INT. J. MATH. EDUC. SCI. TECHNOL., 200?
VOL. ??, NO. ?, 1–7

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Classroom note

Another intuitive approach to Stirling's formula

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(*Received 6 September 2002*)

An intuitive derivation of Stirling's formula is presented, together with a modification that greatly improves its accuracy. The derivation is based on the closed form evaluation of the gamma function at an integer plus one-half. The modification is easily implemented on a hand-held calculator and often triples the number of significant digits calculated. While the modification is not new, it does not seem to have the exposure it deserves.

1. Introduction

Rigorous derivations of Stirling's formula

$$n! \sim \sqrt{2\pi n}\, n^n e^{-n} \tag{1}$$

are difficult, often using advanced machinery from complex variable theory [1], or the Euler summation formula [2–4]. (Here the symbol $\sim$ means that the ratio of the left side to the right side approaches 1 as n approaches infinity.) Often the need for (1) arises in elementary courses, so it is introduced to students with an intuitive derivation [5, 6], involving several steps where the professor must do some 'hand waving'. We present here another non-rigorous derivation. We do not claim that it is shorter, or easier than other methods. We also derive the considerably improved modification

$$n! \sim \sqrt{2\pi n}\, n^n e^{-n} \exp\left(\frac{1}{12n}\right)$$

which often triples the number of correct significant digits calculated.

2. An intuitive derivation of Stirling's formula

We start by assuming that the gamma function

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\,\mathrm{d}t, \quad x > 0 \tag{2}$$

International Journal of Mathematical Education in Science and Technology
ISSN 0020–739X print/ISSN 1464–5211 online © 200? Taylor & Francis Ltd
http://www.tandf.co.uk/journals
DOI: 10.1080/00207390310001639925

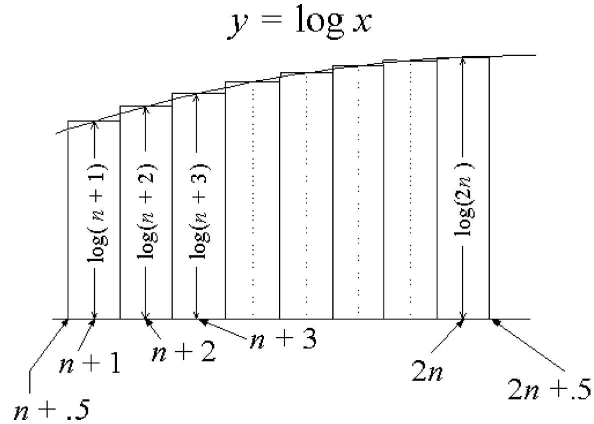

Figure 1. The sum compared to the integral.

has been introduced and that its values at half integers

$$\Gamma(n+1/2) = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n}\sqrt{\pi} = \frac{(2n)!}{2^{2n}n!}\sqrt{\pi}, \quad n = 0, 1, 2, \ldots, \tag{3}$$

have been calculated.

We begin by approximating the expression $(2n)!/n!$ from the right-hand side of equation (3).

$$\begin{aligned}\log\frac{(2n)!}{n!} &= \log(2n)! - \log n! \\ &= \log(n+1) + \log(n+2) + \log(n+3) + \ldots + \log(2n)\end{aligned}$$

From figure 1 we see that the area under the rectangles is nearly the area under the curve $y = \log x$ from $x = n + 0.5$ to $x = 2n + 0.5$. We now have

$$\log\frac{(2n)!}{n!} = \sum_{k=n+1}^{2n} \log k \approx \int_{n+1/2}^{2n+1/2} \log x \,\mathrm{d}x \tag{4}$$

This is our first non-rigorous step, although the picture makes it appear reasonably convincing. Integrating by parts we get

$$\begin{aligned}\log\frac{(2n)!}{n!} &\approx \int_{n+1/2}^{2n+1/2} \log x \,\mathrm{d}x \\ &\approx x\,\log x - x\big|_{n+1/2}^{2n+1/2} \\ &\approx (2n+1/2)\log(2n+1/2) - (n+1/2)\log(n+1/2) - n \\ &\approx \left(2n+\frac{1}{2}\right)\log\left[(2n)\left(1+\frac{1}{4n}\right)\right] - \left(n+\frac{1}{2}\right)\log\left[n\left(1+\frac{1}{2n}\right)\right] - n \\ &\approx \left(2n+\frac{1}{2}\right)\left[\log(2n)+\log\left(1+\frac{1}{4n}\right)\right] - \left(n+\frac{1}{2}\right)\left[\log n+\log\left(1+\frac{1}{2n}\right)\right] - n\end{aligned} \tag{5}$$

Taking the exponential of both sides we get

$$\frac{(2n)!}{n!} \approx \frac{(2n)^{2n+1/2}(1+1/4n)^{2n+1/2}e^{-n}}{n^{n+1/2}(1+1/2n)^{n+1/2}}$$
$$\approx \frac{\sqrt{2}2^{2n}n^n(1+(1/2)/2n)^{2n}\sqrt{1+1/4n}\ e^{-n}}{(1+(1/2)/n)^n\sqrt{1+1/2n}} \tag{6}$$

Recalling that $\lim_{m\to\infty}(1+x/m)^m = e^x$ and $\lim_{n\to\infty}\sqrt{1+a/n} = 1$ it seems reasonable to make the approximations

$$(1+x/m)^m \approx e^x, \quad \text{for large } m \tag{7}$$

and

$$\sqrt{1+a/n} \approx 1, \quad \text{for large } n \tag{8}$$

in the above expression to get

$$\frac{(2n)!}{n!} \sim \sqrt{2}2^{2n}n^n e^{-n} \tag{9}$$

Combining (9) with (3) we have

$$\Gamma(n+1/2) \approx \sqrt{2\pi}n^n e^{-n}$$

Since $x! = \Gamma(x+1)$ when x is a natural number, we can use this expression to define $x!$ when x is any real number. So we can write $(n-1/2)! = \Gamma(n+1/2)$ and we have $(n-1/2)! \approx \sqrt{2\pi}n^n e^{-n}$. While n was an integer in the above argument, the continuity of the gamma function makes it reasonable to replace n with $n+1/2$ and get

$$n! \approx \sqrt{2\pi}\left(n+\frac{1}{2}\right)^{n+1/2} e^{-n-1/2}$$
$$\approx \sqrt{2\pi}n^{n+1/2}\left(1+\frac{1}{2n}\right)^{n+1/2} e^{-n-1/2}$$
$$\approx \sqrt{2\pi n}\, n^n\left(1+\frac{1/2}{n}\right)^n\sqrt{1+\frac{1}{2n}}e^{-n}e^{-1/2}$$

Again using (7) for large n, we can make the substitution

$$\left(1+\frac{1/2}{n}\right)^n \simeq e^{1/2}$$

and also using (8) in the form

$$\sqrt{1+\frac{1}{2n}} \approx 1$$

we get Stirling's approximation.

3. An improvement in Stirling's formula

We now derive an improved version of Stirling's formula. We return to the previous derivation and recall that from figure 1, the area under the rectangles is nearly the area under the curve $y = \log x$ from $x = n + 0.5$ to $x = 2n + 0.5$. We now replace the approximate equation (4) with an exact equation.

The midpoint rule from numerical integration states that

$$\int_a^{a+nd} f(x)dx = d\sum_{k=1}^{n} f(a + kd - d/2) + \frac{d^3}{24}\sum_{k=1}^{n} f''(c_k)$$

where $f(x)$ is a twice differentiable function, and $a + (k-1)d \leqslant c_k \leqslant a + kd$. We now have, using this midpoint rule with $a = n + 1/2$, $f(x) = \log x$ and $d = 1$,

$$\log\frac{(2n)!}{n!} = \sum_{k=n+1}^{2n} \log k = \int_{n+1/2}^{2n+1/2} \log x \, \mathrm{d}x + \sum_{k=1}^{n} \frac{1}{24c_k^2}$$

Here c_k is a point in the interval $[n + k - 1/2, \ n + k + 1/2]$. We can now add the term

$$\sum_{k=1}^{n} \frac{1}{24c_k^2}$$

to the previous approximate equations to convert them to exact equations. Thus approximate equations (5) and (6) are now changed to the exact equations

$$\log\frac{(2n)!}{n!} = \left(2n + \frac{1}{2}\right)\left[\log(2n) + \log\left(1 + \frac{1}{4n}\right)\right] - \left(n + \frac{1}{2}\right)\left[\log n + \log\left(1 + \frac{1}{2n}\right)\right] - n + \sum_{k=1}^{n} \frac{1}{24c_k^2}$$

and

$$\frac{(2n)!}{n!} = \sqrt{2}2^{2n} n^n e^{-n} \sqrt{1 - \frac{1}{4n+2}} \left(1 + \frac{1}{8n(2n+1)}\right)^n \exp\left(\sum_{k=1}^{n} \frac{1}{24c_k^2}\right)$$

Combining this last relation with (3) we have

$$\Gamma(n + 1/2) = \sqrt{2\pi}\, n^n e^{-n} \sqrt{1 - \frac{1}{4n+2}} \left(1 + \frac{1}{8n(2n+1)}\right)^n \exp\left(\sum_{k=1}^{n} \frac{1}{24c_k^2}\right) \quad (10)$$

Up to this point no approximations have been made. Estimating

$$\sum_{k=1}^{n} \frac{1}{24c_k^2}$$

is difficult because all that we know of the numbers c_k is that they are in the interval $[n + k - 1/2, \ n + k/2]$. We make the approximation

$$\sum_{k=1}^{n} \frac{1}{24c_k^2} \approx \int_{n+1/2}^{2n+1/2} \frac{1}{24x^2} \mathrm{d}x = \frac{n}{6(2n+1)(4n+1)}$$

Substituting this last relation into equation (10) we get

$$\Gamma(n + 1/2) \approx \sqrt{2\pi}\, n^n e^{-n} \sqrt{1 - \frac{1}{4n+2}} \left(1 + \frac{1}{8n(2n+1)}\right)^n \exp\left(\frac{n}{6(2n+1)(4n+1)}\right) \quad (11)$$

We will now let $n = z + 1/2$, and think of z as an arbitrary number. Since (10) was derived under the assumption that n was a positive integer, the assumption that (10) is true for non-integer n is a weak point in our derivation. However, the continuity and smoothness of the gamma function make this a reasonable assumption. Now (11) becomes, after the substitution $n = z + 1/2$ and some simplification,

$$\Gamma(z+1) \approx \sqrt{2\pi z}\left(\frac{z}{e}\right)^z \left\{e^{-1/2}\left(1+\frac{1}{2z}\right)^z\right\}\sqrt{\left(1+\frac{1}{2z}\right)\left(1-\frac{1}{4(z+1)}\right)}$$
$$\times\left(1+\frac{1}{16(z+1)(z+1/2)}\right)^{z+1/2}\exp\left(\frac{2z+1}{24(z+1)(4z+3)}\right) \qquad (12)$$

For convenience, we rewrite this last equation as

$$\Gamma(z+1) \approx \sqrt{2\pi z}\, z^z e^{-z} f(z) g(z) h(z) k(z) \qquad (13)$$

where

$$f(z) = e^{-1/2}\left(1+\frac{1}{2z}\right)^z, \quad g(z) = \sqrt{\left(1+\frac{1}{2z}\right)\left(1-\frac{1}{4(z+1)}\right)}$$
$$h(z) = \left(1+\frac{1}{16(z+1)(z+1/2)}\right)^{z+1/2} \quad \text{and} \quad k(z) = \exp\left(\frac{2z+1}{24(z+1)(4z+3)}\right)$$

From (7) and (8) we notice that $f(z)$, $g(z)$, $h(z)$ and $k(z)$ all approach 1 as z approaches infinity. Therefore relation (13) becomes Stirling's formula as in the previous section. If we use the more sensitive approximations $f(z) \approx 1$, $g(z) \approx 1$,

$$h(z) \approx \exp\left(\frac{1}{16(z+1)}\right) \approx \exp\left(\frac{1}{16z}\right), \quad k(z) \approx \exp\left(\frac{1}{48z}\right)$$

we get another very simple but very improved approximation

$$\Gamma(z+1) \approx \sqrt{2\pi z}\left(\frac{z}{e}\right)^z \exp\left(\frac{1}{12z}\right) \qquad (14)$$

It is known that the asymptotic expansion for the gamma function is

$$\Gamma(z+1) = \sqrt{2\pi z}\left(\frac{z}{e}\right)^z\left(1+\frac{1}{12z}+\frac{1}{288z^2}-\frac{139}{51840z^3}-\frac{571}{2488320z^4}+\dots\right).$$

The Taylor's series

$$\exp\left(\frac{1}{12z}\right) = 1+\frac{1}{12z}+\frac{1}{288z^2}+\dots$$

agrees with the first three terms of the above bracketed correction factor to Stirling's formula. Thus we see that expression (15) should be a much improved approximation. (See [3] and [4] for a derivation of a more general expansion of which expression (15) is a special case.)

The following table compares the numerical values of Stirling's formula and our modification (15) with the exact values of the gamma function. We see that the modification is a great improvement. For example, Stirling's formula for 100,000! gives 6 correct significant digits, while expression (15) gives us 17 correct digits!

Number z	$z! = \Gamma(z+1)$	Stirling's Formula $\sqrt{2\pi z}(z/e)^z$	Modified Stirling's Formula $\sqrt{2\pi z}(z/e)^z \exp(1/12z)$
1	1. 00000	0. 92213 70088	1. 00227 44449
10	3 628 800	3 598 695. 61874	3 628 810. 06142
100	9. 33262 15443 94415 E 157	9. 32484 76252 E 157	9. 33262 15703 E 157
1000	4. 02387 26007 70938 E 2677	4. 02353 72920 E 2567	4. 02387 26007 82115 E 2567
10000	2. 84625 96809 17055 E 35659	2. 84623 59621 E 35659	2. 84625 96809 17062 E 35659
100000	2. 82422 94079 60347 87429 E 456573	2. 82422 70544 E 456573	2. 82422 94079 60347 88214 E 456573

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In most of these examples we triple the number of correct digits! This simple modification is not new, and is very easily implemented on a hand-held calculator. However, it does not seem to have the exposure it deserves.

4. Final remarks

There have been many elementary derivations of Stirling's formula and its relatives, some rigorous, some intuitive [7–21].

From 1982 to 1987, a flurry of articles and letters appeared in *The American Journal of Physics* discussing the derivations and merits of various approximations to the factorial [22–30].

Acknowledgment

The author wishes to thank the referee for many valuable suggestions. In particular, the referee's recommendation to use the midpoint rule in section 3 led to the derivation of the improvement in Stirling's formula.

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