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A NOVEL METHOD FOR FINDING $\zeta(2p)$ FROM A PRODUCT OF SINES

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The zeta function $\zeta(z)$ given by the series $\zeta(z)=\sum_{n=1}^{\infty}\frac{1}{n^z}$, valid for $\text{Re}(z)>1$, (see [4] and [9]), was first evaluated in closed form by Euler [5] when z is a positive even integer. The result is

(1) $$\zeta(2p)=\sum_{n=1}^{\infty}\frac{1}{n^{2p}}=\frac{(-1)^{p+1}2^{2p-1}B_{2p}}{(2p)!}\pi^{2p}\ .$$

Here the numbers B_n are called Bernoulli's numbers, and they are all rational. The first few are

$$B_0=1, B_1=-\frac{1}{2}, B_2=\frac{1}{6}, B_4=-\frac{1}{30}, B_6=\frac{1}{42}, \ldots, \text{ and } \quad B_3=B_5=B_7=\ldots=0\ .$$

These can all be calculated recursively by starting with $B_0=1$, and using

$$\binom{n}{0}B_0+\binom{n}{1}B_1+\binom{n}{2}B_2+\cdots+\binom{n}{n-1}B_{n-1}=0$$

for $n=2,3,4,\cdots$. (See Knopp [6], page 237.) Several additional methods of deriving (1) have been given since Euler, some of which are found in [1], [3], [4], [6], [7], and [8]. In this paper we will not derive (1) with the Bernoulli numbers, but we will show how to find $\zeta(2)=\pi^2/6$, $\zeta(4)=\pi^4/90$, and $\zeta(6)=\pi^6/945$ using a method that is relatively simple and appropriate for the undergraduate classroom. We finish by outlining how our method could be used to find $\zeta(2p)$ for any integer p.

We first review Euler's evaluation of $\zeta(2)$ as found in [5]. Start with the infinite product expansion $\sin z = z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{\pi^2 n^2}\right)$. Expanding the product we get

$$\text{(2)} \qquad \sin z = z - \frac{\zeta(2)}{\pi^2}z^3 + \frac{\sum_{n=1}^{\infty}\sum_{m=n+1}^{\infty}\frac{1}{n^2m^2}}{\pi^4}z^5 + \cdots .$$

Writing the Taylor's series for $\sin z$ on the left-hand side of (2) we get

$$z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = z - \frac{\zeta(2)}{\pi^2}z^3 + \frac{\sum_{n=1}^{\infty}\sum_{m=n+1}^{\infty}\frac{1}{n^2m^2}}{\pi^4}z^5 + \cdots .$$

Equating coefficients of z^3 we get $\zeta(2) = \pi^2/6$ at once. We can get $\zeta(4)$ from the coefficients of z^5, but we must first find the auxiliary relation

$\zeta(2)^2 = \zeta(4) + 2\sum_{n=1}^{\infty}\sum_{m=n+1}^{\infty}\frac{1}{n^2m^2}$. The auxiliary relations for $\zeta(6), \zeta(8), \zeta(10), \cdots$ become progressively more complex. Euler found the general nature of these auxiliary relations, and thereby found (1). The method we use avoids the need for auxiliary functions by replacing the function $\sin z$ by an appropriate function which is a product of sines.

To find $\zeta(4)$, consider the product

$$\text{(3)} \qquad z^2\prod_{n=1}^{\infty}\left(1-\frac{z^4}{\pi^4 n^4}\right) = z^2 - \frac{\zeta(4)}{\pi^4}z^6 + \cdots ,$$

which displays $\zeta(4)$ directly as the coefficient of z^6 on the right hand side. Now the product on the left is

(4) $$z^2\prod_{n=1}^{\infty}\left(1-\frac{z^4}{\pi^4 n^4}\right)=z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{\pi^2 n^2}\right)z\prod_{n=1}^{\infty}\left(1+\frac{z^2}{\pi^2 n^2}\right).$$

We recognize this last product as $-i\sin z\sin(iz)$. Thus we have from (3) and (4)

(5) $$-i\sin z\sin(iz)=z^2-\frac{\zeta(4)}{\pi^4}z^6+\cdots.$$

Expanding the left side in Taylor's series and multiplying we get

(6) $$\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots\right)\left(z+\frac{z^3}{3!}+\frac{z^5}{5!}+\cdots\right)=z^2+\left(\frac{2}{5!}-\frac{1}{(3!)^2}\right)z^6+\cdots.$$

Equating coefficients of z^6 in (5) and (6) we get $\zeta(4)=\dfrac{\pi^4}{90}$.

To find $\zeta(6)$, consider the product

(7) $$z^3\prod_{n=1}^{\infty}\left(1-\frac{z^6}{\pi^6 n^6}\right)=z^3-\frac{\zeta(6)}{\pi^6}z^9+\cdots,$$

which displays $\zeta(6)$ directly as the coefficient of z^9 on the right hand side. Now the product on the left is

(8) $$z^3\prod_{n=1}^{\infty}\left(1-\frac{z^6}{\pi^6 n^6}\right)=\omega^{-3}z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{\pi^2 n^2}\right)\omega z\prod_{n=1}^{\infty}\left(1-\frac{(\omega z)^2}{\pi^2 n^2}\right)\omega^2 z\prod_{n=1}^{\infty}\left(1-\frac{(\omega^2 z)^2}{\pi^2 n^2}\right) \quad ,$$

where $\omega=e^{\pi i/3}$. We recognize this last product as $\omega^{-3}\sin z\,\sin(\omega z)\,\sin\left(\omega^2 z\right)$. Thus we have from (7) and (8)

(9) $$-\sin z\,\sin(\omega z)\,\sin\left(\omega^2 z\right)=z^3-\frac{\zeta(6)}{\pi^6}z^9+\cdots \quad .$$

Expanding the left side in Taylor's series and multiplying we get

$$-\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+\cdots\right)\left(\omega z-\frac{\omega^3 z^3}{3!}+\frac{\omega^5 z^5}{5!}-\frac{\omega^7 z^7}{7!}+\cdots\right)\left(\omega^2 z-\frac{\omega^6 z^3}{3!}+\frac{\omega^{10} z^5}{5!}-\frac{\omega^{14} z^7}{7!}+\cdots\right)=$$

$$-\omega^3 z^3+\left(\frac{\omega^3+\omega^5+\omega^7}{3!}\right)z^5-\left(\frac{\omega^5+\omega^7+\omega^9}{(3!)^2}\right)z^7$$

.

$$-\left(-\frac{\omega^5+2\omega^7+2\omega^{11}+\omega^{13}}{3!5!}-\frac{\omega^9}{(3!)^3}-\frac{\omega^3+\omega^9+\omega^{15}}{7!}\right)z^9+\cdots$$

Now $\omega^3+\omega^5+\omega^7=\omega^5+\omega^7+\omega^9=0$, $\omega^5+2\omega^7+2\omega^{11}+\omega^{13}=\omega+\omega^{-1}+2(\omega+\omega^{-1})$ =3,

and $\omega^3+\omega^9+\omega^{11}=-3$ so the above expression becomes

(10) $$z^3-\left(-\frac{3}{3!5!}+\frac{1}{(3!)^3}+\frac{3}{7!}\right)z^9+\cdots=z^3-\left(\frac{1}{945}\right)z^9+\cdots$$

Equating coefficients of z^9 in (9) and (10) we get $\zeta(6)=\dfrac{\pi^6}{945}$.

Looking back at (2), (5), and (9) we saw that

$$\sin z=z-\frac{\zeta(2)}{\pi^2}z^3+\cdots,$$

$$e^{-\pi i/2}\sin z\sin(e^{i\pi/2}z)=z^2-\frac{\zeta(4)}{\pi^4}z^6+\cdots,$$

and that

$$e^{-2i\pi/2}\sin z\ \sin(e^{i\pi/3}z)\ \sin\left(e^{i2\pi/3}z\right)=z^3-\frac{\zeta(6)}{\pi^6}z^9+\cdots.$$

A little thought shows that these generalize to

(11) $$e^{(p-1)\pi i/2}\sin z\ \sin(e^{\pi i/p}z)\sin(e^{2\pi i/p}z)\sin(e^{3\pi i/p}z)\cdots\sin(e^{(p-1)\pi i/p}z)=z^p-\frac{\zeta(2p)}{\pi^{2p}}z^{3p}+\cdots$$

To find $\zeta(2p)$ from (11), we could replace each sine function on the left-hand side of by its Taylor's series and then multiply and equate coefficients of z^{3p} as we did above to determine $\zeta(2)$, $\zeta(4)$ and $\zeta(6)$. Without actually calculating this coefficient of z^{3p}, we can easily see that it is a rational number, since it is a sum of products of rational numbers. Thus $\zeta(2p)$ is a rational number times π^{2p}.

We will not derive (1) (with the Bernoulli numbers) from (11). The reader interested in this derivation will find it in [7].

REFERENCES

[1] Apostol, Tom M., *Another elementary proof of Euler's formula for* $\zeta(2n)$, American Mathematical Monthly, 80(1973), pp. 425-431.

[2] Ayoub, Raymond, *Euler and the Zeta function*, American Mathematical Monthly, 81(1974), pp. 1067-1086.

[3] Berndt, Bruce C., *Elementary evaluation of* $\zeta(z)$, Mathematics Magazine, 48(1975), pp. 148-154.

[4] Edwards, H. M., *Riemann's Zeta Function*, Academic Press, New York, 1974.

[5] Euler, Leonard *Introduction to Analysis of the Infinite, Book I,* (Translated by John D. Blanton) Springer-Verlag, New York, 1988, pp.137-153.

[6] Knopp, Konrad, *Theory and Application of Infinite Series*, Dover Publications, New York, 1990, pp. 236-240. (A translation by R. C. H. Young of the 4th German addition of 1947.)

[7] Osler, Thomas J., *Finding Zeta(2p) from a product of sines.* The American Mathematical Monthly, 111(2004), pp. 52-54.

[8] Williams, Kenneth S., *On* $\sum_{n=1}^{\infty}\left(1/n^{2k}\right)$, Mathematics Magazine, 44(1971), pp. 273-276.

[9] Titchmarsh, E. C. and Heath-Brown, D. R., *The Theory of the Riemann Zeta-function*, (2nd Ed.), Oxford University Press, Oxford, 1986.