Variations on a Theme from Pascal's Triangle

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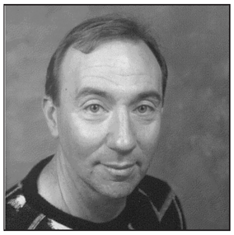

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Introduction

Pascal's triangle,

$$\begin{array}{lllll} 1 & & & & \\ 1 & 1 & & & \\ 1 & 2 & 1 & & \\ 1 & 3 & 3 & 1 & \\ 1 & 4 & 6 & 4 & 1 \\ & & \cdots & & \end{array}$$

is one of the oldest and most important tools in mathematics. It is associated with the powers of $(1+x)$:

$$\begin{aligned} (1+x)^0 &= 1 \\ (1+x)^1 &= 1+\ \ x \\ (1+x)^2 &= 1+2x+\ \ x^2 \\ (1+x)^3 &= 1+3x+3x^2+\ \ x^3 \\ (1+x)^4 &= 1+4x+6x^2+4x^3+x^4. \end{aligned}$$

The simple theme being played here is: *Any number in the triangle is the sum of the number just above it and the number to the left.* If $C(m,n)$ is the number in row m at column n, then this theme is written as

$$C(m,n) = C(m-1,n) + C(m-1,n-1).$$

Unless otherwise stated, we number the rows and columns starting with 0. Thus $C(2,1) = 2$.

In the following sections, we give various number arrays, all in some way a variation of Pascal's triangle. We invite the reader to examine each array and answer the following questions:

(1) What is the theme being played? In other words, if $A(m,n)$ is the number in row m and column n, then what is the recursion relation used to extend the array?

(2) What sequence of algebra problems is solved by using this array?

While listening to these melodies, we will also hear the faint sounds of the Fibonacci and Lucas numbers and polynomials as well as whispers of restricted partitions. We hope these will be interesting problems for students in any course in which Pascal's triangle has been presented.

An example

We now examine the array:

1	2	5				
1	3	7	5			
1	4	10	12	5		
1	5	14	22	17	5	
1	6	19	36	39	22	5

$\cdots$.

A brief observation reveals the familiar theme from Pascal's triangle,

$$A(m, n) = A(m-1, n) + A(m-1, n-1),$$

which answers the first question.

In Pascal's triangle the first row is 1, but here we have three numbers: 1, 2, 5. If the first row is an abbreviation for $1 + 2x + 5x^2$, multiplying by $1 + x$ gives $1 + 3x^2 + 7x^3 + 5x^4$. Notice that the numbers 1, 3, 7, 5 form the second row of our array. Continuing in this way, we see that row m of the array gives us the coefficients for the expansion of $(1 + 2x + 5x^2)(1 + x)^m$. This completes our performance of the tune.

Here are some additional hints before trying our variations.

1. Where there are no numbers in the array, imagine that the number zero is present.
2. In every variation, we imagine that the first row is given. New rows are calculated, number by number, from left to right. New numbers are created from the previous numbers by some simple rule that you are to determine.
3. Each row is an abbreviation for a polynomial or an infinite series. For example, the row 1 2 4 is an abbreviation for the polynomial $1 + 2x + 4x^2$.
4. The polynomial (or infinite series) in row $m + 1$ is obtained from the polynomial (or infinite series) in row m by multiplying it by some factor. (The factor can change in some simple way from row to row.) You must determine this factor. For example, in Pascal's triangle the factor is $(1 + x)$.

The variations

Here are the variations for you to ponder. The answers and comments are given in the next section.

Variation 1.

1								
1	1	1						
1	2	3	2	1				
1	3	6	7	6	3	1		
1	4	10	16	19	16	10	4	1

$\cdots$.

Variation 2.

1	2	5								
1	3	8	7	5						
1	4	12	18	20	12	5				
1	5	17	34	50	50	37	17	5		
1	6	23	56	101	134	137	104	59	22	5

$\cdots$.

Variation 3. Number the first row as $m = -3$, so that the row with only the number 1 occurs as row $m = 0$.

1	3	3	1				
1	2	1					
1	1						
1							
1	−1	1	−1	1	−1	1	⋯
1	−2	3	−4	5	−6	7	⋯

$\cdots$.

Variation 4. Number the first row as $m = -2$, so that the row with only the numbers 1 2 5 occurs as row $m = 0$.

1	4	10	12	5					
1	3	7	5						
1	2	5							
1	1	4	−4	4	−4	4	−4	4	⋯
1	0	4	−8	12	−16	20	−24	28	⋯
1	−1	5	−13	25	−41	61	−85	113	⋯
1	−2	7	−20	45	−86	147	−232	345	⋯

$\cdots$.

Variation 5. Number the first row as $m = -2$, so that the row with only the number 1 occurs as row $m = 0$.

1	2	3	2	1								
1	1	1										
1												
1	−1	0	1	−1	0	1	−1	0	1	−1	0	⋯
1	−2	1	2	−4	2	3	−6	3	4	−8	4	⋯
1	−3	3	2	−9	9	3	−18	18	4	−30	30	⋯

$\cdots$.

Variation 6.

1										
	1	1								
		1	2	1						
			1	3	3	1				
				1	4	6	4	1		
					1	5	10	10	5	1

$\cdots$.

After completing the melody for this variation, additional interesting harmonies can be found by examining the vertical columns as well as the horizontal ones.

Variation 7.

1	2										
	1	3	2								
		1	4	5	2						
			1	5	9	7	2				
				1	6	14	16	9	2		
					1	7	20	30	25	11	2

$\cdots.$

Variation 8.

1														
1	1	1	1	1	1	1	1	1	1	1	1	1	1	$\cdots$
1	1	2	2	3	3	4	4	5	5	6	6	7	7	$\cdots$
1	1	2	3	4	5	7	8	10	12	14	16	19	21	$\cdots$
1	1	2	3	5	6	9	11	15	18	23	27	34	39	$\cdots$
1	1	2	3	5	7	10	13	18	23	30	37	47	57	$\cdots$

$\cdots.$

Answers and comments

Here are some of melodies heard from the variations of the previous section.

Melody from variation 1. The recursion relation is $A(m,n) = A(m-1,n) + A(m-1,n-1) + A(m-1,n-2)$. Row m is an abbreviation for $(1+x+x^2)^m$.

Melody from variation 2. Here we have the same recursion relation, $A(m,n) = A(m-1,n) + A(m-1,n-1) + A(m-1,n-2)$. Since we start with $1+2x+5x^2$, row m is an abbreviation for $(1+2x+5x^2)(1+x+x^2)^m$.

Melody from variation 3. This looks like Pascal's triangle turned upside down. The recursion relation is $A(m,n) = A(m-1,n) - A(m,n-1)$. Now we are generating the coefficients of the power series for $(1+x)^{-m}$ in row m. For example, $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$ is the familiar geometric series with ratio $-x$.

Melody from variation 4. This is the triangle in an example turned upside-down. It has the same recursion relation as the previous variation, $A(m,n) = A(m-1,n) - A(m,n-1)$. The algebraic expression being expanded in row m is $(1+2x+5x^2)(1+x)^{-m}$.

Melody from variation 5. This is the triangle from variation 1 turned upside-down. The recursion relation is $A(m,n) = A(m-1,n) - A(m,n-1) - A(m,n-2)$. Row m is an abbreviation for the expansion of $(1+x+x^2)^{-m}$.

Melody from variation 6. This looks like Pascal's triangle with each row m shifted m places to the right. The recursion relation is $A(m,n) = A(m-1,n-1) + A(m-1,n-2)$. Row m shows the coefficients in the expansion of $x^m(1+x)^m$. The full algebraic expansion looks like

$$
\begin{aligned}
1 &= 1 \\
x(1+x) &= x + x^2 \\
x^2(1+x)^2 &= x^2 + 2x^3 + x^4 \\
x^3(1+x)^3 &= x^3 + 3x^4 + 3x^5 + x^6 \\
x^4(1+x)^4 &= x^4 + 4x^5 + 6x^6 + 4x^7 + x^8 \\
x^5(1+x)^5 &= x^5 + 5x^6 + 10x^7 + 10x^8 + \cdots.
\end{aligned}
$$

Notice what happens when you sum the elements in each column of this triangle. You get the sequence $1, 1, 2, 3, 5, 8, \ldots$. This suggests that these are the Fibonacci numbers [**6**, p. 629] defined by $F_1 = 1$, $F_2 = 1$, and the recursion relation $F_k = F_{k-1} + F_{k-2}$. If we now add all the rows we expect to get

$$\sum_{m=0}^{\infty} x^m(1+x)^m = \sum_{n=0}^{\infty} F_{n+1}x^n.$$

The left-hand side is a geometric series with ratio $x(1+x)$, so

$$\frac{1}{1-x(1+x)} = \sum_{n=0}^{\infty} F_{n+1}x^n.$$

We have found the generating function for the Fibonacci numbers.

We can also imagine our numerical array as an abbreviation for the expansion of $x^m(y+x)^m$. In this case we have

$$
\begin{aligned}
1 &= 1 \\
x(y+x) &= yx + x^2 \\
x^2(y+x)^2 &= y^2x^2 + 2yx^3 + x^4 \\
x^3(y+x)^3 &= y^3x^3 + 3y^2x^4 + 3yx^5 + x^6 \\
x^4(y+x)^4 &= y^4x^4 + 4y^3x^5 + 6y^2x^6 + 4yx^7 + \cdots \\
x^5(y+x)^5 &= y^5x^5 + 5y^4x^6 + 10y^3x^7 + \cdots.
\end{aligned}
$$

Now when we add the columns we get

$$\sum_{m=0}^{\infty} x^m(y+x)^m = 1 + yx + (1+y^2)x^2 + (2y+y^3)x^3 + (1+3y^2+y^4)x^4 + \cdots.$$

The coefficient of x^n is called a Fibonacci polynomial $F_{n+1}(y)$. The Fibonacci polynomials [**6**, p. 633] satisfy the recursion relation $F_{n+1}(y) = yF_n(y) + F_{n-1}$, with the initial values $F_1(y) = 1$, and $F_2(y) = y$. Using the geometric series we have the generating function for these polynomials:

$$\frac{1}{1-x(y+x)} = \sum_{n=0}^{\infty} F_{n+1}(y)x^n.$$

Melody from variation 7. The recursion relation is $A(m,n) = A(m-1,n-1) + A(m-1,n-2)$, the same relation used in the previous variation. Since the first row has changed from 1 to 1 2, we expect the array to be an abbreviation for the expansion of expressions of the form $(1+2x)x^m(1+x)^m$. The full expansions look like

$$\begin{aligned}
1+2x &= 1+2x \\
(1+2x)x(1+x) &= \quad x+3x^2+2x^3 \\
(1+2x)x^2(1+x)^2 &= \qquad x^2+4x^3+5x^4+2x^5 \\
(1+2x)x^3(1+x)^3 &= \qquad\quad x^3+5x^4+9x^5+\cdots \\
(1+2x)x^4(1+x)^4 &= \qquad\qquad x^4+6x^5+\cdots \\
(1+2x)x^5(1+x)^5 &= \qquad\qquad\quad x^5+\cdots.
\end{aligned}$$

If we sum the columns we get

$$(1+2x)\sum_{m=0}^{\infty} x^m(1+x)^m = 1+3x+4x^2+7x^3+11x^4+18x^5+\cdots.$$

The sequence of coefficients $1, 3, 4, 7, 11, 18, \ldots$ are called the Lucas numbers [**6**, p. 1111] and are denoted by $L_1 = 1$, $L_2 = 3$, ... The Lucas numbers satisfy the same recursion relation as do the Fibonacci numbers: $L_n = L_{n-1} + L_{n-2}$. Using the geometric series we can write our last relation as a nice generating function for them:

$$\frac{1+2x}{1-x(1+x)} = \sum_{n=0}^{\infty} L_{n+1}x^n.$$

As in the previous variation, we can enlarge our view of these algebraic expansions by introducing another variable y. In this case we will consider the expansions of the expression $(y+2x)x(y+x)^m$. We get

$$\begin{aligned}
y+2x &= y+2x \\
(y+2x)x(y+x) &= \quad y^2x+3yx^2+2x^3 \\
(y+2x)x^2(y+x)^2 &= \qquad y^3x^2+4y^2x^3+5yx^4+2x^5 \\
(y+2x)x^3(y+x)^3 &= \qquad\quad y^4x^3+5y^3x^4+9y^2x^5+\cdots \\
(y+2x)x^4(y+x)^4 &= \qquad\qquad y^5x^4+6y^4x^5+\cdots \\
(y+2x)x^5(y+x)^5 &= \qquad\qquad\quad y^5x^5+\cdots.
\end{aligned}$$

Now when we add the columns we get

$$(y+2x)\sum_{m=0}^{\infty} x^m(y+x)^m = y+(y^2+2)x+(y^3+3y)x^2+(y^4+4y^2+2)x^3+\cdots.$$

The polynomial coefficients of x^{n-1}, denoted by $L_n(y)$, are called Lucas polynomials [**6**, p. 1112]. They are closely related to the Cardan polynomials [**5**]. We get the generating function for these polynomials using the geometric series

$$\frac{y+2x}{1-x(y+x)} = \sum_{n=0}^{\infty} L_{n+1}(y)x^n.$$

Melody from variation 8. This is a very hard melody. Row number 1 with infinitely many ones gives the coefficients in the expansion of $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$. Row number 2 is obtained from the previous row by summing the number above plus every second number to the left as shown below.

1														
1	1	1	1	1	1	1	1	1	1	1	1	1	1	⋯
1	1	2	2	3	3	4	4	5	5	6	6	7	7	⋯
1	1	2	3	4	5	7	8	10	12	14	16	19	23	⋯

This means that we multiplied the previous row $(1-x)^{-1}$ by $(1-x^2)^{-1} = 1 + 0x + x^2 + 0x^3 + x^4 + \cdots$. We now have the coefficients in the expansion of

$$(1-x)^{-1}(1-x^2)^{-1}.$$

Row number 3 is obtained from the previous row by summing the number above plus every third number to the left as shown below.

1														
1	1	1	1	1	1	1	1	1	1	1	1	1	1	⋯
1	1	2	2	3	3	4	4	5	5	6	6	7	7	⋯
1	1	2	3	4	5	7	8	10	12	14	16	19	23	⋯

This means that we multiplied the previous row $(1-x)^{-1}(1-x^2)^{-1}$ by $(1-x^3)^{-1} = 1 + 0x + 0x^2 + x^3 + 0x^4 + 0x^5 + x^6 + \cdots$. We now have the coefficients in the expansion of $(1-x)^{-1}(1-x^2)^{-1}(1-x^3)^{-1}$.

Continuing in this way we see that row number m gives the coefficients in the expansion of $(1-x)^{-1}(1-x^2)^{-1}(1-x^3)^{-1}\cdots(1-x^m)^{-1}$. This is an important example from the theory of partitions (see [**1**] and [**2**]). If we let $p(n,m)$ denote the number of partitions of the number n using only parts not exceeding m then we have the generating function

$$\prod_{k=1}^{m}\left(1-x^k\right)^{-1} = \sum_{n=0}^{\infty} p(n,m)x^n.$$

For example, the number 3 circled below is located in column $n = 4$ and in row $m = 2$. This means that $p(n,m) = p(4,2) = 3$. Let us now enumerate the partitions of the number 4 using only the numbers 1 and 2. These are $1+1+1+1$, $2+1+1$, and $2+2$. Thus there are 3 partitions of the number 4 using only parts that do not exceed 2.

1														
1	1	1	1	1	1	1	1	1	1	1	1	1	1	⋯
1	1	2	2	3	3	4	4	5	5	6	6	7	7	⋯
1	1	2	3	4	5	7	8	10	12	14	16	19	23	⋯

It was easy to enumerate the partitions in the above example, but consider the number 57 located in column $n = 13$ and row $m = 5$. This means that there are 57 partitions of the number 13 using only the numbers 1, 2, 3, 4, and 5 to form the sum. It would not be so easy to enumerate these 57 partitions. Another easy way to find $p(m,n)$ is found in [**4**], and a computer program for finding the unrestricted partition function $p(n) = p(n,n)$ is found in [**3**].

References

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