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THE UNION OF VIETA'S AND WALLIS' S PRODUCTS FOR PI

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The beautiful infinite product of radicals

(1) $$\frac{2}{\pi}=\sqrt{\frac{1}{2}}\ \sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\ \sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\ \cdots$$

due to Vieta [2] in 1592, is one of the oldest noniterative analytical expressions for π. The Wallis's product [3] dating from 1655

(2) $$\frac{2}{\pi}=\frac{1\cdot 3}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\frac{5\cdot 7}{6\cdot 6}\cdot\frac{7\cdot 9}{8\cdot 8}\cdots$$

is also most remarkable. Both are usually included in any list of interesting expressions for π [1].

The purpose of this short note is to call attention to the following union of Vieta and Wallis-like products:

(3) $$\frac{2}{\pi}=\prod_{n=1}^{p}\underbrace{\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\cdots+\frac{1}{2}\sqrt{\frac{1}{2}}}}}}_{(n\ radicals)}\quad\prod_{n=1}^{\infty}\frac{2^{p+1}n-1}{2^{p+1}n}\cdot\frac{2^{p+1}n+1}{2^{p+1}n}\ .$$

While (1) and (2) seem unrelated, the above expression (3) shows that they are both special cases of a more general "double product". The first product in (3) consists of the first p factors of Vieta's original infinite product (1). The second product in (3) is a Wallis-like product. We say this because the case where p = 0 gives us the original Wallis's product (2), and for other values of p it is the original Wallis's product with factors deleted. Notice also that the Wallis-like product in (3) provides us with the error factor needed to make the Vieta product (1) exact when only a finite number of factors are used .

Relation (3) yields Vieta's product (1) as the limiting case as p goes to infinity, and the Wallis's product (2) as the case p=0. For each intermediate value of p = 1, 2, 3, ... we obtain "united Vieta-Wallis-like products":

p=0: $$\frac{2}{\pi}=\frac{1\cdot 3}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\frac{5\cdot 7}{6\cdot 6}\cdot\frac{7\cdot 9}{8\cdot 8}\cdot\frac{9\cdot 11}{10\cdot 10}\cdot\frac{11\cdot 13}{12\cdot 12}\cdots \quad \text{(original Wallis's product)}$$

p=1: $$\frac{2}{\pi}=\sqrt{\frac{1}{2}}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\frac{7\cdot 9}{8\cdot 8}\cdot\frac{11\cdot 13}{12\cdot 12}\cdot\frac{15\cdot 17}{16\cdot 16}\cdot\frac{19\cdot 21}{20\cdot 20}\cdots$$

p=2: $$\frac{2}{\pi}=\sqrt{\frac{1}{2}}\cdot\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\cdot\frac{7\cdot 9}{8\cdot 8}\cdot\frac{15\cdot 17}{16\cdot 16}\cdot\frac{23\cdot 25}{24\cdot 24}\cdot\frac{31\cdot 33}{32\cdot 32}\cdots$$

p=3: $$\frac{2}{\pi}=\sqrt{\frac{1}{2}}\cdot\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdot\frac{15\cdot 17}{16\cdot 16}\cdot\frac{31\cdot 33}{32\cdot 32}\cdot\frac{47\cdot 49}{48\cdot 48}\cdot\frac{63\cdot 65}{64\cdot 64}\cdots$$

...

$p\to\infty$: $$\frac{2}{\pi}=\sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdots \quad \text{(Vieta's original product).}$$

An examination of the above special cases of (3) shows that each time we increase p by one, we increase the Vieta's product by one new radical factor, and remove alternate factors from the Wallis-like product. The author accidentally discovered our main result (3) while trying to derive Vieta's product (1).

To derive (3) we start by applying the double angle formula for the sine function p times to obtain

$$\sin\theta = 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}$$

$$= 2^2 \cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\sin\frac{\theta}{2^2}$$

$$= 2^3 \cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}\sin\frac{\theta}{2^3}$$

...

(4) $$\sin\theta = 2^p \cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}\cdots\cos\frac{\theta}{2^p}\sin\frac{\theta}{2^p}$$

Next we use the infinite product for the sine function [4], (valid for all x),

$$\sin x = x\prod_{n=1}^{\infty}\left(1-\frac{x^2}{\pi^2 n^2}\right) = x\prod_{n=1}^{\infty}\left(\frac{\pi n - x}{\pi n}\cdot\frac{\pi n + x}{\pi n}\right)$$

with $x = \theta / 2^p$ to replace the last factor in (4). We get after dividing by θ

(5) $$\frac{\sin\theta}{\theta} = \cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}\cdots\cos\frac{\theta}{2^p}\prod_{n=1}^{\infty}\left(\frac{2^p\pi n-\theta}{2^p\pi n}\cdot\frac{2^p\pi n+\theta}{2^p\pi n}\right).$$

We evaluate each of the cosine factors in (5) in terms of $\cos\theta$ by repeated use of the half-angle formula for the cosine. (Here we will assume $-\pi / 2 \le \theta \le \pi / 2$ so that the cosines are never negative.)

$$\cos\frac{\theta}{2} = \sqrt{\frac{1}{2}+\frac{1}{2}\cos\theta}$$

$$\cos\frac{\theta}{2^2} = \sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\cos\theta}}$$

...

(6) $$\cos\frac{\theta}{2^p} = \sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\cdots+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\cos\theta}}}}$$

$$(p \ \ radicals)$$

Combining (6) with (5) we obtain

(7) $$\frac{\sin\theta}{\theta}=\prod_{n=1}^{p}\underbrace{\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\cdots+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\cos\theta}}}}}_{(n\ radicals)}\ \prod_{n=1}^{\infty}\left(\frac{2^{p}\pi n-\theta}{2^{p}\pi n}\cdot\frac{2^{p}\pi n+\theta}{2^{p}\pi n}\right)$$

If we set $\theta=\pi/2$ in (7) and simplify we obtain our united Vieta-Wallis formula (3). This completes our proof.

References

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