An unusual product for $\sin z$ and variations of Wallis's product

1. *Introduction*

When we think of a product expansion for the sine function, we usually think of Euler's product $\sin z = z \prod_{k=1}^{\infty}\left(1 - \frac{z^2}{k^2\pi^2}\right)$. In the Cambridge Mathematical Tripos of 1904 (possibly the oldest university examination in the world), a problem was given which generalises Euler's product. A simplified version of the expression to be verified on the Tripos is

$$\sin z = \exp\left(\frac{z}{\pi}\log\frac{M}{N}\right) z \prod_{k=0}^{\infty}\left(1 - \frac{z/\pi}{kM+1}\right)\left(1 - \frac{z/\pi}{kM+2}\right)\dots\left(1 - \frac{z/\pi}{kM+M}\right)$$
$$\left(1 + \frac{z/\pi}{kN+1}\right)\left(1 + \frac{z/\pi}{kN+2}\right)\dots\left(1 + \frac{z/\pi}{kN+N}\right). \qquad (1)$$

(Here M and N are natural numbers.) This Tripos problem also appears in Whittaker and Watson [1] on page 40 as problem 17.

It is the purpose of this note to prove (1) and to use it to derive variations of Wallis's product.

Special cases of (1) have also appeared in classical texts. In Whittaker and Watson [1], on page 35 we find a very brief derivation of the special case where $M = 2$ and $N = 1$:

$$\sin z = \exp\left(\frac{z}{\pi}\log 2\right) z \left\{\left(1 - \frac{z}{\pi}\right)\left(1 - \frac{z}{2\pi}\right)\left(1 + \frac{z}{\pi}\right)\right\}$$
$$\left\{\left(1 - \frac{z}{3\pi}\right)\left(1 - \frac{z}{4\pi}\right)\left(1 + \frac{z}{2\pi}\right)\right\}\dots. \qquad (2)$$

In Jolly [2, p. 190] we find the case where $M = 3$ and $N = 1$:

$$\sin z = \exp\left(\frac{z}{\pi}\log 3\right) z \left\{\left(1 - \frac{z}{\pi}\right)\left(1 - \frac{z}{2\pi}\right)\left(1 - \frac{z}{3\pi}\right)\left(1 + \frac{z}{\pi}\right)\right\} \times$$
$$\left\{\left(1 - \frac{z}{4\pi}\right)\left(1 - \frac{z}{5\pi}\right)\left(1 - \frac{z}{6\pi}\right)\left(1 + \frac{z}{2\pi}\right)\right\}\dots. \qquad (3)$$

2. *Two lemmas*

We now prove two lemmas needed in our derivation of (1) in the next section. First we will prove a lemma concerning the conditionally convergent series $\frac{1}{1} - \frac{1}{1} + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \frac{1}{4} - \frac{1}{4} + \dots$ and the result of rearranging its terms in a special way. Let M and N be positive integers. Define the infinite series $S(M, N)$ by

$$S(M, N) = \left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{M} - \frac{1}{1} - \frac{1}{2} - \frac{1}{3} - \dots \frac{1}{N}\right) +$$

$$\left(\frac{1}{M+1}+\frac{1}{M+2}+\ \dots\ +\frac{1}{2M}-\frac{1}{N+1}-\frac{1}{N+2}-\ \dots\ \frac{1}{2N}\right)+$$
$$\left(\frac{1}{2M+1}+\frac{1}{2M+2}+\ \dots\ +\frac{1}{3M}-\frac{1}{2N+1}-\frac{1}{2N+2}-\ \dots\ \frac{1}{3N}\right)+$$
$$\vdots \qquad (4)$$

To simplify the notation we introduce

$$h(m,\ d)\ =\ \frac{1}{m\ +\ 1}\ +\ \frac{1}{m\ +\ 2}\ +\ \frac{1}{m\ +\ 3}\ +\ \dots\ +\ \frac{1}{m\ +\ d}.$$

Then our series (4) becomes

$$S(M,\ N)\ =\ \sum_{k=0}^{\infty}\bigl(h(kM,\ N)\ -\ h(kN,\ N)\bigr). \qquad (5)$$

Lemma 1:

$$S(M,\ N)\ =\ \log\frac{M}{N}. \qquad (6)$$

Proof:

Consider the first n terms of the series which defines $S(M,\ N)$.

$$S_n(M,\ N)\ =\ \sum_{k=0}^{n-1}\bigl(h(kM,\ N)\ -\ h(kN,\ N)\bigr).$$

Adding and subtracting $\log(nM)$ and $\log(nN)$ we have

$$S_n(M,N)=\left(\sum_{k=0}^{n-1}h(kM,M)-\log(nM)\right)-\left(\sum_{k=0}^{n-1}h(kN,N)-\log(nN)\right)$$
$$+\log(nM)-\log(nN).$$

This simplifies to

$$S_n(M,\ N)\ =\ \left(\sum_{k=1}^{nM}\frac{1}{k}\ -\ \log(nM)\right)\ -\ \left(\sum_{k=1}^{nN}\frac{1}{k}\ -\ \log(nN)\right)\ +\ \log\frac{M}{N}.$$

Using the fact that $\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}\ -\ \log n\right)\ =\ \gamma$, where γ is Euler's constant we have

$$S(M,\ N)\ =\ \lim_{n\to\infty}S_n(M,\ N)\ =\ \gamma\ -\ \gamma\ +\ \log\frac{M}{N}\ =\ \log\frac{M}{N}.$$

This completes the proof of lemma 1.

Lemma 2:

Let $S(n)=\prod_{k=1}^{Mn-Nn}\left(1-\frac{w}{Nn+k}\right)\exp\left(\frac{w}{Nn+k}\right)$, let w be any complex

M	N	z	Wallis-like Infinite Products
1	1	$\frac{\pi}{2}$	$\frac{2}{\pi} = \left(\frac{1\times3}{2\times2}\right)\left(\frac{3\times5}{4\times4}\right)\left(\frac{5\times7}{6\times6}\right)\left(\frac{7\times9}{8\times8}\right)\ldots$ (Original Wallis product)
1	1	$\frac{\pi}{3}$	$\frac{3\sqrt{3}}{2\pi} = \left(\frac{2\times4}{3\times3}\right)\left(\frac{5\times7}{6\times6}\right)\left(\frac{8\times10}{9\times9}\right)\left(\frac{11\times13}{12\times12}\right)\ldots$
2	1	$\frac{\pi}{2}$	$\frac{\sqrt{2}}{\pi} = \left(\frac{1\times3}{2\times4} \times \frac{3}{2}\right)\left(\frac{5\times7}{6\times8} \times \frac{5}{4}\right)\left(\frac{9\times11}{10\times12} \times \frac{7}{6}\right)\left(\frac{13\times15}{14\times16} \times \frac{9}{8}\right)\ldots$
2	1	$\frac{\pi}{4}$	$\frac{2\sqrt[4]{2}}{\pi} = \left(\frac{3\times7}{4\times8} \times \frac{5}{4}\right)\left(\frac{11\times15}{12\times16} \times \frac{9}{8}\right)\left(\frac{19\times23}{20\times24} \times \frac{13}{12}\right)\left(\frac{27\times31}{28\times32} \times \frac{17}{16}\right)\ldots$
2	1	$\frac{\pi}{3}$	$\frac{3\sqrt{3}}{2\sqrt[3]{2}\pi} = \left(\frac{2\times5}{3\times6} \times \frac{4}{3}\right)\left(\frac{8\times11}{9\times12} \times \frac{7}{6}\right)\left(\frac{14\times17}{15\times18} \times \frac{10}{9}\right)\left(\frac{20\times23}{21\times24} \times \frac{13}{12}\right)\ldots$
2	1	$\frac{2\pi}{3}$	$\frac{3\sqrt{3}}{4\sqrt[3]{4}\pi} = \left(\frac{1\times4}{3\times6} \times \frac{5}{3}\right)\left(\frac{7\times10}{9\times12} \times \frac{8}{6}\right)\left(\frac{13\times16}{15\times18} \times \frac{11}{9}\right)\left(\frac{19\times22}{21\times24} \times \frac{14}{12}\right)\ldots$
2	1	$\frac{r\pi}{s}$	$\frac{s\sin(r\pi/s)}{2^{r/s}r\pi} = \left(\frac{s-r}{s}\ \frac{2s-r}{2s}\ \frac{s+r}{s}\right)\left(\frac{3s-r}{3s}\ \frac{4s-r}{4s}\ \frac{2s+r}{2s}\right)\left(\frac{5s-r}{5s}\ \frac{6s-r}{6s}\ \frac{3s+r}{3s}\right)\ldots$
3	1	$\frac{\pi}{2}$	$\frac{\sqrt{2}}{\sqrt{3}\pi} = \left(\frac{1\times3\times5}{2\times4\times6} \times \frac{3}{2}\right)\left(\frac{7\times9\times11}{8\times10\times12} \times \frac{5}{4}\right)\left(\frac{13\times15\times17}{14\times16\times18} \times \frac{7}{6}\right)\left(\frac{19\times21\times23}{20\times22\times24} \times \frac{9}{8}\right)\ldots$
3	2	$\frac{\pi}{2}$	$\frac{2}{\sqrt{3}\pi} = \left(\frac{1\times3\times5}{2\times4\times6} \times \frac{3\times5}{2\times4}\right)\left(\frac{7\times9\times11}{8\times10\times12} \times \frac{7\times9}{6\times8}\right)\left(\frac{13\times15\times17}{14\times16\times18} \times \frac{11\times13}{10\times12}\right)$

5. *Final remarks*

Our main product (1) was motivated by a problem from a written university examination. When did these examinations start? One authority gives 1851 as the start of the Natural Sciences Cambridge Tripos. Oxford started written exams in the 1820s. We invite readers familiar with this topic to write to the *Mathematical Gazette*.

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References

1. E. T. Whittaker and G. N. Watson, *A course of modern analysis (4th ed.)*, Cambridge University Press (1927).
2. L. B. W. Jolly, *Summation of series (2nd ed.)*, Dover (1961).

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number and let M and N be positive integers with $N < M$. Then $\lim_{n\to\infty} S(n) = 1$.

Proof:

Consider

$$\left|\log S(n)\right| = \left|\sum_{k=1}^{(M-N)n}\left\{\log\left(1 - \frac{w}{Nn+k}\right) + \frac{w}{Nn+k}\right\}\right|.$$

If we take n so large that $\left|\frac{w}{Nn+k}\right| < 1$, then we can expand log in a Taylor's series to get

$$\left|\log S(n)\right| = \left|\sum_{k=1}^{(M-N)n}\left\{\frac{1}{2}\left(\frac{w}{Nn+k}\right)^2 + \frac{1}{3}\left(\frac{w}{Nn+k}\right)^3 + \frac{1}{4}\left(\frac{w}{Nn+k}\right)^4 + \ldots\right\}\right|$$

$$\leqslant \sum_{k=1}^{(M-N)n}\left\{\frac{1}{2}\left|\frac{w}{Nn+k}\right|^2 + \frac{1}{3}\left|\frac{w}{Nn+k}\right|^3 + \frac{1}{4}\left|\frac{w}{Nn+k}\right|^4 + \ldots\right\}.$$

This last expression lists $(M - N)n$ infinite series, one series for each value of the index k. The corresponding terms of each series get smaller as k increases, so the first series ($k = 1$), dominates all the others and we get

$$\left|\log S(n)\right| < (M-N)n\left\{\frac{1}{2}\left|\frac{w}{Nn+1}\right|^2 + \frac{1}{3}\left|\frac{w}{Nn+1}\right|^3 + \frac{1}{4}\left|\frac{w}{Nn+1}\right|^4 + \ldots\right\}$$

$$< (M-N)n\left|\frac{w}{Nn+1}\right|^2\left\{\frac{1}{2} + \frac{1}{3}\left|\frac{w}{Nn+1}\right| + \frac{1}{4}\left|\frac{w}{Nn+1}\right|^2 + \ldots\right\}.$$

It is clear from this last expression that $\lim_{n\to\infty} \log S(n) = 0$, and lemma 2 is proved.

3. *The main theorem*

Next we give a proof of our main product (1).

It will be convenient to introduce some shorthand notation. Let

$$p(m, d, x) = \left(1 - \frac{x}{m+1}\right)\left(1 - \frac{x}{m+2}\right)\left(1 - \frac{x}{m+3}\right)\ldots\left(1 - \frac{x}{m+d}\right),$$

and let

$$P(M, N, z) = z\prod_{k=0}^{\infty}\left(p(kM, M, z/\pi)\,p(kN, N, -z/\pi)\right). \qquad (7)$$

Notice that (7) is a shorthand expression for the infinite product in (1). Let us look at a few special cases. We see that

$$P(1, 1, z) = z\prod_{k=0}^{\infty}\left(1 - \frac{z}{(k+1)\pi}\right)\left(1 + \frac{z}{(k+1)\pi}\right) = \sin z$$

is Euler's original product. Also notice that (2) can be written as

$$\sin z \;=\; \exp\left(\frac{z}{\pi}\ \log 2\right) P(2,\ 1,\ z)$$

and that (3) can be expressed as $\sin z \;=\; \exp\left(\frac{z}{\pi}\ \log 3\right) P(3,\ 1,\ z)$. Another special case is

$$\begin{aligned} P(3,\ 2,\ z) \;=\; z & \left\{\left(1 - \frac{z}{\pi}\right)\left(1 - \frac{z}{2\pi}\right)\left(1 - \frac{z}{3\pi}\right)\left(1 + \frac{z}{\pi}\right)\left(1 + \frac{z}{2\pi}\right)\right\} \\ & \left\{\left(1 - \frac{z}{4\pi}\right)\left(1 - \frac{z}{5\pi}\right)\left(1 - \frac{z}{6\pi}\right)\left(1 + \frac{z}{3\pi}\right)\left(1 + \frac{z}{4\pi}\right)\right\} \\ & \left\{\left(1 - \frac{z}{7\pi}\right)\left(1 - \frac{z}{8\pi}\right)\left(1 - \frac{z}{9\pi}\right)\left(1 + \frac{z}{5\pi}\right)\left(1 + \frac{z}{6\pi}\right)\right\} \\ & \quad \vdots \end{aligned}$$

We now see that each major factor in $P(M,\ N,\ z)$ displays $M + N$ simple factors of the form $\left(1 \pm \frac{z}{k\pi}\right)$. The first M simple factors with negative signs are written, then N factors with positive signs are used to complete each major factor.

Theorem:

Let M and N be positive integers. Then $\sin z = \exp\left(\frac{z}{\pi} \log \frac{M}{N}\right) P(M, N, z)$.

Proof:

Consider the partial product using $n + 1$ major factors

$$P_n(M,\ N,\ z) \;=\; z \prod_{k=0}^{n} \left(p(kM,\ M,\ z/\pi)\, p(kN,\ N,\ -z/\pi)\right). \tag{8}$$

This product contains elementary factors of the form $\left(1 - \frac{z}{k\pi}\right)$. It is standard practice (see [2], page 34) to multiply such factors by the exponential $\exp\left(z/(k\pi)\right)$ so that the combination $\left(1 - \frac{z}{k\pi}\right)\exp\left(\frac{z}{k\pi}\right)$ results in factors that form an absolutely convergent product. Thus we have

$$p(kM, M, z/\pi) = \left(1 - \frac{z/\pi}{kM+1}\right)\left(1 - \frac{z/\pi}{kM+2}\right)\dots\left(1 - \frac{z/\pi}{kM+M}\right).$$

$$\begin{aligned} p(kM, M, z/\pi) = \exp & \left(-\frac{z}{\pi}\left(\frac{1}{kM+1} + \frac{1}{kM+2} + \ \dots\ + \frac{1}{kM+M}\right)\right)\left(1 - \frac{z/\pi}{kM+1}\right) \times \\ & \exp\left(\frac{z/\pi}{kM+1}\right)\left(1 - \frac{z/\pi}{kM+2}\right)\exp\left(\frac{z/\pi}{kM+2}\right)\dots\left(1 - \frac{z/\pi}{kM+M}\right)\exp\left(\frac{z/\pi}{kM+M}\right). \end{aligned} \tag{9}$$

A similar adjustment of $p(kN,\ N,\ -z/\pi)$ yields

$$\begin{aligned} p(kN, N, -z/\pi) = \exp & \left(\frac{z}{\pi}\left(\frac{1}{kN+1} + \frac{1}{kN+2} + \ \dots\ + \frac{1}{kN+N}\right)\right)\left(1 + \frac{z/\pi}{kN+1}\right) \times \\ & \exp\left(\frac{-z/\pi}{kN+1}\right)\left(1 + \frac{z/\pi}{kN+2}\right)\exp\left(\frac{-z/\pi}{kN+2}\right)\dots\left(1 + \frac{z/\pi}{kN+N}\right)\exp\left(\frac{-z/\pi}{kN+N}\right). \end{aligned} \tag{10}$$

Substituting (9) and (10) in the partial product (8) we get

$$\begin{aligned} P_n(M, N, z) = \exp & \left\{-\frac{z}{\pi}\sum_{k=0}^{n}\left(\left(\frac{1}{kM+1} + \dots + \frac{1}{kM+M}\right) - \left(\frac{1}{kN+1} + \dots + \frac{1}{kN+N}\right)\right)\right\} \times \\ & \prod_{k=0}^{n}\left(1 - \frac{z/\pi}{kM+1}\right)\exp\left(\frac{z/\pi}{kM+1}\right)\dots\left(1 - \frac{z/\pi}{kM+M}\right)\exp\left(\frac{z/\pi}{kM+M}\right) \times \\ & \left(1 + \frac{z/\pi}{kN+1}\right)\exp\left(\frac{-z/\pi}{kN+1}\right)\dots\left(1 + \frac{z/\pi}{kN+N}\right)\exp\left(\frac{-z/\pi}{kN+N}\right). \end{aligned}$$

Next we rearrange these factors by combining as many factors as possible of the form $\left(1 - z/k\pi\right)\exp\left(z/k\pi\right)$ with $\left(1 + z/k\pi\right)\exp\left(-z/k\pi\right)$ to get $\left(1 - z^2/k^2\pi^2\right)$. If we assume $M > N$, then $M - N$ factors of the form $\left(1 - z/k\pi\right)\exp\left(z/k\pi\right)$ are left over and we get

$$\begin{aligned} P_n(M, N, z) = \exp & \left\{-\frac{z}{\pi}\sum_{k=0}^{n}\left(\left(\frac{1}{kM+1} + \dots + \frac{1}{kM+M}\right) - \left(\frac{1}{kN+1} + \dots + \frac{1}{kN+N}\right)\right)\right\} \\ & z\prod_{k=1}^{Nn+N}\left(1 - \frac{z^2}{k^2\pi^2}\right) \times \\ & \left\{\left(1 - \frac{z/\pi}{Nn+1}\right)\exp\left(\frac{z/\pi}{Nn+1}\right)\dots\left(1 - \frac{z/\pi}{Mn}\right)\exp\left(\frac{z/\pi}{Mn}\right)\right\}. \end{aligned} \tag{11}$$

We now let n approach infinity in (11). Notice that (11) contains three major factors. The first factor $\exp\{\dots\}$ approaches $\exp\left(-\frac{z}{\pi}\ \log \frac{M}{N}\right)$ by lemma 1. The second factor $z\prod_{k=1}^{Nn+N}\left(1 - \frac{z^2}{k^2\pi^2}\right)$ is Euler's well-known product and approaches $\sin z$. The third factor

$$\left\{\left(1 - \frac{z/\pi}{Nn+1}\right)\exp\left(\frac{z/\pi}{Nn+1}\right)\dots\left(1 - \frac{z/\pi}{Mn}\right)\exp\left(\frac{z/\pi}{Mn}\right)\right\}$$

approaches 1 as n grows large from lemma 2. Thus we have

$$P(M,\ N,\ z) \;=\; \lim_{n\to\infty} P_n(M,\ N,\ z) \;=\; \exp\left(-\frac{z}{\pi}\ \log\frac{M}{N}\right)\sin z.$$

The cases where $M = N$ and $M < N$ are similar. The proof of the theorem is now complete.

4. A table of variations on Wallis's product

We conclude by examining a few special cases of our main infinite product (1).

The following table shows the results of specifying M, N and z.