^P\log t}$$

$$=\sum_{n=0}^{\infty} t^c(\log t)^{n+Q}\frac{(-1)^n x^n t^{Pn}}{n!}.$$

Assuming that we can integrate this series term by term we get

(4) $$F(P,Q,c;x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!}\int_0^1 t^{Pn+c}(\log t)^{n+Q}dt .$$

Next we define the integrals

$$I(m,n+Q)=\int_0^1 t^{Pm+c}(\log t)^{n+Q}dt .$$

Integrating by parts we get

$$I(m,n+Q)=\int_0^1(\log t)^{n+Q}d\,\frac{t^{Pm+c+1}}{Pm+c+1}$$

$$=-\frac{n+Q}{Pm+c+1}\int_0^1 t^{Pm+c}(\log t)^{n+Q-1}dt\,.$$

Thus we have the reduction formula

(5) $$I(m,n+Q)=-\frac{n+Q}{Pm+c+1}I(m,n+Q-1)\,.$$

To determine an explicit relation for (5) we start with $I(m,0)=\int_0^1 t^{Pm+c}dt=\frac{1}{Pm+c+1}$.

Using (5) we then get

$$I(m,1)=-\frac{1}{Pm+c+1}I(m,0)$$

$$=-\frac{1}{\left(Pm+c+1\right)^2}\,.$$

Continuing in this way we get

$$I(m,2)=\frac{2}{\left(Pm+c+1\right)^3}\,,$$

$$I(m,3)=-\frac{2\cdot 3}{\left(Pm+c+1\right)^4}\,,$$

and in general

$$I(m,n+Q)=\frac{(-1)^{n+Q}(n+Q)!}{(Pm+c+1)^{n+Q+1}}\,.$$

Substituting $I(n,n+Q)$ for the integral in (4) we get the desired series (3) .We note that it is easy to show that (3) converges using the ratio test.

3. Second derivation of the series

In our second derivation, we use the change of variables $t = e^{-v}$ in the integral (2) and get

(6) $$F(P,Q,C;x) = (-1)^Q \int_0^\infty e^{xve^{-Pv}-(c+1)v} v^Q dv.$$

Next we expand the exponential in a Taylor's series

$$e^{xve^{-Pv}} = \sum_{n=0}^\infty \frac{(xv)^n e^{-Pnv}}{n!}.$$

Expression (6) now becomes

(7) $$F(P,Q,C;x) = (-1)^Q \int_0^\infty \left\{ \sum_{n=0}^\infty \frac{(xv)^n e^{-Pnv}}{n!} \right\} e^{-(c+1)v} v^Q dv$$

Assuming that we can interchange summation and integration we get after a little simplification

(8) $$F(P,Q,c;x) = (-1)^Q \sum_{n=0}^\infty \frac{x^n}{n!} \int_0^\infty e^{-(Pn+c+1)v} v^{n+Q} dv.$$

Using the standard integral $\int_0^\infty e^{-sv} v^R dv = \frac{R!}{s^{R+1}}$, (8) becomes

(9) $$F(P,Q,c;x) = (-1)^Q \sum_{n=0}^\infty \frac{(n+Q)!x^n}{n!(Pn+c+1)^{n+Q+1}}.$$

(Students of differential equations will recognize the integral $\int_0^\infty e^{-sv} v^R dv = \frac{R!}{s^{R+1}}$ as the Laplace transform of the function x^R.) This completes our second derivation of the series expansion for our function *F*.

4. A functional equation

We will now derive the useful functional equation

$$F(ab,Q,c;x)=\frac{1}{a^{Q+1}}F\left(b,Q,\frac{c-a+1}{a};\frac{x}{a}\right) \tag{10}$$

$$=\frac{1}{b^{Q+1}}F\left(a,Q,\frac{c-b+1}{b};\frac{x}{b}\right).$$

Starting with (2), we let $v=t^a$ and get after a little simplification

$$F(ab,Q,c,x)=\frac{1}{a^{Q+1}}\int_0^1 v^{-\frac{x}{a}v^b+\frac{c-a+1}{a}}(\log v)^Q\,dv\,. \tag{11}$$

This integral is the first formula in (10). The second formula in (10) comes from letting $v=t^b$ in (2) in the same way.

As an example of the use of our transformation formula, let $a=P$ and $b=1$ in (10) and get

$$F(P,Q,c;x)=\frac{1}{P^{Q+1}}F\left(1,Q,\frac{c-P+1}{P};\frac{x}{P}\right). \tag{12}$$

This last expression shows that the power *P* in our integral (2) can always be transformed into 1. Rewriting (12) in terms of our integral we get

$$\int_0^1 t^{-xt^P+c}(\log t)^Q\,dt=\frac{1}{P^{Q+1}}\int_0^1 t^{-\frac{x}{P}t+\frac{c-P+1}{P}}(\log t)^Q\,dt\,.$$

As a second example, take $a=c+1$, $b=\frac{1}{c+1}$ and $Q=0$ in (10) and get

$$F(1,0,c;x)=\frac{1}{(c+1)^{Q+1}}F\left(\frac{1}{c+1},0,0;\frac{x}{c+1}\right).$$

Writing this last equation in integral form we get

$$\int_0^1 t^{-xt+c}dt=\frac{1}{(c+1)^{Q+1}}\int_0^1 t^{-\frac{x}{c+1}t^{\frac{1}{c+1}}}dt\,.$$

This is the transformation used by Glaister in [5] in which the left integral is improper because he uses $-1 < c < 0$. The integral on the right is not improper in this case.

5. A surprising special case

Consider the special case of our function $F(1,1,0;x) = \int_0^1 t^{-xt}(\log t)dt$. Using (3) we get

$$(13) \qquad F(1,1,0;x) = -\sum_{n=0}^{\infty} \frac{(n+1)!x^n}{n!(n+1)^{n+2}} = -\frac{1}{x}\sum_{n=1}^{\infty}\left(\frac{x}{n}\right)^n.$$

Using (3) again with the special case $F(1,0,0;x) = \int_0^1 t^{-xt}dt$ we get

$$(14) \qquad F(1,0,0;x) = \sum_{n=0}^{\infty} \frac{n!x^n}{n!(n+1)^{n+1}} = \frac{1}{x}\sum_{n=1}^{\infty}\left(\frac{x}{n}\right)^n.$$

Comparing (13) and (14) we see that

$$(15) \qquad F(1,1,0;x) = -F(1,0,0;x).$$

This last relation expressed in integral form is

$$(16) \qquad \int_0^1 t^{-xt}(\log t)dt = -\int_0^1 t^{-xt}dt.$$

We think (16) is a surprising result. We invite the reader to try to prove (16) without the use of the series (3).

6. Final remarks

We note that the series (3) converges very rapidly and provides an excellent way to obtain numeric values for the integral (2) when numeric values are given for the parameters *P*, *Q*, *c*, and *x*. There is much more that can be said concerning the series and integrals studied in this paper. For example, The reader might want to show that equation (15) is a special case of the more general relation

$$F(P,Q,PQ-1;x) = -\frac{1}{P}F(P,Q-1,PQ-1;x)\,.$$

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References

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