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Classroom note

Variations in the solution of linear first-order differential equations

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A special project which can be given to students of ordinary differential equations is described in detail. Students create new differential equations by changing the dependent variable in the familiar linear first-order equation $(dv/dx) + p(x)v = q(x)$ by means of a substitution $v = f(y)$. The student then creates a table of the new equations and describes how they are solved. Applications are also given.

1. Introduction

We describe a student research project that has been used successfully here at Rowan University in an introductory course in differential equations. In this project, a student is able to obtain the solutions to several first-order differential equations, whose solutions are not easily recognized, even with the help of a computer algebra system such as *Mathematica*. The goal is to transform the equation that a student is given, through a suggested change of variable, into the familiar first-order differential equation

$$\frac{dv}{dx} + p(x)v = q(x) \tag{1.1}$$

which is a standard item ([1], pp. 91–95) in elementary courses. The solution of equation (1.1) is

$$v = \exp\left[-\int p(x)\,dx\right]\left[\int \exp\left[\int p(x)\,dx\right]q(x)\,dx + C\right] \tag{1.2}$$

So, if a student is given a differential equation, say,

$$y' = F(x, y) \tag{1.3}$$

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and is able to transform it into equation (1.1) through some transformation, say,

$$v = f(y) \tag{1.4}$$

then the solution of equation (1.3) is readily given by equations (1.2) and (1.4). With this in mind, we have considered several possible transformations f in equation (1.4) and transformed equation (1.1) with each of these transformations to create three tables of differential equations, which students may come across in exercises or applications.

Since there is no limit to the variety of functions f which one might consider for equation (1.4), this project can be worked on by more than one student, and different results are likely and can provide interesting points of comparison. We show the tables and examples created by one Rowan student in this paper.

2. Presenting the project in class

Once our class has become familiar with the linear first-order equation (1.1) and its solution (1.2), we are ready to present this project to them. We begin by first introducing a transformation and then applying it to a particular example.

Suppose we change the dependent variable v to y using the substitution

$$v = f(y) = y^N \tag{2.1}$$

then we get $v' = Ny^{N-1}y'$, and equation (1.1) becomes

$$Ny^{N-1}y' + p(x)y^N = q(x)$$

Dividing by Ny^{N-1} we get a form of Bernoulli's equation ([1], p. 95)

$$y' + \frac{p(x)}{N}y = \frac{q(x)}{N}y^{1-N} \tag{2.2}$$

From equations (1.2) and (2.1) we see that the solution to equation (2.2) is

$$y^N = \exp\left[-\int p(x)\,\mathrm{d}x\right]\left[\int \exp\left[\int p(x)\,\mathrm{d}x\right]q(x)\,\mathrm{d}x + C\right].$$

Next, we show how equations (2.2) and (2.1) are used to solve the specific differential equation

$$y' - \frac{1}{x}y = xy^2 \tag{2.3}$$

Comparing equations (2.3) and (2.2), we see that $1 - N = 2$, so $N = -1$. We also see that

$$\frac{p(x)}{N} = -p(x) = -\frac{1}{x}, \quad \text{so } p(x) = \frac{1}{x}$$

Finally, we note that

$$\frac{q(x)}{N} = -q(x) = x, \quad \text{so } q(x) = -x$$

Using these functions in equation (1.1), we get the linear first-order equation

$$v' + \frac{1}{x}v = -x \tag{2.4}$$

whose solution by equation (1.2) is

$$v = x^{-1}\left[-\frac{x^3}{3} + C\right] = -\frac{x^2}{3} + \frac{C}{x} \tag{2.5}$$

Since the original substitution equation (2.1) is $v = y^n = y^{-1}$, equation (2.4) becomes

$$\frac{1}{y} = -\frac{x^2}{3} + \frac{C}{x}$$

Therefore the solution of problem (2.3) is

$$y = \frac{3x}{C - x^2}$$

3. Constructing the tables

1 The following tables show the result of using different transformation functions f in equation (1.4) to obtain a new differential equations (1.3) from the linear first-order differential equation (1.1).

	Differential equation	Transformation $v = f(y)$
1	$y' + \frac{p(x)}{N} y = \frac{q(x)}{N} y^{1-N}$	$v = y^N$
2	$y' + (2dq(x) - p(x))y = -cq(x)y^2 + \frac{d(p(x) - dq(x))}{c}$	$v = \frac{1}{cy + d}$
3	$y' + \frac{(ad + bc)p(x) - 2cdq(x)}{ad - bc} y = \frac{c(cq(x) - ap(x))}{ad - bc} y^2 + \frac{d(dq(x) - bp(x))}{ad - bc}$	$v = \frac{ay + b}{cy + d}$
4	$y' - \frac{p(x) - 3dq(x)}{2b} y = -\frac{q(x)}{2bd} y^3 - 3\frac{q(x)}{2b} y^2 + \frac{dp(x) - d^2 q(x)}{2b}$	$v = \left(\frac{b}{y + d}\right)^2$
5	$y' - \frac{p(x)(ay + b)(cy + d)}{2(ad - bc)} = \frac{q(x)(cy + d)^3}{2(ad - bc)(ay + b)}$	$v = \left(\frac{ay + b}{cy + d}\right)^2$
6	$y' + \frac{p(x)}{aM}(ay + b) = \frac{q(x)}{aM}(ay + b)^{1-M}$	$v = (ay + b)^M$
7	$y' + \frac{p(x)(ay + b)(cy + d)}{M(ad - bc)} = \frac{q(x)(cy + d)^{M+1}}{M(ad - bc)(ay + b)^{M-1}}$	$v = \left(\frac{ay + b}{cy + d}\right)^M$
8	$y' = \frac{c(cq(x) - ap(x))}{N(ad - bc)} y^{N+1} + \frac{(2cdq(x) - (ad + bc)p(x))}{N(ad - bc)} y + \frac{d(dq(x) - bp(x))}{N(ad - bc)} y^{1-N}$	$v = \frac{ay^N + b}{cy^N + d}$
9	$y' + \frac{p(x)(ay^N + b)(cy^N + d)}{MN(ad - bc)} y^{1-N} = \frac{q(x)(cy^N + d)^{M+1}}{MN(ad - bc)(ay^N + b)^{M-1}} y^{1-N}$	$v = \left(\frac{ay^N + b}{cy^N + d}\right)^M$

Table 1. Rational functions of y.

	Differential equation	Transformation $v = f(y)$
1	$y' + \frac{p(x)}{aN} y^{1-N} = \frac{q(x)}{aN} y^{1-N} e^{-ay^N}$	$v = e^{ay^N}$
2	$y' + \frac{p(x)}{a}(ay+b)\ln(ay+b) = \frac{q(x)}{a}(ay+b)$	$v = \ln(ay+b)$
3	$y' + \frac{p(x)}{1+ay} y = \frac{q(x)}{1+ay} e^{-ay}$	$v = ye^{ay}$
4	$y' + \frac{p(x)}{1+aMy^M} y = \frac{q(x)}{1+aMy^M} e^{-ay} y^{1-N}$	$v = y^N e^{ay^M}$
5	$y' + p(x)\frac{\cos y}{\cos y - \sin y} = q(x)\frac{e^{-y}}{\cos y - \sin y}$	$v = e^y \cos y$

Table 2. Exponential functions of y.

	Differential Equation	Transformation $v = f(y)$
1	$y' - \frac{p(x)}{2}\cot y = -\frac{q(x)}{2}\sec y \csc y$	$v = \cos^2 y$
2	$y' + \frac{p(x)}{2}\tan y = \frac{q(x)}{2}\sec y \csc y$	$v = \sin^2 y$
3	$y' + \frac{p(x)}{2}\sin y \cos y = \frac{q(x)}{2}\cot y \cos^2 y$	$v = \tan^2 y$
4	$y' + p(x)\cot y = q(x)\cot y \cos y$	$v = \sec y$
5	$y' - p(x)\tan y = -q(x)\sin y \tan y$	$v = \csc y$
6	$y' - p(x)\sin y \cos y = -q(x)\sin^2 y$	$v = \cot y$
7	$y' - \frac{p(x)}{2}\sin y \cos y = -\frac{q(x)}{2}\sin^2 y \tan y$	$v = \cot^2 y$
8	$y' + \frac{p(x)}{2}\cot y = \frac{q(x)}{2}\cos^2 y \cot y$	$v = \sec^2 y$
9	$y' - \frac{p(x)}{2}\tan y = -\frac{q(x)}{2}\sin^2 y \tan y$	$v = \csc^2 y$
10	$y' - \frac{p(x)}{aN} y^{1-N}\cot ay^N = -\frac{q(x)}{aN} y^{1-N}\csc ay^N$	$v = \cos ay^N$
11	$y' + \frac{p(x)}{aN} y^{1-N}\tan ay^N = \frac{q(x)}{aN} y^{1-N}\sec ay^N$	$v = \sin ay^N$

Table 3. Trigonometric functions of y.

4. How the tables are used

From one of the three tables, we select the equation that matches most closely the one that we are given to solve. We then determine algebraically the necessary constants and use equations (1.2) and (1.4) to find the solution. We now illustrate the use of our tables with two examples.

Example 1. Suppose we wish to solve the differential equation

$$y' - \frac{1}{x} y^{-4}\cot 3y^5 = y^{-4}\csc 3y^5$$

Attempts to use *Mathematica* prove futile. Comparing this equation with entry 10 of table 3, we see that a match is found if we take $p(x)=15/x$ and $q(x)=15$. The corresponding first-order linear differential equation

$$v' + p(x)v = q(x) \text{ is } v' + \frac{15}{x}v = 15$$

The solution to this equation from (1.2) is

$$v = \frac{15x}{16} + \frac{C}{x^{15}}$$

Since the transformation is given in the table by $v = \cos 3y^5$, the final solution is

$$\cos 3y^5 = \frac{15x}{16} + \frac{C}{x^{15}}$$

or in explicit form

$$y = \sqrt[5]{\frac{1}{3}\cos^{-1}\left[\frac{15x}{16} + \frac{C}{x^{15}}\right]}$$

In our second example, a problem from physics is solved.

Example 2. Let $y(t)$ be the velocity of a particle moving in a uniform gravitational field with acceleration $-g$ and with friction. The deceleration caused by the friction is given by a term of the form $\alpha(t)y + \beta(t)y^2$, where the coefficients of friction vary with time as $\alpha(t) = 0.01(2g + t^{-1})$, and $\beta(t) = 0.0001(g + t^{-1})$. Suppose also that the velocity is zero when $t = 1$. Then the equation for the acceleration is

$$y' = -g - \alpha(t)y - \beta(t)y^2$$

Thus, we are required to solve the differential equation

$$y' + 0.01(2g + t^{-1})y = -0.0001(g + t^{-1})y^2 - g \tag{4.1}$$

with the initial condition $y(1) = 0$. We note that *Mathematica* was unable to solve this problem.

Comparing equation (4.1) with entry 3 of table 1 (and replacing x by t), we see that the constant term is

$$\frac{d(dq(t) - bp(t))}{ad - bc} = -g$$

Solving for $q(t)$ we get

$$q(t) = \frac{bp(t)}{d} - \frac{g(ad - bc)}{d^2} \tag{4.2}$$

Substituting this value for $q(t)$ into all the other terms of entry 3 of table 1 and simplifying we get the differential equation

$$y' + \frac{c}{d}\left(2g + \frac{d}{c}p(t)\right)y = -\frac{c^2}{d^2}\left(g + \frac{d}{c}p(t)\right)y^2 - g \tag{4.3}$$

Comparing equations (4.3) and (4.1) we see that we can take $c=1$, $d=100$, $p(t)=0.01t^{-1}$. We are free to select a and b as we wish (so long as $ad-bc\neq 0$). We take $a=1$ and $b=99$ so that $ad-bc=1$ conveniently. From equation (4.2) we now have $q(t)=0.0001(99t^{-1}-g)$.

The associated linear first-order differential equation from (1.1) is

$$v' + 0.01t^{-1}v = 0.0001(99t^{-1} - g),$$

which has the solution given by equation (1.2) as

$$v = t^{-0.01}\left[0.0001\int t^{0.01}(99t^{-1} - g)\,\mathrm{d}t + C\right]$$

This simplifies to

$$v = 0.99 - \frac{g}{10100}t + Ct^{-0.01}$$

Since the transformation is given by

$$v = \frac{y+99}{y+100}$$

we determine that

$$y = \frac{99 - 100v}{v - 1}$$

Therefore, we have

$$y = \frac{100gt - 1010000Ct^{-0.01}}{10100Ct^{-0.01} - gt - 101}$$

To determine the constant C, we recall that the velocity y is zero when $t=1$. We get $C=g/10100$, and so finally

$$y = -\frac{100g(t^{1.01} - 1)}{g(t^{1.01} - 1) + 101t^{0.01}} \qquad \text{for } 1 \leqslant t$$

Notice that as $t\to\infty$, the velocity y approaches the constant -100.

This completes our examples.

5. Final remarks

All of the differential equations in these tables can be solved using other methods. For example, there is an appropriate integrating factor for each one, although it might not be obvious to the solver. It can also be noted that many of the equations created in the tables can be related to Bernoulli-like equations that can then be solved. This can be expected since both the equations from the table and Bernoulli's equation are both derived from the first-order linear differential equation. Students can be asked to demonstrate their use of as many methods as they are familiar with for solving selected equations from tables of their own making.

Our second example in the previous section involved a problem in mechanics. Students could be asked to invent applications that their new differential equations will solve. Even though the resulting applications might be somewhat bizarre, this is good experience for the student.

Extensive tables of differential equations that can be solved are not usually found in most reference material. A short table of this type can be found in [2]. We found that many of our equations could not be solved using *Mathematica*. Although modern technology has advanced considerably, a little algebraic manipulation and transformations can cause the solution to become apparent, where sophisticated software is unable to solve it.

This entire project was built around transforming the linear first-order differential equation (1.1). However, we could have started with many other differential equations whose solution is familiar to us. For instance, the general solution of $v'' + \omega^2 v = 0$ is $v(t) = c_1 \cos \omega t + c_2 \sin \omega t$. Now the transformation $v = f(y)$ gives us the new (in general, non-linear) differential equation

$$y'' + \frac{f''(y)}{f'(y)}(y')^2 + \frac{\omega^2}{f'(y)} f(y) = 0$$

Its solution is $f(y) = c_1 \cos \omega t + c_2 \sin \omega t$. By selecting specific transformations $v = f(y)$, or even $v = f(x, y)$, students can construct new tables of unfamiliar differential equations and demonstrate their solutions.

We hope that instructors teaching differential equations will find this material suitable to serve as class or individual projects.

References

[1] Tenenbaum, M., and Pollard, H., 1985, *Ordinary Differential Equations* (New York: Dover Publications).

[2] Spiegel, M. R., and Liu, J., 1999, *Mathematical Handbook of Formulas and Tables* (New York: McGraw-Hill).

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1	Table 2 not cited. Can ‘The following tables show…’ be changed to ‘Tables 1, 2 and 3 show …’. Please check.