UNEXPECTED CONSTRUCTIBLE NUMBERS

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One of the three classical problems of antiquity is to "duplicate the cube (Burton, 1999). Given the side *s* of a cube, construct, using only straightedge and compass, the side of a cube having twice the volume. (Note that the straightedge must be unmarked.) Thus, to duplicate the cube, one must construct the length (number) $\sqrt[3]{2}\,s$. It is known that the only possible constructions are formed by adding, subtracting, multiplying, dividing and taking the square root of previously found lengths. These constructions may only be performed a finite number of times. (Courant and Robbins,1996, chapter 3) has a thorough discussion of constructible numbers, while the popular websites of (Weisstein) give an overview. While the square root of any given length can be constructed, it is impossible to construct the cube root of an arbitrarily given length. In (Courant and Robbins, 1996, pages 134-5), it is shown using an elementary argument, that the cube root of 2 is not constructible. Of course, the cube roots of certain special lengths are constructible, such as segments of length 1, 8, 27, 64, …, since these are exact cubes.

Which of the following numbers looks more complicated, $\sqrt[3]{2}$ or $\sqrt[3]{2+\sqrt{5}}$? Of course, your first reaction is to say that $\sqrt[3]{2+\sqrt{5}}$ is more complicated than $\sqrt[3]{2}$. However, it is easy to show that

$$\sqrt[3]{2+\sqrt{5}} = \frac{1+\sqrt{5}}{2}. \tag{1}$$

This last number is the “golden section,” often denoted by ϕ . Now ϕ is constructible, since it involves only addition, division and taking a square root, while $\sqrt[3]{2}$ is not constructible!

To see the truth of (1), simply cube $\frac{1+\sqrt{5}}{2}$ and you will get $2+\sqrt{5}$. This last computation suggests a way to generate more examples of this type. Let *a* and *b* be rational numbers with $b > 0$. Then $\left(a+\sqrt{b}\right)^3 = \left(a^3+3ab\right)+\left(3a^2+b\right)\sqrt{b}$, so

$$\sqrt[3]{\left(a^3+3ab\right)+\left(3a^2+b\right)\sqrt{b}} = a+\sqrt{b}. \tag{2}$$

The complicated looking lion of a cube root on the left is really the constructible lamb on the right. Our example (1) is the special case of (2) in which $a = \frac{1}{2}$, and $b = \frac{5}{4}$. For a more thorough discussion of interesting radicals see (Osler, 2001).

The book (Courant and Robbins, 1996) is a classic and is now available in an inexpensive paperback edition. A very readable discussion of constructible numbers, with complete proofs that require only a precalculus background, is found in chapter 3. It is strongly recommended.

References

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