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SOME UNUSUAL EXPRESSIONS FOR THE INRADIUS OF A TRIANGLE

Thomas J. Osler, Mathematics Department
and
Tirupathi R. Chandrupatla, Mechanical Engineering

Rowan University
Glassboro, NJ 08028

Osler@rowan.edu
Chandrupatla@rowan.edu

Abstract

Several formulae for the inradius of various types of triangles are derived. Properties of the inradius and trigonometric functions of the angles of Pythagorean and Heronian triangles are also presented. The entire presentation is elementary and suitable for classes in geometry, precalculus mathematics and number theory.

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SOME UNUSUAL EXPRESSIONS FOR THE INRADIUS OF A TRIANGLE

The *incircle* of a triangle is that circle which just touches all three sides of the triangle. Figure 1 shows the incircle for a triangle. It is easy to see that the center of the incircle (*incenter*) is at the point where the angle bisectors of the triangle meet.

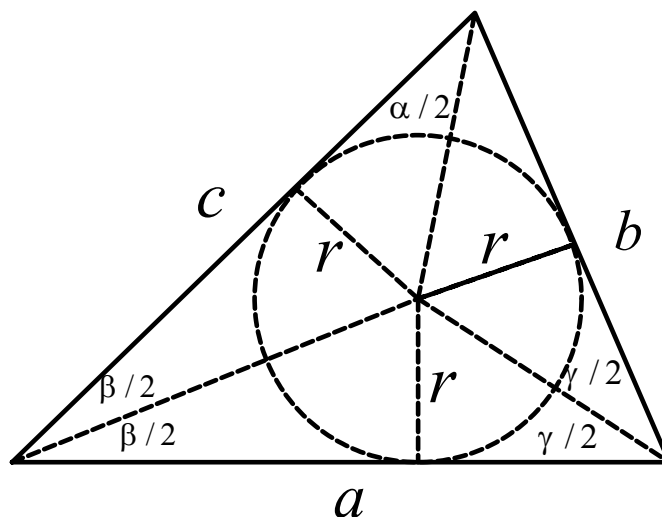


Figure 1

The purpose of this short note is to derive several relations for the inradius r of a triangle. In addition we use these to explore some facts regarding Pythagorean and Heronian triangles. Theorem [1] is found in [2]. We were unable to locate Theorems 2, 6, 7 and 8 in the literature. Theorem 2A is proved in [3], Theorems 3 and 4 are found in [1], and Theorem 5 is essentially proved in [4]. We included them and their proofs to make the presentation complete. The entire presentation is elementary and suitable for classes in geometry, precalculus mathematics and number theory.

Theorem 1: *Let the sides of a triangle be a , b and c . Then the inradius r is given by*

$$(1) \quad r = \frac{2A}{a+b+c}.$$

Proof: Refer to Figure 1. The area A of the triangle is the sum of the areas of the three interior triangles made by the angle bisectors. This sum is

$$A = \frac{ar}{2} + \frac{br}{2} + \frac{cr}{2} = \frac{a+b+c}{2}r$$

Solving this last relation for r we get (1).

Theorem 2: *Let the sides of a triangle be a , b and c , and let the opposite angles be α , β and γ . Then the inradius r is given by the following three expressions.*

$$(2) \quad r = \frac{b+c-a}{2} \tan\left(\frac{\alpha}{2}\right)$$

$$(3) \quad r = \frac{a+c-b}{2} \tan\left(\frac{\beta}{2}\right)$$

$$(4) \quad r = \frac{a+b-c}{2} \tan\left(\frac{\gamma}{2}\right)$$

Proof: The altitude of the triangle is $h = c \sin \beta$, so twice the area is $2A = ac \sin \beta$, and

(1) becomes

$$r = \frac{ac \sin \beta}{a+b+c}.$$

Multiplying and dividing by $(a+c-b)$ we get

$$r = \frac{ac(a+c-b) \sin \beta}{(a+c)^2 - b^2} = \frac{ac(a+c-b) \sin \beta}{a^2 + c^2 - b^2 + 2ac}.$$

From the law of cosines we have $a^2 + c^2 - b^2 = 2ac \cos \beta$, and the last expression becomes

$$r = \frac{ac(a+c-b) \sin \beta}{2ac \cos \beta + 2ac} = \frac{(a+c-b) \sin \beta}{2(1+\cos \beta)}.$$

But $\frac{\sin \beta}{1+\cos \beta} = \tan\left(\frac{\beta}{2}\right)$, and the expression above becomes (3). Relations (2) and (4)

follow from the symmetry of the triangle.

Theorem 2A: *The inradius r of a right triangle with legs a and b and hypotenuse c is*

$$(5) \quad r = \frac{a+b-c}{2}.$$

Proof: Since we have a right triangle, angle $\gamma = \pi/2$. The result (5) follows immediately from (4).

The following theorems follow easily from Theorems 2 and 2A. Recall that a Pythagorean triangle is a right triangle with all three sides integers, and a Heronian triangle has all three sides and area given by integers. A primitive Pythagorean triangle is one in which all three sides have no common factor.

Theorem 3: *The inradius of a Pythagorean triangle is an integer.*

Proof: This follows from (5) and the fact that either one side, or all three sides of the Pythagorean triangle are even numbers.

Theorem 4: *The inradius of a Heronian triangle is a rational number.*

Proof: This follows immediately from (1).

If the inradius of a Heronian triangle is the rational number $r = m/n$, we can multiply all three sides by n to obtain a similar Heronian triangle with *integer inradius* $r = m$.

Theorem 5: *The trigonometric functions of the interior angles α, β and γ of a Heronian triangle are all rational numbers. Also $\tan(\alpha/2)$, $\tan(\beta/2)$ and $\tan(\gamma/2)$ are rational.*

Proof: Refer to Figure 1. From (2), (3) and (4) it follows that the tangents of the half angles are all rational. Since $\tan\alpha = \frac{2\tan(\alpha/2)}{1-\tan^2(\alpha/2)}$, it follows that $\tan\alpha$ is rational. In

the same way $\tan\beta$ and $\tan\gamma$ are rational. From the law of cosines we get

$\cos\alpha = \frac{b^2 + c^2 - a^2}{2bc}$, and thus $\cos\alpha$ is rational. The same must be true for $\cos\beta$ and

$\cos\gamma$. Since the tangents and cosines of all three angles of a Heronian triangle are rational, it follows that the remaining trigonometric functions of these angles are rational.

We can construct a Heronian triangle by joining together two Pythagorean triangles using the following steps:

1. Select any two Pythagorean triangles and place them as shown in Figure 2.

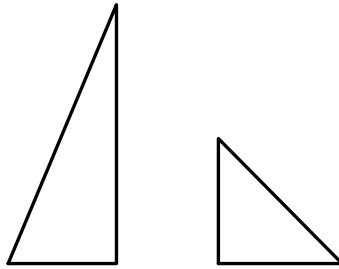


Figure 2

2. Multiplying each triangle by appropriate integers we obtain two new similar Pythagorean triangles with identical altitudes as shown in Figure 3.

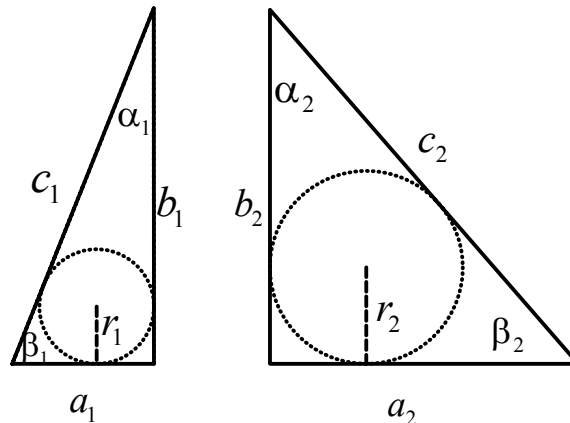


Figure 3

3. Now place the triangles side by side to form a new Heronian triangle as shown in Figure 4. Notice that we have set $h = b_1 = b_2$.

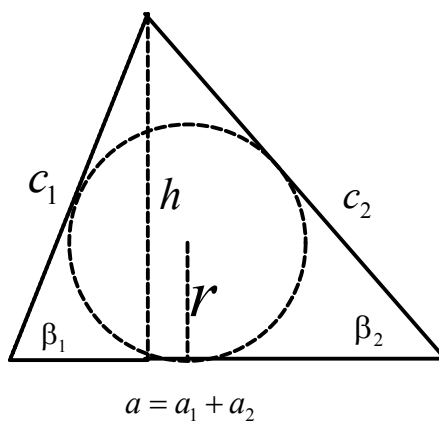


Figure 4

By reversing the steps we can start with any Heronian triangle, and decompose it into two Pythagorean triangles. To see this look at Figure 4 and notice that the right triangle on the left side has altitude $h = c_1 \sin \beta_1$ and base $a_1 = c_1 \cos \beta_1$. Now c_1 is an integer, and by Theorem 5, $\sin \beta_1$ and $\cos \beta_1$ are rational numbers. Thus the right triangle on the left side has rational sides and thus is similar to a Pythagorean triangle. The same is true of the triangle on the right side.

For the moment, let the two triangles in Figure 3 be any two right triangles (not necessarily Pythagorean) with identical altitudes ($h = b_1 = b_2$). The following theorem

relates the inradius r of their union (shown in Figure 4) with the inradii r_1 and r_2 of the individual right triangles.

Theorem 6: *With the notation of Figures 3 and 4 we have*

$$(6) \quad r = (h - r_1 - r_2) \tan\left(\frac{\alpha}{2}\right), \text{ where } \alpha = \alpha_1 + \alpha_2.$$

Proof: From (2) we have

$$\begin{aligned} r &= \frac{c_1 + c_2 - (a_1 + a_2)}{2} \tan \frac{\alpha}{2} \\ &= \frac{c_1 - a_1 + c_2 - a_2}{2} \tan \frac{\alpha}{2}. \end{aligned}$$

Adding and subtracting b_1 and b_2 we get

$$r = \frac{b_1 + b_2 + (c_1 - a_1 - b_1) + (c_2 - a_2 - b_2)}{2} \tan \frac{\alpha}{2}.$$

Using (5) for the inradius of a right triangle we see that $r_1 = \frac{a_1 + b_1 - c_1}{2}$ and

$r_2 = \frac{a_2 + b_2 - c_2}{2}$. Also, $h = b_1 = b_2$ and the above expression becomes (6). The theorem is

proved.

Now consider the case in which the two right triangles used above are congruent. Their resulting union is an isosceles triangle with $c = c_1 = c_2$, $a_1 = a_2$, $\alpha_1 = \alpha_2 = \alpha / 2$, and $r_1 = r_2$. The inradius of this isosceles triangle can now be calculated from (6) as

$$r = (h - 2r_1) \frac{a_1}{h},$$

where we have used $\tan \frac{\alpha}{2} = \tan \alpha_1 = \frac{a_1}{h}$. We can use (5) to get $2r_1 = a_1 + h - c_1$.

Replacing $2r_1$ in the above expression we get $r = (h - (a_1 + h - c_1)) \frac{a_1}{h}$. This simplifies to

$r = (c_1 - a_1) \frac{a_1}{h}$, and we have proved the following theorem.

Theorem 7: *Given an isosceles triangle with base $2a_1$, altitude h and two sides equal to c , the inradius r is given by*

$$(7) \quad r = \frac{(c - a_1)a_1}{h}.$$

Theorem 8: *Suppose a Heronian isosceles triangle is formed by the union of two identical primitive Pythagorean triangles with legs a_1 and h and hypotenuse c . Then the incircle cannot have integer radius.*

Proof: In a primitive Pythagorean triangle, the sides a_1 , h , and c have no common factor.

If the inradius were an integer, then (7) shows that h divides $(c - a)$. If h divides $(c - a)$,

then $\frac{c - a_1}{h} = m$, an integer. But $c^2 - a_1^2 = h^2$, and so $\frac{c - a_1}{h} = \frac{h}{c + a_1}$. This means that

$\frac{h}{c + a_1} = m$ which is impossible since $c + a_1 > h$.

References

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