

## Classroom notes

### A simple proof $e^2$ is irrational

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It is well known that  $e^2$  is irrational: this note presents a simple proof of it. The arguments stay within the realms of a first proof course in mathematical analysis offered for undergraduates.

#### 1. Introduction

It is very well known that both  $\pi^2$  and  $e^2$  are irrational numbers. Irrationality of  $\pi^2$  has an elementary proof that does not assume the fact that  $\pi$  is transcendental [1]. Likewise we present a simple proof of the irrationality of  $e^2$  without using the transcendentalism of  $e$ .

#### 2. Preliminaries

Let us begin with some definitions, notation and a review of relevant results.

##### Definition 2.1.

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

**Theorem 2.2.**  $e$  is irrational.

**Remark.** For a proof of Theorem 2.2, see [2,3].

**Definition 2.3.** A number  $\alpha$  is said to be algebraic if there exist a positive integer  $n$  and integers  $a_0, a_1, a_2, \dots, a_n$  with  $a_n \neq 0$  and  $\sum_{i=0}^n a_i \alpha^i = 0$ . A number  $\alpha$  is said to be transcendental if  $\alpha$  is not algebraic.

**Theorem 2.4.**  $e$  is transcendental.

**Remark.** For a proof of Theorem 2.4, see [1].

As usual let  $R$  denote the set of real numbers.

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**Theorem 2.5.**

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad x \in \mathbb{R}$$

**Remark.** For a proof of Theorem 2.5, see [2].

**Remark.** The following result is a direct application of Theorem 2.5.

**Corollary 2.6.**

$$e^2 = \sum_{n=0}^{\infty} \frac{2^n}{n!}.$$

**3.  $e^2$  is irrational**

Irrationality of  $e^2$  immediately follows from Theorem 2.4 and the fact that the square root of any rational number is algebraic. Also this follows as a special case from problem 3.1.48 [4].

We present a simple proof of the irrationality of  $e^2$  based on the following elementary results.

**Lemma 3.1.** For each  $n = 1, 2, 3, \dots$  there is an integer  $q_n$  such that

$$(2^n + 1)(2^n + 2)(2^n + 3) \cdots (2^n + 2^n - 1)(2^n + 2^n) = 2^{2^n} q_n.$$

**Proof.** When  $n = 1$ , take  $q_1 = 3$ . Assume that the result is true for some integer  $n$ . Then

$$\begin{aligned} & (2^{n+1} + 1)(2^{n+1} + 2)(2^{n+1} + 3) \cdots (2^{n+1} + 2^{n+1} - 1)(2^{n+1} + 2^{n+1}) \\ &= 2^{2^n} (2^n + 1)(2^n + 2)(2^n + 3) \cdots (2^n + 2^n - 1)(2^n + 2^n)(2^{n+1} + 1) \\ & \quad \times (2^{n+1} + 3) \cdots (2^{n+1} + 2^{n+1} - 1) \\ &= 2^{2^{n+1}} q_n (2^{n+1} + 1)(2^{n+1} + 3) \cdots (2^{n+1} + 2^{n+1} - 1). \end{aligned}$$

Proof follows from induction.

Lemma 3.1 leads to the next simple corollary.

**Corollary 3.2.** For each  $n = 1, 2, 3, \dots$  there is an integer  $p_n$  such that  $(2^n)! = 2^{2^n - 2} p_n$ .

**Proof.** When  $n = 1$ , take  $p_1 = 2$ . Assume that the result is true for some integer  $n$ . Then

$$\begin{aligned} (2^{n+1})! &= (2^n)!(2^n + 1)(2^n + 2)(2^n + 3) \cdots (2^n + 2^n - 1)(2^n + 2^n) \\ &= 2^{2^n - 2} p_n 2^{2^n} q_n = 2^{2^{n+1} - 2} p_n q_n. \end{aligned}$$

Proof follows from induction.

The next lemma is a very basic result.

**Lemma 3.3.** If  $2^k|n!$ , where  $n, k$  are positive integers, then  $k < n$ .

**Proof.** Suppose there is an integer  $n$  such that  $2^k|n!$  with  $k \geq n$ . Let  $m$  be the least such integer. Assume  $m$  is even. Therefore,  $m=2n$  for some integer  $n$ . Thus  $(2n)! = 2^k r$  for some integers  $k, r$  with  $k \geq 2n$ .

However,  $(2n)! = 2n(2n-1)(2n-2) \cdots 3 \cdot 2 \cdot 1 = 2^n n!(2n-1)(2n-3) \cdots 3 \cdot 1$ . Hence,  $2^{k-n} r = n!(2n-1)(2n-3) \cdots 3 \cdot 1$ . Since  $(2n-1)(2n-3) \cdots 3 \cdot 1$  is odd  $2^{k-n}|n!$  and  $k-n \geq n$ .

This is a contradiction, since  $n < m=2n$ . In case  $m$  is odd we obtain a similar contradiction.

Our next result is a consequence of Corollary 3.2 and Lemma 3.3.

**Corollary 3.4.**

$$2^{2^n-2} \left| \frac{2^k(2^n)!}{k!} \right| \text{ for } n = 1, 2, 3, \dots \text{ and for } k = 0, 1, 2, \dots, 2^n.$$

Now we present our simple proof that  $e^2$  is irrational.

**Theorem 3.5.**  $e^2$  is irrational.

**Proof.** Suppose  $e^2 = p/q$ . Choose an integer  $n$  such that  $2^n > 16q + 1$ . Then

$$\frac{p}{q} = 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \cdots + \frac{2^{2^n}}{(2^n)!} + \frac{2^{2^n+1}}{(2^n+1)!} + \frac{2^{2^n+2}}{(2^n+2)!} + \frac{2^{2^n+3}}{(2^n+3)!} + \cdots$$

Therefore,

$$\begin{aligned} (2^n)!p &= q \left\{ (2^n)! + 2(2^n)! + \frac{2^2(2^n)!}{2!} + \frac{2^3(2^n)!}{3!} + \cdots + 2^{2^n} \right\} \\ &+ q2^{2^n+1} \left\{ \frac{1}{2^n+1} + \frac{2}{(2^n+1)(2^n+2)} + \frac{2^2}{(2^n+1)(2^n+2)(2^n+3)} + \cdots \right\}. \end{aligned}$$

From Corollary 3.4 there is an integer  $s_n$  such that

$$\begin{aligned} 2^{2^n-2} p_n p &= q2^{2^n-2} s_n \\ &+ q2^{2^n+1} \left\{ \frac{1}{2^n+1} + \frac{2}{(2^n+1)(2^n+2)} + \frac{2^2}{(2^n+1)(2^n+2)(2^n+3)} + \cdots \right\}. \end{aligned}$$

This leads to

$$p_n p = q s_n + 8q \left\{ \frac{1}{2^n+1} + \frac{2}{(2^n+1)(2^n+2)} + \frac{2^2}{(2^n+1)(2^n+2)(2^n+3)} + \cdots \right\}.$$

Notice that

$$\begin{aligned} &\frac{1}{2^n+1} + \frac{2}{(2^n+1)(2^n+2)} + \frac{2^2}{(2^n+1)(2^n+2)(2^n+3)} + \cdots \\ &\leq \frac{1}{2^n+1} + \frac{2}{(2^n+1)^2} + \frac{2^2}{(2^n+1)^3} + \cdots = \frac{1}{2^n-1}. \end{aligned}$$

Hence,

$$0 < p_n p - q s_n < \frac{8q}{2^n - 1} < \frac{1}{2}.$$

The contradiction that  $p_n p - q s_n$  is an integer proves the theorem.

Theorem 3.5 leads to our final result, which is of course very well known.

**Corollary 3.6.**  $e$  is irrational.

**Proof.** This follows from the fact that the square of any rational number is rational.

## References

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## Parametric integrals and Catalan numbers

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Catalan numbers are defined by combinatorial properties, in numerous situations. Here, an integral representation for these Catalan numbers is computed.

### 1. An infinite sequence of parametric integrals

Let  $a$  be a positive real number. For every non-negative integer  $n$ , we consider the following definite integral

$$I_n = \int_0^a x^n \sqrt{a^2 - x^2} \, dx \quad (1)$$

We will study the sequence  $(I_n)$ .

The first two integrals of the sequence are easy to compute; we have:

$$I_0 = \int_0^a \sqrt{a^2 - x^2} \, dx = \left[ \frac{a^2}{2} \arcsin \frac{x}{a} + \frac{1}{2} x \sqrt{a^2 - x^2} \right]_0^a = \frac{1}{2} a^2 \arcsin 1 = \frac{a^2 \pi}{4}$$

and, by an integration by parts,

$$I_1 = \int_0^a x \sqrt{a^2 - x^2} \, dx = \left[ -\frac{1}{3}(a^2 - x^2)^{3/2} \right]_0^a = \frac{a^3}{3}$$

Now let us compute an induction formula for the sequence  $(I_n)$ . Define

$$u(x) = x^n \quad \text{and} \quad v(x) = \frac{1}{2}x \sqrt{a^2 - x^2} + \frac{1}{2} a^2 \arcsin \frac{x}{a}$$

The first derivatives are given by

$$u'(x) = n x^{n-1} \quad \text{and} \quad v'(x) = \sqrt{a^2 - x^2}$$

An integration by parts provides:

$$\begin{aligned} I_n &= \left[ \frac{1}{2} x^{n+1} \sqrt{a^2 - x^2} + \frac{1}{2} a^2 x^n \arcsin \frac{x}{a} \right]_0^a \\ &\quad - \frac{n}{2} \int_0^a x^n \sqrt{a^2 - x^2} \, dx - \frac{1}{2} a^2 n \int_0^a x^{n-1} \arcsin \frac{x}{a} \, dx \\ &= \frac{1}{2} a^{n+2} \frac{\pi}{2} - \frac{n}{2} I_n - \frac{1}{2} a^2 n \int_0^a x^{n-1} \arcsin \frac{x}{a} \, dx \end{aligned}$$

Before the next computational step, we wish to make the following remark: exercises of this type are generally built with functions for which the fully integrated term of the integration by parts is equal to 0. Here this does not occur, but this non-zero term will be cancelled after the next step. Denote

$$K_n = \int_0^a x^{n-1} \arcsin \frac{x}{a} \, dx$$

Thus, the following identity holds:

$$\left(1 + \frac{n}{2}\right) I_n = \frac{\pi}{4} a^{n+2} - \frac{1}{2} a^2 n K_n \quad (2)$$

We compute  $K_n$  by an integration by parts; let

$$u_1(x) = x^{n-1} \quad \text{and} \quad v_1(x) = x \arcsin \frac{x}{a} + \sqrt{a^2 - x^2}$$

thus

$$u_1'(x) = (n-1)x^{n-2} \quad \text{and} \quad v_1'(x) = \arcsin \frac{x}{a}$$

and we obtain:

$$\begin{aligned} K_n &= \left[ x^n \arcsin \frac{x}{a} + x^{n-1} \sqrt{a^2 - x^2} \right]_0^a \\ &\quad - (n-1) \int_0^a x^{n-1} \arcsin \frac{x}{a} \, dx - (n-1) \int_0^a x^{n-2} \sqrt{a^2 - x^2} \, dx \\ &= a^n \frac{\pi}{2} - (n-1)K_n - (n-1)I_{n-2}. \end{aligned}$$

Move terms from side to side:

$$nK_n = a^n \frac{\pi}{2} - (n-1) I_{n-2} \quad (3)$$

Now substitute the right-hand side of equation (3) into equation (2):

$$\begin{aligned} \left(1 + \frac{n}{2}\right)I_n &= \frac{\pi}{4}a^{n+2} - \frac{1}{2}a^2\left(a^n \frac{\pi}{2} - (n-1)I_{n-2}\right) \\ &= \frac{\pi}{4}a^{n+2} - \frac{\pi}{4}a^{n+2} + \frac{1}{2}a^2(n-1)I_{n-2} \\ \frac{2+n}{2}I_n &= \frac{1}{2}a^2(n-1)I_{n-2} \end{aligned}$$

Finally we obtain the relation:

$$I_n = \frac{a^2 (n-1)}{n+2} I_{n-2} \quad (4)$$

## 2. A closed form and its interpretation

Equation (4) defines the sequence  $(I_n)$  with an induction relation whose step is equal to 2; recall that the first terms of the sequence have already been computed. We distinguish two subsequences, according to whether the index is even or odd.

### 2.1. $n$ even

Suppose that  $n$  is even; we denote  $n = 2p$ ,  $p \in \mathbb{N}$ . Equation (4) becomes:

$$I_{2p} = \frac{a^2 (2p-1)}{2(p+1)} I_{2p-2} \quad (5)$$

We use telescopic methods [1,2]:

$$\begin{aligned} I_{2p} &= \frac{a^2 (2p-1)}{2(p+1)} I_{2p-2} \\ &= \frac{a^2 (2p-1)}{2(p+1)} \cdot \frac{a^2 (2p-3)}{2p} \cdot I_{2p-4} \\ &= \frac{a^2 (2p-1)}{2(p+1)} \cdot \frac{a^2 (2p-3)}{2p} \cdot \frac{a^2 (2p-5)}{2(p-1)} \cdot I_{2p-6} \\ &= \dots \\ &= (a^2)^p \cdot \frac{2p-1}{2(p+1)} \cdot \frac{2p-3}{2p} \cdot \frac{2p-5}{2(p-1)} \dots \frac{a^2}{2 \cdot 2} I_0 \\ &= (a^2)^p \cdot \frac{2p-1}{2(p+1)} \cdot \frac{2p-3}{2p} \cdot \frac{2p-5}{2(p-1)} \dots \frac{a^2}{2 \cdot 2} \cdot \frac{\pi a^2}{4} \\ &= \frac{a^{2p+2}}{4 \cdot 2^p} \cdot \frac{(2p-1)!}{(p+1)! 2^{p-1} (p-1)!} \cdot \pi \\ &= \frac{(a^2)^{p+1}}{2^{2p+1}} \cdot \frac{2p \cdot (2p-1)!}{(p+1)! 2^p (p-1)!} \cdot \pi \end{aligned}$$

thus obtaining the following closed form:

$$I_{2p} = \left(\frac{a}{2}\right)^{2p+2} \cdot \frac{(2p)!}{p!(p+1)!} \cdot \pi \quad (6)$$

The number

$$C_p = \frac{(2p)!}{p!(p+1)!}$$

is called the  $p^{\text{th}}$  ‘Catalan number’ (see [3,4]).

Catalan numbers are known for their numerous combinatorial occurrences. We mention three of them:

- (i) Consider a convex polygon with  $p+2$  vertices. The number of decompositions of this polygon into triangles by non-intersecting diagonals is equal to  $C_p$  (Euler). The appearance of Catalan numbers in Euler’s work is described in [6], together with connections with Pascal’s triangle and other combinatorial problems. Segner’s formula giving the solution to Euler’s polygon division problem is described in [7].
- (ii) When multiplying  $n$  numbers, the number of ways in which parentheses can be placed in order to compute products of two numbers at each step is equal to  $C_n$ .
- (iii) The number of planar trees with  $n+1$  leaves is equal to  $C_n$ ; diagrams are displayed in [4].

Note that for  $a=2$ , the coefficient of  $\pi$  is the Catalan number itself. We propose here an integral presentation for Catalan numbers:

**Proposition 2.1.** For any positive integer  $p$ ,

$$C_p = \frac{1}{\pi} \int_0^2 x^{2p} \sqrt{4-x^2} \, dx$$

Another integral presentation for Catalan numbers is derived in [8], namely:

$$C_n = \frac{1}{2\pi} \int_0^4 x^n \sqrt{\frac{4-x}{x}} \, dx \quad (7)$$

using complex variables and Mellin transform. This integral presentation and ours can be obtained, each from the other, by substitution.

## 2.2. $n$ odd

Suppose now that  $n$  is odd, i.e.  $n=2p+1$  for some non-negative integer  $p$ . The sequence  $(I_{2p+1})$  shows also an occurrence of Catalan numbers, but in a strange fashion. We have:

$$I_{2p+1} = a^2 \frac{2p}{2p+3} I_{2p-1} \quad (8)$$

We write now our telescopic equations:

$$\begin{aligned}
 I_{2p+1} &= a^2 \frac{2p}{2p+3} \cdot a^2 \frac{2p-2}{2p+1} I_{2p-3} \\
 &= a^2 \frac{2p}{2p+3} \cdot a^2 \frac{2p-2}{2p+1} \cdot a^2 \frac{2p-4}{2p-1} I_{2p-5} \\
 &= \dots \\
 &= a^2 \frac{2p}{2p+3} \cdot a^2 \frac{2p-2}{2p+1} \cdot a^2 \frac{2p-4}{2p-1} \dots \frac{a^2 \cdot 2}{5} I_1
 \end{aligned}$$

and by telescopic computations we finally obtain

$$I_{2p+1} = \frac{2^{2p+1} \cdot a^{2p+3} \cdot p! \cdot (p+1)!}{(2p+3)!} = \frac{2^{2p+1} \cdot a^{2p+3}}{(2p+3)(2p+2)(2p+1)} \cdot \frac{p! \cdot (p+1)!}{(2p)!} \quad (9)$$

Note that here the inverses of Catalan numbers appear; this was expected because of the form of equation (8). The author did not meet such a phenomenon elsewhere.

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## How to 'check the result'? Discourse revisited

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The problem of comparison of answers in trigonometric equations arises frequently when different solution strategies are encouraged in the classroom. This paper shows how such a problem can be put in context structured by classic

and contemporary perspectives on teaching problem solving, including the use of technology.

## 1. Introduction

Problem solving, reasoning, communication, connections, and representation are foundational processes underlying the development of mathematical knowledge. Recently, the National Council of Teachers of Mathematics [1] has elevated these processes to the status of *Standards* for school mathematics in North America. Whereas the first standard – Problem Solving – can be viewed as a basis for the reform of mathematics curriculum and instruction [2], the other four (process) standards may provide a meaningful structure for promoting what Dewey [3] has called ‘reflective inquiry’ – a problem-solving method that blurs the distinction between knowing and doing. In other words, appropriately designed interplay of reasoning, communication, connections and representation in mathematics classroom enables one to acquire knowledge through applying it.

This paper reflects the author’s past experience in teaching challenging trigonometry curriculum at the secondary level where all the processes mentioned above can come into play. It shows how this past experience can be put in contemporary educational context that, in particular, includes the appropriate use of technology. In what follows the main focus is on the problem of comparison of answers in trigonometric equations. Such a problem emerges frequently when allowing for the diversity of thinking as an alternative to the pure ‘production of correct answers’ [4, p. 207], brings about not only different solution strategies for a particular equation but different (in terms of arc functions) forms of answers that these strategies yield. A readiness for the acceptance of multiple problem-solving strategies and more than one ‘correct answer’ on the part of a teacher constitutes what Comiti and Ball [5] have called ‘pedagogical understandings of mathematics’ (p. 1147). For example, in the context of in-service teacher education, Hershkowitz, Arcavi and Bruckheimer [6] discussed the problem of counting the number of individual line segments that make up a square grid through the use of a variety of strategies. However, different problem-solving strategies proposed by in-service teachers resulted either in the same answers, or (algebraic) transformations required to demonstrate their sameness were simple.

The following two sections will show how complex situations might be in the context of solving trigonometric equations and comparing answers resulting from the use of different strategies. More specifically, a classroom discourse associated with the equation

$$2 + \cos^2 2x = (2 - \sin^2 x)^2 \quad (1)$$

for which students came up with different problem-solving strategies and consequently with different answers, will be presented. This will include the resolution of a didactical situation filled with a sense of uncertainty among the students who could not ‘check the result’ [7, p. 59] by matching each other’s answers. Finally, the appropriate use of technology in the discourse will be discussed.

## 2. Students' contribution to the discourse

Four students who all had different answers (let Alan, Betsy, Christina and Dave be their pseudonyms) volunteered sharing their solutions with the class. Alan was the first to present his work.

Alan's solution. By using the formula

$$\cos 2x = 1 - 2 \sin^2 x$$

Alan replaced equation (1) by

$$2 + (1 - 2 \sin^2 x)^2 = (2 - \sin^2 x)^2$$

and after simple algebraic manipulations (the class was carefully assessing their correctness) arrived at a surprisingly simple equation

$$\sin^4 x = \frac{1}{3}$$

Further, having written that

$$\sin x \pm \frac{1}{\sqrt[3]{3}}$$

Alan completed his solution with the formula

$$x = \pm \arcsin \frac{1}{\sqrt[3]{3}} + \pi n \quad (2)$$

(from here onwards  $n$  means any integer).

Explained Alan: *'I solved equation (1) by reducing it to an algebraic equation in respect to  $\sin x$ .*

*'We reduced equation (1) to algebraic equation with respect to  $\cos x$ '* – exclaimed Betsy and Christina almost simultaneously – *'but even our answers don't coincide'*. The class looked first at

Betsy's solution. Using the formula

$$\cos 2x = 2 \cos^2 x - 1$$

Betsy transformed equation (1) into the form

$$3 \cos^4 x - 6 \cos^2 x + 2 = 0,$$

solved it as a quadratic equation to obtain

$$\cos^2 x = \frac{3 - \sqrt{3}}{3} \quad (3)$$

(at that point Christina uttered that she did exactly the same transformation), and having extracted the square root from both sides of equation (3), Betsy completed her solution with the formula

$$x = \pm \arccos \left( \pm \sqrt{\frac{3 - \sqrt{3}}{3}} \right) + 2\pi n \quad (4)$$

'I decreased the power of cosine in (3)' – Christina said, and the class looked at her solution.

Christina's solution. As was mentioned above, Christina reduced (1) to (3) also, however, then, by using the formula

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

equation (3) was transformed to the form

$$\cos 2x = \frac{3 - 2\sqrt{3}}{3}$$

whence

$$x = \pm \frac{1}{2} \arccos \frac{3 - 2\sqrt{3}}{3} + \pi n \quad (5)$$

'I reduced equation (1) to a homogeneous equation' – said Dave appearing slightly confused. So the class turned to

Dave's solution. Using the Pythagorean relation

$$\sin^2 x + \cos^2 x = 1$$

Dave transformed equation (1) into the form

$$2(\sin^4 x + 2 \sin^2 x \cos^2 x + \cos^4 x) + (\cos^2 x - \sin^2 x)^2 = (\sin^2 x + 2 \cos^2 x)^2 \quad (6)$$

and then by dividing each term of (6) by  $\cos^4 x$  ('one does not miss any solution here' – noted Dave with confidence) obtained the equation

$$2 \tan^4 x - 2 \tan^2 x - 1 = 0$$

(once again, the class was assessing carefully his transformations) and then presented the final result in the form

$$x = \pm \arctan \sqrt{\frac{1 + \sqrt{3}}{2}} + \pi n \quad (7)$$

### 3. Teacher's contribution to the discourse

Classroom interaction described in the previous section motivated students to learn how mathematical reasoning can be utilized in finding connections among different, yet apparently correct representations of solutions to a trigonometric equation. In other words, the class was prepared to discover a proof that such connections do exist. As Almeida [8] argued, 'pre-condition for fostering mathematical proof is social interaction orchestrated by the teacher in order to catalyse (and gain acceptance for) new roles for proving' (p. 60). In accord with this view, it was the teacher's turn to say something in response to diverse and sophisticated demonstrations through which four distinct forms of roots' representation for equation (1), i.e. series (2), (4), (5), and (7), were obtained by the students. To this end, the teacher drew a sketch (figure 1) designed to demonstrate that the series

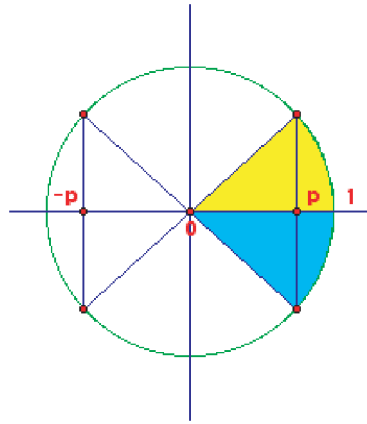


Figure 1. Connecting  $\arccos p + \pi n$  and  $\arccos(\pm p) + 2\pi n$  through geometrization.

$\pm \arccos(\pm p) + 2\pi n$ , and  $\pm \arccos p + \pi n$  are, in fact, identical. Therefore series (4) can be rewritten in the form

$$x = \pm \arccos \sqrt{\frac{3 - \sqrt{3}}{3}} + \pi n \quad (4')$$

Now all four series (2), (4'), (5), (7) have identical structure leaving one to verify that the following chain of equalities

$$\arcsin \frac{1}{\sqrt{3}} = \arccos \sqrt{\frac{3 - \sqrt{3}}{3}} = \frac{1}{2} \arccos \frac{3 - 2\sqrt{3}}{3} = \arctan \sqrt{\frac{1 + \sqrt{3}}{2}} \quad (8)$$

holds true.

Taking a viewpoint that ‘to find a lucid geometric representation for non geometrical problem could be an important step toward the solution’ [7, p. 108] one can show that each term of chain (8) represents the same acute angle of a right triangle. In other words, one has to demonstrate that the difference among these terms resides in notation used to represent the angles only. To this end, because ‘generalization may be useful in solution of problems’ [7, p. 108], one can attempt to generalize chain (8) enabling a representation associated with an arbitrary right triangle. It should be noted that such a classic perspective on problem solving has been reiterated recently by Ollerton and Shannon [9] in terms of the importance of teaching generalization skills as foundations for mathematical understanding who suggested that having a proof of a more general statement ‘allows the original result to be better understood and seen in a wider context’ (p. 136).

With this in mind, one can construct a right triangle with unit length for the hypotenuse  $AB$  and legs  $AC = p$ ,  $BC = q$  (figure 2). Note that  $\arcsin p$  is the acute angle whose sine equals  $p$ ,  $\arccos q$  is the acute angle whose cosine equals  $q$ , and  $\arctan p/q$  is the acute angle whose tangent equals  $p/q$ . As figure 2 shows, this is the same (acute)  $\angle ABC$ , therefore

$$\arcsin p = \arccos q = \arctan p/q, \quad \text{where } p^2 + q^2 = 1.$$

Next, connecting point  $C$  with the midpoint  $O$  of the hypotenuse  $AB$  yields the relations  $AO = CO = 1/2$ .

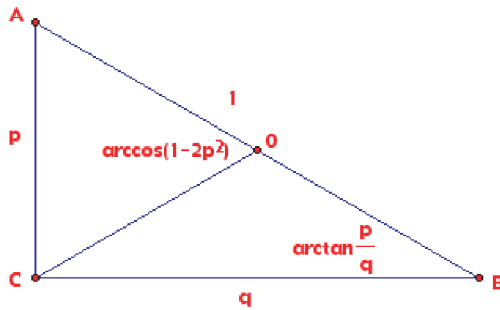


Figure 2. Geometrization of the identity  $2 \arctan(p/q) = \arccos(1 - 2p^2)$ .

In order to find  $\angle AOC$ , one can apply the law of cosines to triangle AOC. This results in the equality  $p^2 = 1/2 - 1/2 \cos(\angle AOC)$ , whence  $\angle AOC = \arccos(1 - 2p^2)$ . From the relation

$\angle ABC = 1/2 \angle AOC$ , it follows that

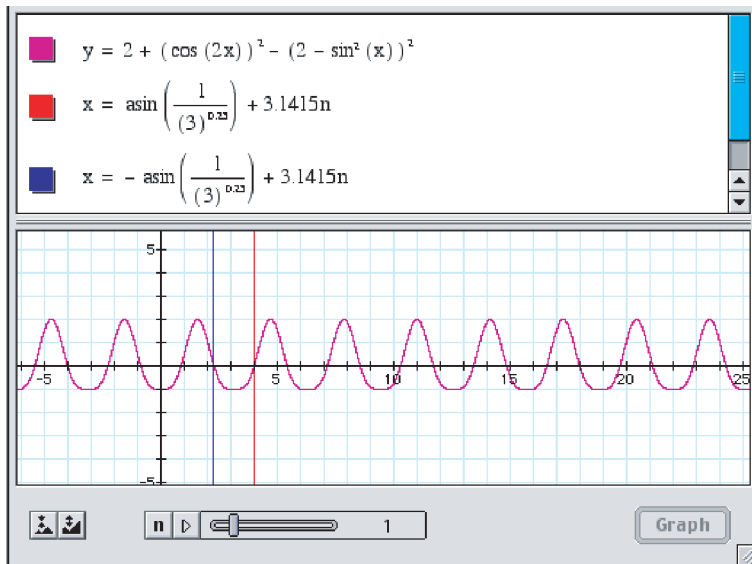
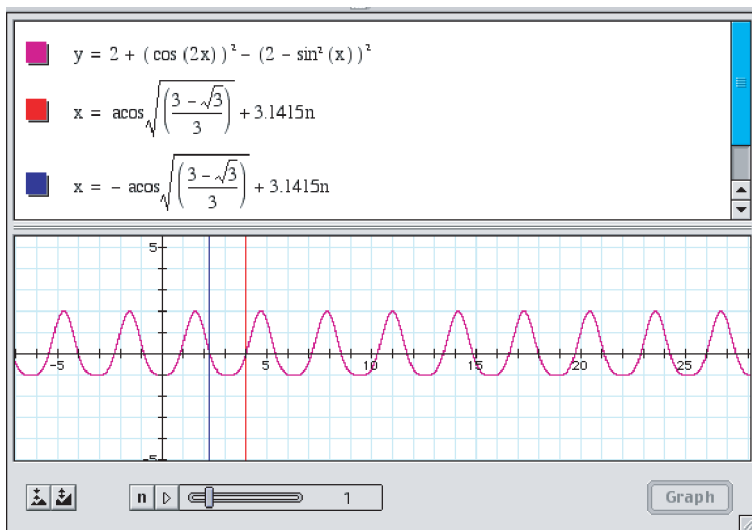
$$\arcsin p = \arccos q = \frac{1}{2} \arccos(1 - 2p^2) = \arctan p/q. \quad (9)$$

Now one can easily verify that chain (9) turns into chain (8) when  $p = 1/\sqrt[4]{3}$ . Thus due to the validity of (8) there should be no reason for disagreement among the students.

#### 4. Technology as an amplifier of the discourse

The above use of geometric representation by the teacher as a way of communicating hidden connections among different notations in trigonometry is a non-routine activity that may be difficult for some students to comprehend. An alternative to this pure mathematical approach to checking results could be to take a contemporary position – ‘technology is essential in teaching and learning mathematics’ [1, p. 24] – and use the example of this paper to illustrate the effectiveness of technology in the resolution of a non-routine problematic situation by students [10]. To this end, one can use computer graphing software that, according to Hennesy, Fung and Scanlon [11], ‘present[s] a unique opportunity to help mathematics students ... develop concepts and skills in a traditionally difficult curriculum area’ (p. 282).

An appropriate form of technology that can be used effectively in checking the results through graphing is the Graphing Calculator 3.2 (GC) produced by Pacific Tech [12]. Its operational capability to graph functions (and, more generally, relations) depending on parameters can be utilized in the context of graphing equation (1) concurrently with series (2), (4), (5), (7) where integer  $n$  is set as a slider-controlled variable. For example, the graphs shown in figures 3 and 4 indicate that for  $n=1$  all roots that belong to a segment of length  $\pi$  (the period of the function  $f(x) = 2 + \cos^2 2x - (2 - \sin^2 x)^2$  and generated by series (2) and (4)) are identical regardless of the notation used to represent them. Furthermore, one can change the value of  $n$  in the slider to see that this identity remains true as  $n$  changes. The same kind of empirical evidence can be provided through graphing series (5) and (7).

Figure 3. Series (2) for  $n = 1$ .Figure 4. Series (4') for  $n = 1$ .

Another possible use of technology in the discourse is worth noting. The point is that technology could be utilized not only in generating empirical evidence for those struggling with a challenging curriculum. Technology can be used to extend the curriculum and provide a motivation for all students ‘to justify by logical reasoning rather than by empirical evidence’ [8, p. 53]. In particular, one can use the GC to investigate a family of trigonometric equations rather than an individual equation alone. The idea of that kind of investigation can be prompted by the following

inquiry: is there any significance in the constant that appears in both sides of equation (1)? In order to address this (reflective) inquiry, the equation

$$r + \cos^2 2x = (r - \sin^2 x)^2 \quad (10)$$

where  $r$  is a real parameter, can be brought to bear as a parametric version of equation (1). Such a direction in which technology-enhanced discourse can move is consistent with standards-based expectations for secondary students that include fostering one's ability to analyse mathematical situations described by 'parametric equations' [1, p. 269]. Also, it supports Kaput's [13] prediction of significant curricular implications (resulting from the use of technology) in shifting the focus of secondary mathematics activities from the study of specific functions to those dependent on parameters. Furthermore, as Zaslavsky [14] argued in the context of professional development of secondary mathematics teachers, this shift provides the teachers with a variety of open-ended tasks made possible by extending curricula to include the study of functions depending on parameters.

Figure 5 shows the locus of equation (10) in which  $r$  is replaced by a custom variable  $y$ . It exhibits several hidden properties of the equation such as the dependence of the number of solutions on parameter  $r$ , the absence of such solutions not only for sufficiently large  $r$  (in absolute value), but for relatively small values of  $r$  as well. In particular, in the locus environment one can formulate new problems such as: (1) Solve equation (10) for  $r=3$  using methods demonstrated by Alan, Betsy, Christina and Dave. Do answers so obtained have different symbolic representations? (2) It appears that for  $r=0$  equation (10) has three roots on the interval  $[0, \pi)$ . Could this empirical evidence be confirmed by formal mathematics? (3) Find empirically a value of parameter  $r$  which provides four intersections of the corresponding level line with the locus of equation (10) and solve it for the value of  $r$  found using any method. (4) What is the smallest/largest value of  $r$  for which equation (10) has real roots? Could these values be found mathematically rather than through cursor pointing?

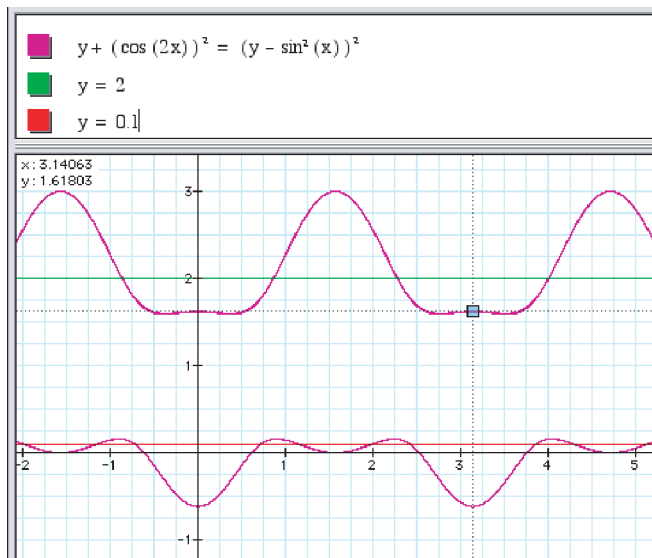


Figure 5. Visualizing locus of equation (10) and its level lines.

The last question may lead to the (parametric) equation  $3z^2 + 2(r - 2)z + 1 + r - r^2 = 0$  from which boundaries for those values of parameter  $r$  that provide at least one solution  $z \in [0, 1]$  can be found using a combination of algebraic reasoning and GC-based computational experiments. Such experiments may include the comparison of two loci – of the last equation and equation (10) – and finding connections among different representations of the same mathematical situation. Furthermore, one can be advised to avoid dealing with irrational inequalities in resolving the situation algebraically, and instead use reasoning based on geometric/graphic representations. In such a way, the shift in focus from solving equation (1) to the study of (a more general) equation (10) made possible by using the GC enables one to appreciate how the use of technology could not only facilitate an already challenging mathematics curriculum but, better still, push against it.

## 5. Conclusions

This article has demonstrated how a combination of classic and contemporary perspectives on problem solving put in the context of current standards for school mathematics can provide a framework for teaching topics from challenging trigonometry curriculum. It was shown that allowing students to be ‘flexible and resourceful problem solvers’ [1, p. 3] fosters their disposition towards reflective inquiry. Students’ readiness to ‘check the result’ through mathematical reasoning provided a milieu for communicating connections that exist among different representations of angular quantities. In particular, this gave rise to a numerical problem that was effectively generalized to a problem ‘in letters’ [7, p. 16] thus supporting the notion that ‘more general problem may be easier to solve’ [7, p. 109]. Finally, it was argued that developing one’s ability to see the general behind the particular in a technology-enhanced environment has the potential to unfold new avenues for open-ended explorations in a seemingly limited problem-solving context.

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## **The rolling can investigation: towards an explanation**

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This paper presents a context lead approach to rotational dynamics. By using nothing more than two cans of cola the basic notions of linear velocity, angular velocity, moments of inertia and conservation of energy can be explored. The approach can be used equally well as both a demonstration or an investigative assignment. The same starting point can, by the nature of the outcome, be used with students at differing levels of mathematical development.

### **1. Introduction**

If two identical cola cans are rolled down an incline they, not unexpectedly, reach the bottom at the same time. However this is not the case if one can is shaken and the other not.

The results are unexpected: the shaken can always rolls down more slowly, and an extensive literature search failed to provide an answer. The results presented here are for an initial investigation in search of a solution.

### **2. The experimental set-up**

Shaken and unshaken cans were rolled down a smooth incline with a drop that varied from 0.02 m to 0.07 m. These small values were chosen to limit any slipping of the can on the runway.

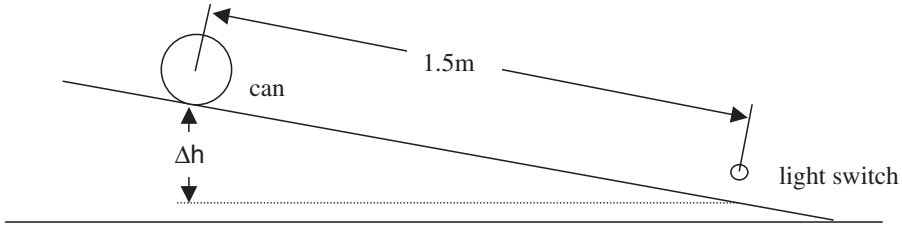


Figure 1. The experiment set-up.

Table 1. Velocity at the end of the runway for shaken and unshaken cans.

$\Delta h$ , m	Shaken velocity, $\text{m s}^{-1}$	Unshaken velocity, $\text{m s}^{-1}$
0.02	0.18	0.53
0.03	0.49	0.60
0.04	0.54	0.71
0.05	0.66	0.82
0.06	0.75	0.89
0.07	0.86	1.02

After rolling a distance of 1.5 m along the runway the cans passed through a light switch which records the velocity by setting the beam to pass through the diameter of the can, 6.57 cm, and recording the time the beam is broken (figure 1)

Five runs were carried out at each of six values of  $\Delta h$  with both the shaken and unshaken cans to provide an average value of velocity for each  $\Delta h$ . These data are presented in table 1.

### 3. Towards a theory

If the can is considered to be a thin walled hollow cylinder with two discs, its moment of inertia,  $I$ , can be considered to be the sum of these three components:

For the wall of the can

$$I_w = m_w r^2 \quad (1)$$

For each end cap

$$I_c = \frac{1}{2} m_c r^2 \quad (2)$$

Therefore

$$\begin{aligned} I &= m_w r^2 + \frac{1}{2} m_c r^2 + \frac{1}{2} m_c r^2 \\ I &= m_w r^2 + m_c r^2 \\ I &= m r^2 \end{aligned} \quad (3)$$

where  $m$  is the total mass of the empty can.

The liquid can be treated, at the two extremes, as a solid cylinder rotating with the can at equal rotational velocity or a mass sliding down the runway without rotating. If the liquid is taken to be a rotating cylinder then the moment of inertia,  $I_1$ , would be given by

$$I_1 = \frac{1}{2}m_1r^2 \quad (4)$$

where  $m_1$  is the mass of the liquid.

By using a conservation of energy argument, assuming no slipping on the runway, we can write:

1. For the case of no rotation of the liquid:

$$\begin{aligned} Mg\Delta h &= \frac{1}{2}m_1v^2 + \frac{1}{2}m_wv^2 + \frac{1}{2}m_cv^2 + \frac{1}{2}I\omega^2 \quad \text{where } M = \text{mass of the full can} \\ Mg\Delta h &= \frac{1}{2}Mv^2 + \frac{1}{2}mv^2 \quad \text{where } v^2 = \omega^2r^2 \\ v^2 &= 2Mg\Delta h/(M + m) \end{aligned} \quad (5)$$

2. For the case of the liquid rolling with the can:

$$\begin{aligned} Mg\Delta h &= \frac{1}{2}m_1v^2 + \frac{1}{2}m_wv^2 + \frac{1}{2}m_cv^2 + \frac{1}{2}I\omega^2 + \frac{1}{2}I_1\omega^2 \\ Mg\Delta h &= \frac{1}{2}Mv^2 + \frac{1}{2}mv^2 + \frac{1}{2}m_1v^2 \\ Mg\Delta h &= \frac{1}{2}Mv^2 + \frac{1}{2}Mv^2 \end{aligned}$$

which is the equation for a 'solid' cylinder

$$v^2 = g\Delta h \quad (6)$$

This leads to  $v^2 = kg\Delta h$  where, in this model with  $M = 0.360$  kg and  $m = 0.027$  kg,  $k$  varies from 1 to 1.9.

Jackson *et al.* [1] report on rolling cans with increasing volumes of water, stating that, 'as more fluid is added its behaviour approaches that of a mass sliding down a frictionless incline.' Using the notation above the sliding mass model would give:

$$\begin{aligned} Mg\Delta h &= \frac{1}{2}Mv^2 \\ v^2 &= 2g\Delta h \end{aligned}$$

which appears to be supportive of  $k = 1.9$ .

The system with the larger value of  $k$  will always roll down the slope faster.

Any observation should then allow us to plot  $v^2$  against  $g\Delta h$  to find a value for  $k$ .

#### 4. Testing the model

The predicted values, using  $k = 1$  and  $k = 1.9$ , are given in table 2.

Table 2. Predicted values of velocity using  $k = 1$  and  $k = 1.9$ .

$\Delta h$ , m	'Rotating' velocity, $\text{m s}^{-1}$	'Non-rotating' velocity, $\text{m s}^{-1}$
0.02	0.44	0.60
0.03	0.54	0.74
0.04	0.63	0.85
0.05	0.70	0.96
0.06	0.77	1.05
0.07	0.83	1.13

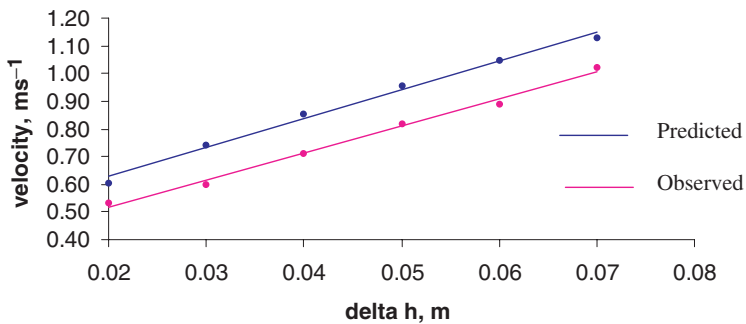


Figure 2. Predicted and observed velocity for the unshaken can.

The suggestion offered here is that in the unshaken can the liquid slides down the runway with the can rotating around it whilst in the shaken can the liquid is 'dragged' around with the can. The predicted and observed values for the shaken and unshaken cans are given in figures 2 and 3.

Removal of the 'outlier' at  $\Delta h = 0.02$  m (see discussion below) allows us to plot the graph shown in figure 4.

In order to further explore the notion that in a shaken can the liquid rotates with the can the investigation was run using a frozen can, which one would expect to behave as a 'solid cylinder'. The results are shown in table 3 and plotted in figure 5.

## 5. Discussion

While a fair measure of agreement between the predicted and observed values can be seen, two points need further comment:

1. The unshaken can is always measured to have a velocity lower than that predicted and this may be explained by some rotation of the liquid which is not taken into account in this model. Indeed a phenomenon given the name 'spin-up' is discussed by Nickas [2] as the time taken for any fluid to reach the rotational velocity of the cylinder.
2. The outlier shown in figure 2 may be explained by the liquid rotating counter to the rotation of the can. This phenomena, while not fully explored here, has been observed with a suspension of fine particles in a rotating liquid [3].

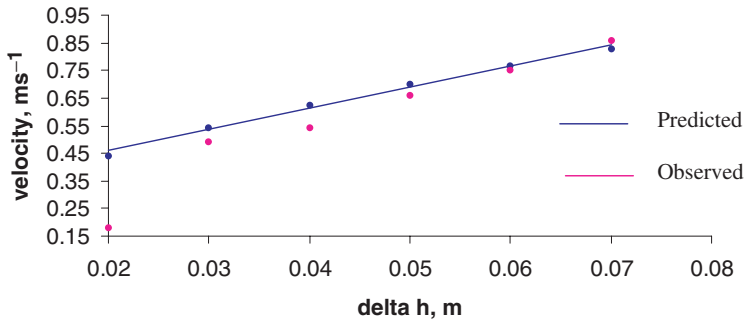


Figure 3. Predicted and observed velocity for the shaken can.

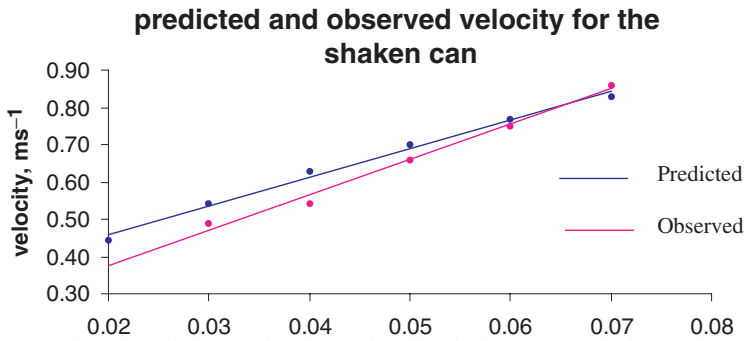


Figure 4. Predicted and observed velocity for the shaken can with the 'outlier' removed.

Table 3. Predicted and observed velocities for the frozen can.

$\Delta h$ , m	Predicted velocity for solid cylinder, m s <sup>-1</sup>	Observed velocity for the frozen can, m s <sup>-1</sup>
0.02	0.44	0.33
0.03	0.54	0.51
0.04	0.63	0.63
0.05	0.70	0.76
0.06	0.77	0.85
0.07	0.83	0.93

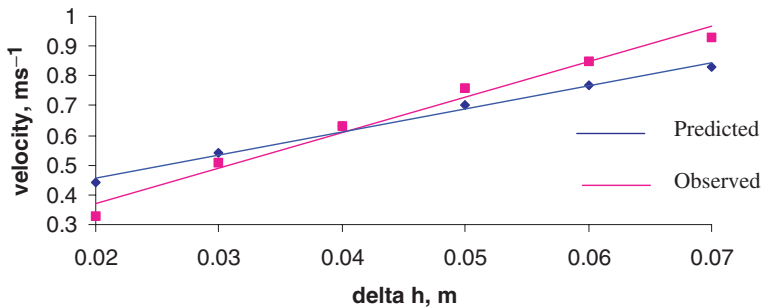


Figure 5. Predicted and observed velocities for the frozen can.

## 6. Conclusions

The work presented here provides, we feel, an initial explanation of the phenomena observed when shaken and unshaken drinks cans are rolled down a runway with the assumption of no slipping. It is felt that this could be utilized as a student investigation at both the pre-university and initial undergraduate levels allowing students opportunity to develop both investigative and modelling skills. The mathematical description may also provide an alternative to the typical ‘ice-skater’ problem.

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## How not to formulate multiple choice problems

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This note discusses some of the shortcomings of multiple choice tests in Mathematics given to undergraduate engineering students. Examples are presented, where the disadvantages of a multiple choice test are pointed out and suggestions how to overcome the difficulties are given.

## 1. Introduction

In this paper we consider several mathematical concepts taught in the mathematics courses taken by engineering students, and present examples of test problems aimed at checking the students’ understanding of the concepts and their ability to use them in problem solving.

The best tool for checking the students’ understanding of the subject is open problems tests. However, there are many cases when one has to use multiple choice tests. In these cases, one has to be careful in constructing and formulating the multiple choice problems. Multiple choice tests are discussed in [1–9].

In this paper we want to share with the readers our experience with ill-formulated multiple choice problems and suggest ways of constructing ‘good’ ones. Shortcomings of multiple choice tests are mentioned in [8], but they are of a different kind.

The paper consists of seven examples. Each example starts with a mathematical concept and an open problem that is used to check how the concept was understood by the students. This is followed by a poorly stated multiple choice problem and by an explanation why such a problem does not show us if the concept was understood. The examples end by suggestions of a better formulated multiple choice problem.

For convenience we use ‘OP’ to denote open problems and ‘MCP’ to denote multiple choice problems.

There are several disadvantages of an MCP. These disadvantages are pointed out in the examples. We also use them to classify the MCPs in the discussion that ends the paper.

We conclude the introduction by remarking that by an MCP we mean a problem with five given answers where only one of them is correct and this answer has to be marked.

## 2. Example 1

Concept: Limit at infinity of the form

$$\lim_{n \rightarrow \infty} (\sqrt{P(n)} + Q(n))$$

where  $P$  and  $Q$  are polynomials.

OP: Given that the limit is finite,

$$\lim_{n \rightarrow \infty} (\sqrt{n^2 + n + 1} + \alpha n)$$

what are the values of the parameter  $\alpha$ ?

Solution: By direct computation the limit is  $+\infty$  or  $-\infty$  if  $\alpha \neq -1$ . For  $\alpha = -1$

$$\lim_{n \rightarrow \infty} (\sqrt{n^2 + n + 1} - n) = \frac{1}{2}.$$

To solve this problem, the student has to distinguish between several cases, compute the general limit for  $\alpha \neq -1$  and compute the limit for  $\alpha = -1$  by multiplying by the conjugate. By giving the open problem we can check that he is capable of doing it.

MCP 1: Given that the limit

$$\lim_{n \rightarrow \infty} (\sqrt{n^2 + n + 1} + \alpha n)$$

is finite, the value of the parameter  $\alpha$  is:

a.  $-1$ ; b.  $0$ ; c.  $1$ ; d.  $2$ ; e.  $3$ .

In all the non-negative options (0, 1, 2, 3) the limit is seen to be  $+\infty$  without any computation. This leaves only the first option, which can be chosen by elimination (not from the above...).

An improved version is:

MCP 2: Given that the limit

$$\lim_{n \rightarrow \infty} (\sqrt{n^2 + n + 1} + \alpha n)$$

is finite, the value of the parameter  $\alpha$  is:

a.  $-2$ ; b.  $-1$ ; c.  $0$ ; d.  $1$ ; e.  $2$ .

Here it is not obvious that  $(-2)$  is not the solution and the student has to calculate the limits in (a) and (b). However, we cannot conclude from his answer whether he can classify all the different possibilities. To remedy this problem we offer another version:

MCP 3: Given that the limit

$$\lim_{n \rightarrow \infty} \left( \sqrt{n^2 + n + 1} + \alpha n \right)$$

is finite, the limit is equal to:

a.  $-1$ ; b.  $-1/2$ ; c.  $0$ ; d.  $1/2$ ; e.  $1$ .

Here, the difficulty is that the student can calculate that  $\alpha = -1$  and the limit is indeed  $1/2$  and by mistake mark solution (a). Thus our final suggestion is:

MCP 4: Given that the limit

$$\lim_{n \rightarrow \infty} \left( \sqrt{n^2 + n + 1} + \alpha n \right)$$

is finite, the limit is equal to:

a.  $-1/2$ ; b.  $0$ ; c.  $1/2$ ; d.  $1$ ; e.  $3/2$ .

### 3. Example 2

Concept: To check if students can compute an improper integral on an infinite interval, by using the definition.

OP: For what values of the parameter  $\alpha$  does the integral

$$\int_1^{\infty} \frac{dx}{x^\alpha}$$

converge.

Compute the integral for such  $\alpha$ s.

Solution: It is well known that the integral converges for  $\alpha > 1$  and diverges for  $\alpha \leq 1$ . If the integral converges, it is equal to  $(1/(\alpha - 1))$ .

MCP 1: The integral

$$\int_1^{\infty} \frac{dx}{x^\alpha}$$

converges for  $\alpha$  equal to:

a.  $-1/2$ ; b.  $0$ ; c.  $1/2$ ; d.  $3/2$ ; e.  $-3/2$ .

The student does not have to analyse the problem. He could substitute  $\alpha = -1/2, 0, 1/2, 3/2, 5/2$  and compute the integral.

We suggest two improved versions.

MCP 2: If the integral

$$\int_1^{\infty} \frac{dx}{x^\alpha}$$

converges for the parameter  $\alpha$ , its value is

a.  $\alpha$ ; b.  $2/\alpha$ ; c.  $\alpha^2$ ; d. does not depend on  $\alpha$ ; e.  $(1/(\alpha - 1))$ .

MCP 3: If the integral

$$\int_1^{\infty} \frac{dx}{x^\alpha}$$

converge ( $\alpha$  parameter) then  $\alpha$  belongs to the interval (where the given interval does not necessarily contain all the possible values of  $\alpha$ ):

- a.  $[0, 2]$ ; b.  $(0, 1)$ ; c.  $(0, 1]$ ; d.  $(1, 2)$ ; e.  $[1, 2]$ .

In MCP 2 and MCP3 the student must analyse the problem and solve it in general.

#### 4. Example 3

Concept: Finding a tangent plane to a given surface under given geometrical conditions.

OP: Find all tangent planes to the surface  $x^2 + 2y^2 + 3z^2 = 6$  that contain both points  $M(0, 0, 2)$  and  $N(0, 3, 0)$ .

Solution: Suppose the tangent point is  $(x_0, y_0, z_0)$ . Then the tangent plane is

$$x_0x + 2y_0y + 3z_0z = 6$$

where  $x_0^2 + 2y_0^2 + 3z_0^2 = 6$ . Since  $M$  and  $N$  lie on the plane,  $6z_0 = 6$ ,  $6y_0 = 6$ , so  $y_0 = z_0 = 1$  and  $x_0^2 = 1$ , so there are two tangent planes:

$$x + 2y + 3z - 6 = 0$$

and

$$-x + 2y + 3z - 6 = 0.$$

MCP 1: Let  $Ax + By + Cz + D = 0$  be the tangent plane to  $x^2 + 2y^2 + 3z^2 = 6$  that contains both points  $M(0, 0, 2)$  and  $N(0, 3, 0)$ . Then:

- $A + B + C + D = -2$
- $A + B + C + D = -1$
- $A + B + C + D = 0$
- $A + B + C + D = 2$
- There is no such plane.

Since the equation  $Ax + By + Cz + D = 0$  can be multiplied by any non-zero constant, the question is clearly ill-defined, and all the first four solutions are correct.

MCP 2: Let  $Ax + By + Cz + D = 0$  be the tangent plane to  $x^2 + 2y^2 + 3z^2 = 6$  that contains both points  $M(0, 0, 2)$  and  $N(0, 3, 0)$ . If  $B = 2$  then:

- $A + B + C + D = -2$
- $A + B + C + D = -1$
- $A + B + C + D = 0$
- $A + B + C + D = 2$
- There is no such plane.

Here solutions (a) and (c) are correct.

MCP 3: Let  $Ax + By + Cz + D = 0$  be the tangent plane to  $x^2 + 2y^2 + 3z^2 = 6$  that contains both points  $M(0, 0, 2)$  and  $N(0, 3, 0)$ . If  $B = 2$  then:

- $A + B + C + D = -1$
- $A + B + C + D = 0$
- $A + B + C + D = 1$
- $A + B + C + D = 2$
- There is no such plane.

Here (b) is the correct solution. A more compact version is:

MCP 4: Let  $Ax + By + Cz + D = 0$  be the tangent plane to  $x^2 + 2y^2 + 3z^2 = 6$  that contains both points  $M(0, 0, 2)$ ,  $N(0, 3, 0)$ . Then:  $((C + D)/B) =$

a.  $-1/2$ ; b.  $-3/2$ ; c.  $-5/2$ ; d.  $-3$ ; e.  $-7/2$ .

However, we miss the point that in fact there are two tangent planes. This is not the case in:

MCP 5: Let  $Ax + By + Cz + D = 0$  be the tangent plane to  $x^2 + 2y^2 + 3z^2 = 6$  that contains both points  $M(0, 0, 2)$ ,  $N(0, 3, 0)$ . Then:

number of solutions  $\times ((C + D)/B) =$

a.  $-3$ ; b.  $-1$ ; c.  $1$ ; d.  $2$ ; e.  $3$ .

## 5. Example 4

Concept: To check if the student knows how to use the definition of a partial derivative.

The following problem is frequently used in teaching calculus of several variables.

OP: Let

$$f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

Find  $f'_x(0, 0)$ .

Solution: Clearly  $f'_x(0, 0) = \lim_{\Delta x \rightarrow 0} ((f(\Delta x, 0) - f(0, 0))/(\Delta x)) = 0$ .

MCP 1: Let

$$f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

Then

a.  $f'_x(0, 0) = 0$ ; b.  $f'_x(0, 0) = 1$ ; c.  $f'_x(0, 0) = 2$ ; d.  $f'_x(0, 0) = 3$ ; e.  $f'_x(0, 0)$  does not exist.

The problem with this wording is that the student can choose the right answer ( $f'_x(0, 0) = 0$ ) for wrong reasons, for example  $(d/dt)0 = 0$ . Our suggestion is to replace the function, so the derivative will not be zero. For example, in the following version the derivative is 1.

MCP 2: Let

$$f(x, y) = \begin{cases} \frac{xy + x^3}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

Then

a.  $f'_x(0, 0) = 0$ ; b.  $f'_x(0, 0) = 1$ ; c.  $f'_x(0, 0) = 2$ ; d.  $f'_x(0, 0) = 3$ ; e.  $f'_x(0, 0)$  does not exist.

**6. Example 5**

OP: Find the extremum points of:

$$g(x) = \int_0^{x^3-3x^2} e^{t^2} dt.$$

Main concept: Using the theorem: if  $f$  is continuous, then

$$\frac{d}{dx} \left( \int_{a(x)}^{b(x)} f(t) dt \right) = f(b(x))b'(x) - f(a(x))a'(x)$$

Minor concept: Computing minimum and maximum.

Solution:

$$g'(x) = e^{(x^3-3x^2)^2} (3x^2 - 6x)$$

$g'(x) = 0$  for  $x=0$  or  $x=2$  and it can be checked that  $x=0$  is the maximum point and  $x=2$  is the minimum point.

MCP 1: If

$$g(x) = \int_0^{x^3-3x^2} e^{t^2} dt$$

then

- $x=2$  is a minimum point of  $g(x)$ ;
- $x=2$  is a maximum point of  $g(x)$ ;
- $x=0$  is a maximum point of  $g(x)$ ;
- $x=-2$  is a maximum point of  $g(x)$ ;
- $g(x)$  has no extremum point.

The problem with this multiple choice version is that, in the case of a wrong solution, it is not clear whether the student made a mistake in determining the extremum points or in differentiation of the integral.

Our suggestion is to give up the minor aim.

MCP 2: If

$$g(x) = \int_0^{x^3-3x^2} e^{t^2} dt$$

then  $g'(1)$  is equal to:

- 0; b.  $-2$ ; c. 1; d.  $-3e^4$ ; e.  $4e^3$ .

Obviously, the correct answer is (d).

**7. Example 6**

Concept: Green's Theorem and conservative field.

OP: compute the integral

$$\int_C \vec{G} d\vec{r}$$

where  $C$  is the circle  $x^2 + y^2 = 100$  and

$$\vec{G} = \left( -\frac{y}{x^2 + y^2} - \frac{y-2}{(x-2)^2 + (y-2)^2} \right) \vec{i} + \left( \frac{x}{x^2 + y^2} + \frac{x-2}{(x-2)^2 + (y-2)^2} \right) \vec{j}$$

Solution: Let

$$P_a = -\frac{y-a}{(x-a)^2 + (y-a)^2}, \quad Q_a = \frac{x-a}{(x-a)^2 + (y-a)^2}$$

$\vec{F}_a = P_a \vec{i} + Q_a \vec{j}$  and  $L_{a,\epsilon} : (x-a)^2 + (y-a)^2 = \epsilon^2$ . Using the parametrization

$$x = a + \epsilon \cos t$$

$$y = a + \epsilon \sin t \quad 0 \leq t \leq 2\pi$$

one gets

$$\int_{L_{a,\epsilon}} \vec{F}_a d\vec{r} = 2\pi$$

Since  $\vec{G} = \vec{F}_0 + \vec{F}_2$ ,

$$\int_C (\vec{F}_0 + \vec{F}_2) d\vec{r} = \int_{L_{0,10}} \vec{F}_0 d\vec{r} + \int_{L_{2,(1/2)}} \vec{F}_2 d\vec{r} = 2\pi + \int_{L_{0,10}} \vec{F}_2 d\vec{r}$$

The second integral can be computed using Green's Theorem. Let  $D$  be the domain bounded by  $L_{0,10}$  and  $L_{2,(1/2)}$ . Since  $(\partial P_2 / \partial y) = (\partial Q_2 / \partial x)$  on  $D$ ,

$$\int_{L_{0,10}} \vec{F}_2 d\vec{r} = \int_{L_{2,(1/2)}} \vec{F}_2 d\vec{r} = 2\pi$$

so

$$\int_C \vec{G} d\vec{r} = 4\pi$$

MCP 1: Let  $C$  be the circle  $x^2 + y^2 = 100$  and

$$\vec{G} = \left( -\frac{y}{x^2 + y^2} - \frac{y-2}{(x-2)^2 + (y-2)^2} \right) \vec{i} + \left( \frac{x}{x^2 + y^2} + \frac{x-2}{(x-2)^2 + (y-2)^2} \right) \vec{j}$$

Then

$$\int_C \vec{G} d\vec{r} =$$

a. 0; b.  $\pi$ ; c.  $2\pi$ ; d.  $3\pi$ ; e.  $4\pi$ .

Here is a wrong solution which yields the correct answer (e): the disc bounded by  $C$  contains two singular points of  $\vec{G}$  and  $2 \cdot 2\pi = 4\pi$ . To overcome this problem we recommend a slight change in  $\vec{G}$ :

MCP 2: Let  $C$  be the circle  $x^2 + y^2 = 100$  and

$$\vec{G} = \left( -\frac{y}{2(x^2 + y^2)} - \frac{y-2}{(x-2)^2 + (y-2)^2} \right) \vec{i} + \left( \frac{x}{2(x^2 + y^2)} + \frac{x-2}{(x-2)^2 + (y-2)^2} \right) \vec{j}$$

Then

$$\int_C \vec{G} d\vec{r} =$$

a. 0; b.  $\pi$ ; c.  $2\pi$ ; d.  $3\pi$ ; e.  $4\pi$ .

Of course, here the right answer is (d).

### 8. Example 7

Concept: The rank of a linear operator. The dependence of a linear operator on the underlying field.

OP: Let  $T$  be a linear operator from  $R^3$  to  $R^3$  defined by

$$T(x_1, x_2, x_3) = ((a-1)x_1 + 2x_2 - x_3, 2x_1 + (a+1)x_2 + 2x_3, \\ 2(a+1)x_1 + (a+2)x_2 + (a+2)x_3)$$

where  $a$  is a parameter. What is  $\text{rank}T$  (as a function of the parameter  $a$ )?

Solution: Let

$$A = \begin{pmatrix} a-1 & 2 & -1 \\ 2 & a+1 & 2 \\ 2a+2 & a+2 & a+2 \end{pmatrix}$$

be the matrix representation of  $T$  in the standard basis, so  $\text{rank}T = \text{rank}A$ . The determinant of  $A$  is equal to  $a(a^2 + 2a + 3)$  and the only real value for which it is zero is  $a=0$ . (For  $a = 1 \pm i\sqrt{2}$ ,  $T$  is not a real operator).

For  $a=0$ ,  $\text{rank}A = 2$  so

$$\text{rank}T = \begin{cases} 3, & a \neq 0 \\ 2, & a = 0 \end{cases}$$

MCP 1: Let  $T$  be a linear operator from  $R^3$  to  $R^3$  defined by

$$T(x_1, x_2, x_3) = ((a-1)x_1 + 2x_2 - x_3, 2x_1 + (a+1)x_2 + 2x_3, \\ 2(a+1)x_1 + (a+2)x_2 + (a+2)x_3)$$

where  $a$  is a parameter. Then:

- For  $a=0$ ,  $\text{rank}T = 3$ .
- For  $a=1$ ,  $\text{rank}T = 2$ .
- For  $a=2$ ,  $\text{rank}T = 2$ .
- For  $a=0$ ,  $\text{rank}T = 2$ .
- For every  $a$ ,  $\text{rank}T = 3$ .

We have two problems with this version. First, it is possible to substitute the suggested values of  $a$  and the student does not need to observe, and use the fact, that  $T$  is a real operator.

A more severe problem is that the student may make mistakes in the (not very simple) calculations even if he knows how to solve the problem. Since this is a multiple choice problem he will not get any credit for his work since the grader has no way to see what caused the mistake.

The calculations in the following version are much simpler but it does not change at all the aims of the problem.

MCP 2: Let  $T$  be a linear operator from  $R^3$  to  $R^3$  defined by

$$T(x_1, x_2, x_3) = (ax_1, (a+1)x_2 - 5x_3, x_2 + (a-1)x_3)$$

where  $a$  is a parameter. Then:

- a. For  $a=0$ ,  $\text{rank}T=3$ .
- b. For  $a=1$ ,  $\text{rank}T=2$ .
- c. For  $a=2$ ,  $\text{rank}T=2$ .
- d. For  $a=0$ ,  $\text{rank}T=2$ .
- e. For every  $a$ ,  $\text{rank}T=3$ .

This leaves us with the first problem (it is possible to substitute the suggested values of  $a$  and the student does not need to observe, and use the fact, that  $T$  is a real operator). Part of the problem is taken care of in the following version:

MCP 3 : Let  $T$  be a linear operator from  $R^3$  to  $R^3$  defined by

$$T(x_1, x_2, x_3) = (ax_1, (a+1)x_2 - 5x_3, x_2 + (a-1)x_3)$$

where  $a$  is a parameter, such that  $\text{rank}T < 3$ . Then  $a + \text{rank}T$  is equal to:

- a.  $2 - 2i$ .
- b. 2.
- c.  $2 + 2i$ .
- d. 3.
- e.  $3i$ .

Here the correct solution is 2 and the simplest way to see it is the way that the OP version is solved.

## 9. Discussion

Most of the examples presented in this paper are from Calculus. We also have a problem in Linear Algebra. Some of the problems are quite simple, others are more advanced.

There are several possible difficulties in using multiple choice problems. We tried to show how these difficulties can be overcome.

Difficulty No. 1: Not all the abilities that we want to check, can be detected.

An example of such ability is the student's appreciation of a proof. In this paper we do not discuss proofs. The reader is referred to [3].

Difficulty No. 2: The correct answer can easily be guessed.

Difficulty No. 3: The correct answer can be obtained in a wrong way.

Difficulty No. 4: The student marks a wrong solution and it is not clear why.

This happens, for example, when one tries to combine several concepts in the same problem.

Difficulty No. 5: The problem has more than one solution.

Difficulty No. 6: The computations are unnecessarily long.

We now discuss the examples given in the paper having in mind the difficulties mentioned above.

The difficulties in Example 1 are Difficulty 1 and Difficulty 2. We still have Difficulty 1 in MCP 2 and both difficulties are taken care of by MCP 3, but then Difficulty 3 appears. It seems that there are no problems in MCP 4.

The difficulty in Example 2, MCP 1 is Difficulty 1.

In Example 3, MCP 1 is poorly stated and has more than one solution (Difficulty 5). MCP 2 still has two correct solutions. The difficulty in MCP 3 and 4 is Difficulty 1.

The difficulty in Example 4 is Difficulty 5.

Two concepts play a role in Example 5 which causes Difficulty 4.

The correct answer to MCP 1 in Example 6 could be obtained by a wrong solution, so this is Difficulty 3.

The difficulties in Example 7, MCP 1 are Difficulty 1 and Difficulty 6. The second difficulty is taken care of in MCP 2 and we hope that there are no unnecessary difficulties in MCP 3.

We would appreciate the readers' suggestions of other difficulties that were not treated in this paper.

### Acknowledgment

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## Why $\sin 80x$ looks like $\sin x$ on some graphing calculators

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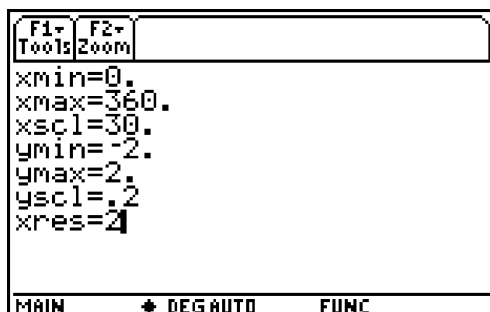
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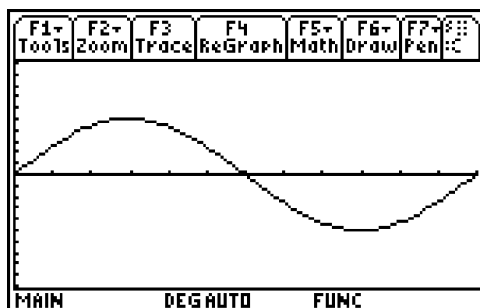
In rare cases, the use of advanced mathematical calculators can give incorrect results. One such error occurs with graphing calculators because the screen is not continuous, but a rectangular array of pixels. On some frequently used calculators, the graph of  $\sin 80x$  looks like  $\sin x$ . We also study other related examples.

## 1. Introduction

Suppose you set your TI 89 calculator in degree mode and the window settings



as shown. When you enter the function  $y = \sin 80x$ , you get the graph



Since the range is  $0 \leq x \leq 360$ , this is the graph of  $y = \sin x$ , and not the graph of  $y = \sin 80x$ . What is going wrong? Some graphs of this type are mentioned in [1], without details. It is the purpose of this short note to explain this and other related misleading graphs that calculators can create.

This material can be used in any precalculus or higher class in which a graphing calculator is used. Only familiarity with trigonometric functions and the addition formula for  $\sin x$  is required. While the discussion centres on the TI-89, we show how to adopt the material to any graphing calculator in the final section.

## 2. A picture by pixels

The graphing calculator screen is composed of a rectangular array of tiny specks called *pixels*. By colouring each pixel black or white, the image on the screen is created. The TI-89 Guidebook [2, p. 222] states that the array contains 159 pixel columns (horizontal direction), and 77 pixels in each column (vertical direction). This will prove to be the ultimate source of the incorrect graphs. Actually, the TI-89 screen contains 160 pixel columns but only 159 are used for graphing. Also, the TI-83 or TI-83 Plus contain 96 pixel columns, but only 95 are used for graphing. The extra column, on the right, is used for something else. For example, it is for menu display in the TI-89 and for the busy signal in the TI-83.

When creating a graph, the calculator starts with the first column of pixels on the left edge of the screen, and decides which pixel on this column corresponds to the appropriate point on the graph. It then moves to the next column and decides which pixel in this column is appropriate, and so on. Since our range is  $0 \leq x \leq 360$ , and there are 159 columns of pixels, the distance between pixel columns is  $\Delta x = 360/158 = 180/79$ . We call this number the *step size*. Thus the values of  $x$  at which the graph is calculated are given by

$$x(n) = \Delta x n = \frac{180}{79} n \quad \text{where } n = 0, 1, 2, \dots, 158 \quad (1)$$

There is one further complication. To decrease the time spent in creating the graph, the TI-89 can select to calculate only at every other pixel column, or every third pixel column, etc. This is determined by the last setting in the window screen called 'xres'. Xres can be set to 1, 2, 3, ..., or 10. If  $xres = 2$  (the default value) as it is in the above window, then the calculations are made at every other pixel column, and the values of  $x$  at which  $y$  values are calculated is now given by

$$x(n) = \Delta x(x \text{ res})n = \frac{180}{79} 2n = \frac{360}{79} n \quad \text{where } n = 0, 1, 2, \dots, 79 \quad (2)$$

### 3. Why $\sin 80x$ looks like $\sin x$

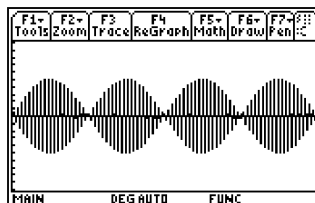
We can now unlock the mystery behind the incorrect graph. The function  $y = \sin 80x$  was evaluated only at the values of  $x$  given by equation (2). Thus we have

$$\begin{aligned} \sin(80x) &\Rightarrow \sin\left(80 \frac{360}{79} n\right) = \sin\left((79 + 1) \frac{360}{79} n\right) \\ &= \sin\left(360n + \frac{360}{79} n\right) = \sin\left(\frac{360}{79} n\right) \end{aligned}$$

The last step follows from the fact that the sine function has period 360 degrees. (We have used the notation ' $\Rightarrow$ ' to mean 'will be plotted as'.) But by equation (2)  $x(n) = (360/79) n$ , so we get  $\sin(80x) \Rightarrow \sin(x(n))$ . This is why the graph of  $y = \sin(80x)$  looks like the graph of  $y = \sin x$  on the TI-89.

### 4. Another example

Now suppose we use the same window as before, but change the value of the last setting to  $xres = 1$ . Now we enter the function  $y = \sin(77x)$  and get the graph below.



This is certainly not the graph we expected! The values of  $x$  at which the values of the function are calculated are now given by

$$x(n) = \Delta x (x \text{ res}) n = \frac{180}{79} n \quad \text{where } n = 0, 1, 2, \dots, 159 \quad (3)$$

We now have

$$\begin{aligned} \sin(77x) &\Rightarrow \sin\left(77 \frac{180}{79} n\right) \\ &= \sin\left((79 - 2) \frac{180}{79} n\right) \\ &= \sin\left(180n - 2 \frac{180}{79} n\right) \end{aligned}$$

Using the addition formula  $\sin(\alpha - \beta) = \sin \alpha \cos \beta - \sin \beta \cos \alpha$  we get

$$\begin{aligned} \sin(77x) &\Rightarrow \sin(180n) \cos\left(2 \frac{180}{79} n\right) - \sin\left(2 \frac{180}{79} n\right) \cos(180n) \\ &= -\sin\left(2 \frac{180}{79} n\right) \cos\left(79 \frac{180}{79} n\right) \end{aligned}$$

Since

$$x(n) = \frac{180}{79} n$$

by equation (3), we get

$$\sin(77x) \Rightarrow -\sin(2x(n)) \cos(79x(n)) \quad \text{where } n = 0, 1, 2, \dots, 158$$

We see that the graph of  $y = \sin(77x)$  looks like the graph of  $y = -\sin(2x) \cos(79x)$  on the TI-89.

## 5. Some problems

In the following tables, we list more functions whose graphs appear incorrect on both the TI-89 and the TI-83. The reader might enjoy using the above methods to explain the results.

	Function entered	Function appearing on the TI-89	Range	$x \text{ res}$
1	$y = \sin(79x)$	$y = 0$	$0 \leq x \leq 360$	2
2	$y = \sin(39.5x)$	$y = 0$	$0 \leq x \leq 360$	2
3	$y = \sin(81x)$	$y = \sin(2x)$	$0 \leq x \leq 360$	2
4	$y = \cos(79x)$	$y = 1$	$0 \leq x \leq 360$	2
5	$y = \tan(79x)$	$y = 0$	$0 \leq x \leq 360$	1
6	$y = \sin(79x)$	$y = 1$	$-90 \leq x \leq 270$	2
7	$y = \sin(76x)$	$y = -\sin(3x)\cos(79x)$	$0 \leq x \leq 360$	1

	Function entered	Function appearing on the TI-83	Range	$x$ res
1	$y = \sin(48x)$	$y = \sin x$	$0 \leq x \leq 360$	2
2	$y = \sin(49x)$	$y = \sin(2x)$	$0 \leq x \leq 360$	2
3	$y = \sin(47x)$	$y = 0$	$0 \leq x \leq 360$	2
4	$y = \cos(46x)$	$y = \cos x$	$0 \leq x \leq 360$	2
5	$y = \cos(94x)$	$y = 1$	$0 \leq x \leq 360$	1
6	$y = \cos(47x)$	$y = 1$	$0 \leq x \leq 360$	2
7	$y = \sin(48x)$	$y = -\sin(x) \cdot \cos(47x)$	$0 \leq x \leq 360$	1

## 6. Final remarks

To adopt this material to another graphing calculator, you must first know the number  $P$  of pixel columns that compose the viewing screen. If this number is not available in the manual, try making graphs of  $y = \sin(Nx)$ . Find the smallest value of  $N$  (a natural number or a natural number plus  $1/2$ ) whose graph looks like  $y = 0$ . Then the number of pixel columns is  $P = 2N + 1$ . Using this method we found that  $P = 96$  on the TI-81,  $P = 95$  on the TI-82 and TI-83,  $P = 127$  on the TI-85 and  $P = 239$  on the TI-92. On the Sharp EL-9600c we found  $P = 127$ .

In general, the step size will be

$$\Delta x = \frac{x_{\max} - x_{\min}}{P - 1}$$

and the values of  $x$  at which the function is calculated will be given by

$$x(n) = x_{\min} + \Delta x \cdot (x \text{ res})n \quad \text{where } n = 0, 1, 2, \dots, [(P - 1)/x \text{ res}]$$

If  $x$ res is not known, use  $x \text{ res} = 1$ .

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