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FINDING $\zeta(2p)$ FROM A PRODUCT OF SINES

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The zeta function $\zeta(z)$ given by the series $\zeta(z)=\sum_{n=1}^{\infty}\frac{1}{n^z}$ (valid for $\operatorname{Re}(z)>1$) was first evaluated in closed form by Euler [5] when z is a positive even integer. The result is

$$(1)\qquad \zeta(2p)=\sum_{n=1}^{\infty}\frac{1}{n^{2p}}=\frac{(-1)^{p+1}2^{2p-1}B_{2p}}{(2p)!}\pi^{2p}\ .$$

Here the numbers B_n are called Bernoulli's numbers, and they are all rational. The first few are

$$B_0=1, B_1=-\frac{1}{2}, B_2=\frac{1}{6}, B_4=-\frac{1}{30}, B_6=\frac{1}{42},\ldots,\text{ and }\quad B_3=B_5=B_7=\ldots=0\,.$$

These can all be calculated recursively by starting with $B_0=1$, and using

$$B_n=\sum_{k=0}^{n}\binom{n}{k}B_k\,,$$

for $n=2,3,4,\cdots$. When $n = 2$, this gives $B_1=-1/2$. (See Knopp [6], page 183, for an equivalent formula.) Several additional methods of deriving (1) have been given since Euler, some of which are found in [1], [3], [4], and [6]. We present a method here that we were unable to locate in the literature.

We begin our evaluation of $\zeta(2p)$ by considering the following two lemmas.

Lemma 1: *If* $\sum_{n=1}^{\infty} a_n$ *is an absolutely convergent series, then the product* $\prod_{n=1}^{\infty}\left(1+a_n z\right)$ *is an entire function of z whose power series expansion about the origin begins as follows:*

$$\prod_{n=1}^{\infty}\left(1+a_n z\right)=1+\left(\sum_{n=1}^{\infty} a_n\right)z+\cdots .$$

This Lemma is an immediate consequence of [6, p.439, example 1], where explicit formulas are also given for the coefficients of $z^2, z^3, \cdots$. (Euler also used this Lemma but he required the coefficients of the higher powers. In this paper, we use only the constant and linear term.)

The next lemma is the well known factorization of $1-t^p$ in terms of pth roots of unity.

Lemma 2: *If* $\omega = e^{\pi i/p}$*, where p is a positive integer, the following algebraic identity is valid for all t*:

$$\prod_{k=0}^{p-1}\left(1-\omega^{2k} t\right)=1-t^p .$$

When $t=\dfrac{z^2}{\pi^2 n^2}$, Lemma 2 gives

$$\prod_{k=0}^{p-1}\left(1-\frac{\left(\omega^k z\right)^2}{\pi^2 n^2}\right)=1-\frac{z^{2p}}{\pi^{2p} n^{2p}} . \tag{2}$$

Next use the infinite product representation $\sin z = z\prod_{n=1}^{\infty}\left(1-\dfrac{z^2}{\pi^2 n^2}\right)$ with *z* replaced by $\omega^k z$ and form the finite product of sines

$$\prod_{k=0}^{p-1}\sin\left(\omega^k z\right)=\omega^{p(p-1)/2} z^p \prod_{n=1}^{\infty}\prod_{k=0}^{p-1}\left(1-\frac{\left(\omega^k z\right)^2}{\pi^2 n^2}\right)$$

$$= \omega^{p(p-1)/2} z^p \prod_{n=1}^{\infty} \left(1 - \frac{z^{2p}}{\pi^{2p} n^{2p}} \right),$$

where in the last step we used (2). Now define

$$g(z) = \omega^{p(1-p)/2} \prod_{k=0}^{p-1} \sin\left(\omega^k z\right) = z^p \prod_{n=1}^{\infty} \left(1 - \frac{z^{2p}}{\pi^{2p} n^{2p}} \right).$$

Apply Lemma 1 to get the power series expansion

$$g(z) = \omega^{p(1-p)/2} \prod_{k=0}^{p-1} \sin\left(\omega^k z\right) = z^p - \frac{\zeta(2p)}{\pi^{2p}} z^{3p} + \cdots. \tag{3}$$

To find $\zeta(2p)$ from (3), we could replace each sine function by its Taylor's series and then multiply and equate coefficients of z^{3p}. However, to derive (1), it is easier if we first take the derivative of $g(z)$. From (3) we get

$$g'(z) = pz^{p-1} - 3p \frac{\zeta(2p)}{\pi^{2p}} z^{3p-1} + \cdots. \tag{4}$$

Another form of expansion (4) can be obtained by logarithmic differentiation of the finite product defining $g(z)$. We have from (3)

$$\log\left|g(z)\right| = \sum_{k=0}^{p-1} \log\left|\sin\left(\omega^k z\right)\right|,$$

whose derivative is

$$\frac{g'(z)}{g(z)} = \sum_{k=0}^{p-1} \omega^k \cot\left(\omega^k z\right),$$

and hence

$$g'(z) = g(z) \sum_{k=0}^{p-1} \omega^k \cot\left(\omega^k z\right) = \left(z^p - \frac{\zeta(2p)}{\pi^{2p}} z^{3p} + \cdots \right) \sum_{k=0}^{p-1} \omega^k \cot\left(\omega^k z\right). \tag{5}$$

To evaluate $\zeta(2p)$ we equate the coefficient of z^{3p-1} in (4) with that in (5).

Contributions to this coefficient in (5) come from two sources arising from the Laurent expansion of the sum of cotangents (from the coefficient of z^{-1}), and from the coefficient of z^{2p-1}. Because

$$\cot z = \frac{1}{z} + \sum_{r=1}^{\infty} c_r z^{2r-1} \text{, where } c_r = (-1)^r \frac{2^{2r} B_{2r}}{(2r)!},$$

we have

$$\omega^k \cot\left(\omega^k z\right) = \frac{1}{z} + \omega^k \sum_{r=1}^{\infty} c_r \left(\omega^k z\right)^{2r-1} = \frac{1}{z} + \sum_{r=1}^{\infty} c_r \omega^{2rk} z^{2r-1}.$$

When this is summed over *k* the total contribution from z^{-1} is *p*, while that from z^{2p-1} is pc_p because $\omega^{2kp} = 1$. Equating the coefficient of z^{3p-1} in (4) with that in (5) we find

$$-3p \frac{\zeta(2p)}{\pi^{2p}} = -p \frac{\zeta(2p)}{\pi^{2p}} + pc_p.$$

This gives $\zeta(2p) = -\frac{1}{2} c_p \pi^{2p} = (-1)^{p+1} \frac{2^{2p-1} B_{2p}}{(2p)!}$ as required. □

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