January 12, 1999

The Tautochrone Under Arbitrary Potentials Using Fractional Derivatives

American Journal of Physics, (Am. J. Phys.), 67(1999), pp. 718-722.

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Abstract

The classical tautochrone problem involves motion along curves caused by the special potential $V(y) = mgy$. We use fractional derivatives to find tautochrone curves under arbitrary potentials $V(y)$. We generalize these further to potentials that are functions of two variables $V(x, y)$. An appendix gives intuitive motivation for the fractional calculus employed.

1. Introduction

The classical tautochrone problem is to find the curve $x = x(y)$, passing through the origin, along which a point mass will descend without friction, in the same time regardless of the point $(x(Y), Y)$ at which it starts. We assume that the initial velocity is zero, the mass is unity, and that the potential is $V(y) = gy$. One solution to this classical problem is due to Abel (1823) in which he initiated the theory of integral equations. His solution is a cycloid and can be found in Miller and Ross' book[1] pages 255-260, Oldham and Spanier's book[2] pages 183-186 and in Wheeler's notes[3] pages 19-22.

In this paper we show how the classical solution, using fractional derivatives, can easily be extended to motion under the influence of arbitrary potentials $V(y)$. The advantage of the fractional calculus over other methods of solution is that the notation itself suggests the appropriate manipulations. For this reason we will manipulate formally and at times ignore the needs of mathematical rigor. By consulting the references, all steps can be justified. Finally, we show how our one-dimensional potentials $V(y)$ can be transformed into two-dimensional potentials with the same tautochrone curves.

2. Ideas needed from the fractional calculus

To solve this tautochrone problem, we will need a few ideas from the fractional calculus. The first is the notion of the Riemann-Liouville integral

$$D_x^\alpha f(x) = \frac{1}{\Gamma(-\alpha)} \int_0^x \frac{f(t)dt}{(x-t)^{\alpha+1}} \ , \tag{1}$$

This integral enables us to find the derivative of arbitrary order α of the function $f(x)$. While this integral converges only for $\alpha < 0$, the notion of fractional derivative can be extended to all orders in a simple way. Oldham and Spanier's book[2] (page 49), shows the details.

We will need the fractional derivative of $f(x) = x^p$ which is

$$D_x^\alpha x^p = \frac{\Gamma(p+1)x^{p-\alpha}}{\Gamma(p-\alpha+1)} \tag{2}$$

The expression (2) can be obtained from (1) using the beta integral (Oldham and Spanier[2], page 66).

The familiar law of indexes

$$D_x^{\alpha} D_x^{\beta} f(x) = D_x^{\alpha+\beta} f(x) \tag{3}$$

is in general true for fractional derivatives. There are however, special cases of the function in which this law must be modified. We need not concern ourselves with this complication here, but the details can be found in Oldham and Spanier's book[2], pages 82 to 87.

Finally we need the notion of the fractional derivative of $f(x)$ with respect to another function $g(x)$. A simple change of variable converts (1) to

$$D_{g(x)}^{\alpha} f(x) = \frac{1}{\Gamma(-\alpha)} \int_{g^{-1}(0)}^{x} \frac{f(t)g'(t)dt}{(g(x)-g(t))^{\alpha+1}} \tag{4}$$

This idea is considered in the work of Lavoie, Osler, and Trembley[4], on page 257.

In addition to the references given above, a brief intuitive explanation of the above four results is given in the appendix to this paper.

Fractional calculus has been an invaluable tool in physics. Fractional calculus is the mathematical tool that made possible dimensional regularization[5] [5]. Dimensional regularization is a way to handle the divergent integrals of the "Standard Model" of particle physics. By integrating in a dimension slightly smaller than 4, we can separate finite and infinite terms in the integral and we can assign physical significance to the finite terms. The relativistic tautochrone problem for a constant force field is another example in physics where fractional calculus has been used to find a solution[6].

3. The tautochrone under a potential V(y)

We are now ready to solve the tautochrone problem under a general potential $V(y)$. Suppose we have a bead of unit mass, which starts from rest at the point (X,Y)

on the curve $x = x(y)$. Without loss of generality, we assume the curve passes through the origin. If the bead descends without friction, then conservation of energy gives us

$$\frac{v^2}{2} = V(Y) - V(y) \tag{5}$$

where $v = -\frac{ds}{dt}$ is the velocity and s is the arc length from the origin to the point $\left(x(y), y\right)$. Now (5) becomes

$$-\frac{ds}{\sqrt{V(Y) - V(y)}} = \sqrt{2}dt \tag{6}$$

Integrating (6) from the starting point at $t = 0$ where $y = Y$ to the final point where $t = T$ and $y = 0$ we get

$$\int_0^Y \frac{ds}{\sqrt{V(Y) - V(y)}} = \sqrt{2}T \tag{7}$$

Next we modify the integrand and divide by $\Gamma(1/2) = \sqrt{\pi}$ to get

$$\frac{1}{\Gamma(1/2)} \int_0^Y \frac{\frac{ds}{dV(y)} V'(y)}{\sqrt{V(Y) - V(y)}} dy = \sqrt{2/\pi}\; T \tag{8}$$

Comparing (8) with (4) we see that the left side is the fractional derivative of order $-1/2$ of the function $\frac{ds}{dV(y)}$ with respect to $V(y)$. (Notice that the lower limit of integration requires that $V^{-1}(0) = 0$. This can always be achieved by adding an appropriate constant to the potential that has no effect on the motion.) Thus, equation (8) in fractional derivative notation becomes

$$D_{V(y)}^{-1/2} \frac{ds}{dV(y)} = \sqrt{2/\pi}\, T \tag{9}$$

Since $\frac{ds}{dV(Y)}$ can also be written as $D^{1}_{V(Y)}s$ equation (9) becomes

$$D^{-1/2}_{V(Y)}D^{1}_{V(Y)}s = \sqrt{2/\pi}\,T \tag{10}$$

Using the law of indexes (3) and substituting y for Y we now have

$$D^{1/2}_{V(y)}s = \sqrt{2/\pi}\,T$$

Recall that T, the time of descent, is a constant. Next we solve for s by operating on both sides of (11) with $D^{-1/2}_{V(y)}$ to get

$$s = \sqrt{2/\pi}\,T\ D^{-1/2}_{V(y)}1 \tag{11}$$

The fractional derivative of a constant is not in general zero. We use (2) with $p = 0$ to evaluate this last fractional derivative to get

$$s = \frac{2\sqrt{2V(y)}}{\pi}T \tag{12}$$

Finally we solve (12) for the potential $V(y)$ in terms of the arc length s to get

$$V(y) = \frac{\pi^2}{8T^2}s^2 \tag{13}$$

This last relation will allow us to find several tautochrone curves for potentials $V(y)$ that differ from the usual $V(y) = gy$. The easiest way to do this is to start with known curves, which pass through the origin, whose arc lengths s are given by elementary expressions. We can then use (13) to find the potential $V(y)$ that makes the given curve a tautochrone. We have actually solved our problem in reverse. We started with a mathematically nice curve, and calculated the potential, which makes that curve a tautochrone.

4. Examples of tautochrone curves under a potential V(y)

The following table shows the results of our calculations. To check our results the reader should first graph the curve given in the first column. Next check the formula for the arc length from the origin to the point (x, y) on the given curve. Finally, the potential under which this curve acts as a tautochrone is calculated from equation (13) and is listed in the last column. The reader can check that the curves are truly tautochrones by showing directly that the integral (7) is a constant.

Remarks on the table:

The first three curves in the table are circles of radius R passing through the origin. The centers of the circles are $(0, R)$, $(R,0)$ and $(R\sin(\alpha), R\cos(\alpha))$ respectively. Curve 4 is the standard inverted cycloid that yields the solution to the classical tautochrone problem.

Notice also that all the tautochrone curves pass through the origin of coordinates and that all the potentials satisfy V(0)=0. Both these conditions were a consequence of our fractional derivative solution. Adding an appropriate constant to the potential can always satisfy the later condition. The former condition can be satisfied by replacing y by $y - y_0$ in the potential so as to move the y intercept (y_0) of the tautochrone to the origin. Entry 9 in the Table illustrates this point. By replacing y by $y - a$, the tautochrone curve becomes $y = a\cosh(x/a)$ and the potential becomes $\frac{\pi^2}{8T^2}\left[y^2 - a^2\right]$.

These are algebraically nicer results than those in the Table are.

Table 1: Examples of Potentials and Corresponding Tautochrone Curves

	Tautochrone Curve	**Arc length s**	**Potential V(y)**
1	$x = R\sin\theta$, $y = R - R\cos\theta$ **circle: center (0,R), radius R**	$s = R\theta = R\cos^{-1}\left(\frac{R-y}{R}\right)$	$\frac{\pi^2 R^2}{8T^2}\left\{\cos^{-1}\left(\frac{R-y}{R}\right)\right\}^2$
2	$x = R - R\cos\theta$ $y = R\sin\theta$ **circle: center (R,0), radius R**	$s = R\theta = R\sin^{-1}(y/R))$	$\frac{\pi^2 R^2}{8T^2}\left\{\sin^{-1}(y/R)\right\}^2$
3	$x = R\sin(\alpha+\theta) - R\sin\alpha$ $y = R\cos\alpha - R\cos(\alpha+\theta)$, R *and* α *fixed* **circle: radius R** **center (-R sinα , R cos α)**	$s = R\theta$ $\theta = \cos^{-1}\left(\frac{R\cos(\alpha) - y}{R}\right) - \alpha$	$\frac{\pi^2 R^2}{8T^2}\left\{\cos^{-1}\left(\frac{a-y}{R}\right) - \alpha\right\}^2$ *where* $a = R\cos(\alpha)$
4	$x = R(\theta + \sin\theta)$ $y = R(1 - \cos\theta)$ **inverted cycloid: base line y=2R**	$s = 4R\sin(\theta/2)$ $s = 2\sqrt{2Ry}$	$\frac{\pi^2 R}{T^2}y$
5	$x = R\theta - R\sin\theta$ $y = R - R\cos\theta$ **cycloid**	$s = 4R(1 - \cos(\theta/2))$ $s = 4R\left(1 - \sqrt{1 - \frac{y}{2R}}\right)$	$\frac{2\pi^2 R^2}{T^2}\left(1 - \sqrt{1 - \frac{y}{2R}}\right)^2$
6	$x = ay$	$s = \sqrt{1+a^2}\,y$	$\frac{\pi^2(1+a^2)}{8T^2}y^2$
7	$x = 2\sqrt{ay^3}/3$	$s = \frac{2}{3a}\left[(1+ay)^{3/2} - 1\right]$	$\frac{\pi^2}{18a^2T^2}\left[(1+ay)^{3/2} - 1\right]^2$
8	$x = ay^2/2$	$s = (ay\sqrt{1+a^2y^2} +$ $\ln(ay + \sqrt{1+a^2y^2}))/2a$	$\frac{\pi^2}{32a^2T^2}(ay\sqrt{1+a^2y^2} +$ $\ln(ay + \sqrt{1+a^2y^2}))^2$
9	$y = a\cosh(x/a) - a$	$s = \sqrt{(y+a)^2 - a^2}$	$\frac{\pi^2}{8T^2}\left[(y+a)^2 - a^2\right]$

5. The tautochrone under arbitrary potentials V(y).

We will now find an expression for the tautochrone curve x = x(y) under an arbitrary potential V(y). Differentiate (13) with respect to y to obtain

$$\frac{ds}{dy} = \frac{\sqrt{2}T}{\pi\sqrt{V(y)}} V'(y) .$$

Since $ds/dy = \sqrt{1 + x'(y)^2}$ this last expression becomes (after squaring)

$$1 + x'(y)^2 = \frac{2T^2}{\pi^2}\frac{V'(y)^2}{V(y)} .$$

Solving for $x'(y)$ we get

$$\frac{dx}{dy} = \sqrt{\frac{2T^2}{\pi^2}\frac{V'(y)^2}{V(y)} - 1} .$$

Integrating from the origin to any point on the tautochrone curve we get

$$x(y) = \int_0^y \sqrt{\frac{2T^2}{\pi^2}\frac{V'(u)^2}{V(u)} - 1}\, du \tag{14}$$

This is our tautochrone curve x = x(y) which passes through the origin under the influence of the arbitrary potential V(y). We recall that without loss of generality, it may be necessary to add a constant to V(y) so that V(0) = 0.

The simplest example of the use of (14) comes from the potential $V(y) = ky^2$. In this case the expression under the radical in (14) is a constant, and the solution is a straight line through the origin. This checks with entry 6 in our Table. All the tautochrones in our Table could have been derived from (14) if we started with the potential V(y) as given. The difficulty with (14) is that for most potentials, the resulting integral can not be evaluated in terms of known functions.

There are several interesting results that can be derived from (14). In particular, for a given potential V(y), a given time of descent T and a given initial height Y, (14) can determine if a tautochrone exists for these conditions. We require that the expression under the radical be non-negative. For example a potential of the form $V = cy^3$ will not produce a tautochrone curve since the radical will be imaginary near the origin. We will not explore this matter further here.

6. The tautochrone under two dimensional potentials V(x,y)

The potentials we obtained in the above table are only functions of y, and not x. However, for each of the tautochrone curves $x = x(y)$ we obtained above for a potential $V(y)$, it is easy to obtain potentials $V(x,y)$ for which this curve is still a tautochrone. Suppose we have a function $F(x,y)$ for which $x = x(y)$ is a level line. Then $F\left(x(y),y\right)$ is a constant for all y. Now consider the new two-dimensional potential $V(x,y) = V(y) + F(x,y)$. Since this new potential $V(x,y)$ is a constant plus the old potential $V(y)$ along the curve $x = x(y)$, the acceleration is exactly the same under the influence of the potentials $V(y)$ and $V(x,y)$. Therefore, the curve $x = x(y)$ is a tautochrone curve under the new potential $V(x,y)$.

As an example, consider entry 6 in our table. Here we can write $x = ay$ for the tautochrone curve and $V(y) = b^2y^2$ for the potential. To simplify matters we have written $b^2 = \pi^2(1+a^2)/T^2$. Suppose we add to the potential $V(y)$ the function $F(x,y) = c^2(x^2 - a^2y^2)$. Notice that the function $F(x,y) = 0$ along the tautochrone curve $x = ay$. Thus we have

$$V(x,y) = V(y) + F(x,y)$$

$$= b^2y^2 + c^2(x^2 - a^2y^2)$$

$$= c^2x^2 + (b^2 - a^2c^2)y^2 .$$

Since b and c are arbitrary this last result can be written simply as

$$V(x,y) = c^2x^2 + d^2y^2$$

for arbitrary c and d. With $c = d$ we have the important case of a spring with one end fixed at the origin.

This completes our examination of the tautochrone curves under arbitrary potentials.

Appendix: A quick look at the fractional calculus

Since most readers are unfamiliar with the fractional calculus, we present a brief intuitive introduction to the topics needed in this paper. This introduction is informal and non-rigorous. Here we wish to gain quick insight into the four relations from section 2.

In the fractional calculus we extend the meaning of

$$\frac{d^n f(x)}{dx^n} = D_x^n f(x) = D^n f(x)$$

where the order of the derivative is $n = 0,1,2,3,\cdots$, to $D^\alpha f(x)$ where now α can be any number, integer, rational, irrational or complex.

The Riemann-Liouville integral:

Since D^n means repeat differentiation n times, it is not unreasonable to imagine that D^{-n} would mean integration repeated n times. We could call D^{-1} the integral where

$$D^{-1}f(x) = \int f(x)dx .$$

This integral involves an arbitrary constant since it is indefinite. To fix this constant we will instead use the notation

$$D^{-1}f(x) = \int_0^x f(t)dt .$$

The second integral will then be

$$D^{-2}f(x) = \int_0^x \int_0^{t_2} f(t_1)dt_1dt_2 . \tag{15}$$

This is a double iterated integral that is studied in calculus. Let us look at the limits of integration. First, we integrate in t_1 from 0 to t_2. Then we integrate in t_2 from 0 to x. We will interchange the order of integration to produce

$$\int_0^x \int_{t_1}^x f(t_1)dt_2dt_1 .$$

Since $f(t_1)$ is not a function of t_2, it can be moved outside the inner integral. We now have

$$\int_0^x f(t_1)\int_{t_1}^x dt_2dt_1 .$$

The inner integral is equal to $x - t_1$, and making this substitution we can write (15) as a single integral:

$$D^{-2}f(x) = \int_0^x f(t)(x-t)dt .$$

Using the same procedure we can show that

$$D^{-3}f(x)=\frac{1}{2}\int_0^x f(t)(x-t)^2\,dt$$

$$D^{-4}f(x)=\frac{1}{2\cdot 3}\int_0^x f(t)(x-t)^3\,dt\,.$$

At this point you can see the factorial starting to emerge. Using the above arguments, we can write the nth iterated integral as a single integral denoted

$$D^{-n}f(x)=\frac{1}{(n-1)!}\int_0^x f(t)(x-t)^{n-1}\,dt\,. \tag{16}$$

An interesting side note here is the fact that (16) is the “ghost” of Cauchy's integral formula in the real domain. Now, it is not unreasonable to replace the $-n$ with arbitrary α. To this end, we use the gamma function to replace the factorial and get

$$D_x^{\alpha}f(x)=\frac{1}{\Gamma(-\alpha)}\int_0^x \frac{f(t)\,dt}{(x-t)^{\alpha+1}} \tag{17}$$

This is our relation (1) and it is known as the *Riemann-Liouville integral*. It gives us the general formula from which to find the fractional derivative of a wide class of functions. We note that the integral diverges if $0\le\alpha$. There are ways around this restriction, but we will not dwell on the matter further here.

The fractional derivative of x to a power:

Let $f(x)=x^p$ in (17) and get

$$D^{\alpha}x^{p}=\frac{1}{\Gamma(-\alpha)}\int_0^x \frac{t^p\,dt}{(x-t)^{\alpha+1}} \tag{18}$$

If we let $t=xu$ in (18) and use the well known beta integral

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}=\int_0^1 \frac{u^{a-1}du}{(1-u)^{1-b}} \tag{19}$$

we obtain

$$D^{\alpha} x^{p} = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} x^{p-\alpha} . \tag{20}$$

This is the relation (2) in the introduction. It is very important because it enables us to find the fractional derivative of any function of the form $x^{p} g(x)$ where p is any number and $g(x)$ can be expanded in a power series

$$g(x) = a_0 + a_1 x + a_2 x^2 + \cdots . \tag{21}$$

After differentiating fractionally term by term we get

$$D^{\alpha} x^{p} g(x) = \sum_{n=0}^{\infty} \frac{a_n \Gamma(p+n+1)}{\Gamma(p-\alpha+n+1)} x^{p-\alpha+n} . \tag{22}$$

This last relation is very useful.

The law of exponents:

Next we examine the validity of the law of exponents $D^{\alpha} D^{\beta} f(x) = D^{\alpha+\beta} f(x)$. Suppose that $f(x) = x^{p} g(x)$, where $g(x)$ can be expanded in a power series (21). Then

$$D^{\alpha} D^{\beta} f(x) = D^{\alpha} (D^{\beta} x^{p} g(x))$$

Using (22) we get

$$D^{\alpha} D^{\beta} f(x) = D^{\alpha} \left(\sum_{n=0}^{\infty} \frac{a_n \Gamma(p+n+1)}{\Gamma(p-\beta+n+1)} x^{p-\beta+n} \right). \tag{23}$$

Again we use (22) to evaluate this last fractional derivative to get

$$D^{\alpha} D^{\beta} f(x) = \sum_{n=0}^{\infty} \frac{a_n \Gamma(p+n+1) \Gamma(p-\beta+n+1)}{\Gamma(p-\beta+n+1) \Gamma(p-\alpha-\beta+n+1)} x^{p-\alpha-\beta+n} \tag{24}$$

In the above relation, two gamma functions cancel to give us

$$D^{\alpha} D^{\beta} f(x) = \sum_{n=0}^{\infty} \frac{a_n \Gamma(p+n+1)}{\Gamma(p-\alpha-\beta+n+1)} x^{p-\alpha-\beta+n} . \tag{25}$$

Comparing this last relation with (22) we see that

$$D^{\alpha} D^{\beta} f(x) = D^{\alpha+\beta} f(x) \tag{26}$$

and we have verified the law of exponents.

There is however a possible flaw in the above argument. The gamma function $\Gamma(z+1)$ has singularities where z is a negative integer. This means we can think of the gamma function as equating "infinity" at these singularities. Returning to (23) we see that when $p-\beta = -N$, a negative integer, then $\Gamma(p-\beta+n+1) = \infty$ for $n = 0,1,2,\cdots, N-1$. This will make the first N terms in the series in (23) vanish. Because of this, correction terms must then be added to (26) when $p-\beta$ is a negative integer. We will get

$$D^{\alpha+\beta} f(x) = D^{\alpha} D^{\beta} f(x) + \sum_{n=0}^{N-1} \frac{\Gamma(p+n+1) a_n x^{p-\alpha-\beta+n}}{\Gamma(p-\alpha-\beta+n+1)}.$$

We will have no need to discuss this complication further here.

The derivative of f(x) with respect to g(x):

Finally we derive the expression (4)

$$D^{\alpha}_{g(x)} f(x) = \frac{1}{\Gamma(-\alpha)} \int_{g^{-1}(0)}^{x} \frac{f(t) g'(t) dt}{(g(x)-g(t))^{\alpha+1}} \tag{27}$$

from the Riemann-Liouville integral (17) which we write below with a simple change of variables

$$D^{\alpha}_{z} F(z) = \frac{1}{\Gamma(-\alpha)} \int_{0}^{z} \frac{F(u) du}{(z-u)^{\alpha+1}}. \tag{28}$$

Set $z = g(x)$, $u = g(t)$ and $F(z) = f(g^{-1}(z)) = f(x)$ in (28) to get

$$D^{\alpha}_{g(x)} f(x) = \frac{1}{\Gamma(-\alpha)} \int_{0}^{z} \frac{f(g^{-1}(u))du}{(z-u)^{\alpha+1}}$$

$$= \frac{1}{\Gamma(-\alpha)} \int_{g^{-1}(0)}^{x} \frac{f(t)g'(t)dt}{(g(x)-g(t))^{\alpha+1}}$$

This completes our derivation of the derivative of f(x) with respect to g(x).

The reader should consult the references for a full discussion of the fractional calculus.

Acknowledgment:

The authors wish to thank the referees for improving the paper and for adding example 9 in the Table.

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