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FINDING ζ(2p) FROM A PRODUCT OF SINES
(Original Version with Motivational Material)

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The zeta function $\zeta(z)$ given by the series $\zeta(z)=\sum_{n=1}^{\infty}\frac{1}{n^z}$ (valid for $\mathrm{Re}(z)>1$) was first evaluated in closed form by Euler [5] when z is a positive even integer. The result is

(1) $$\zeta(2p)=\sum_{n=1}^{\infty}\frac{1}{n^{2p}}=\frac{(-1)^{p+1}2^{2p-1}B_{2p}}{(2p)!}\pi^{2p} .$$

Here the numbers B_n are called Bernoulli's numbers, and they are all rational. The first few are

$$B_0=1, B_1=-\frac{1}{2}, B_2=\frac{1}{6}, B_4=-\frac{1}{30}, B_6=\frac{1}{42},\ldots, \text{ and } \quad B_3=B_5=B_7=\ldots=0 .$$

These can all be calculated recursively by starting with $B_0=1$, and using

$$\binom{n}{0}B_0+\binom{n}{1}B_1+\binom{n}{2}B_2+\cdots+\binom{n}{n-1}B_{n-1}=0$$

for $n=2,3,4,\cdots$. (See Knopp [6], page 237.) Several additional methods of deriving (1) have been given since Euler, some of which are found in [1], [3], [4], and [6]. We present a method here that we were unable to locate in the literature.

To understand our method, we first review Euler's evaluation of $\zeta(2)$ as found in [5]. Start with the infinite product expansion $\sin z = z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{\pi^2 n^2}\right)$. Expanding the product we get

(2) $$\sin z = z - \frac{\zeta(2)}{\pi^2}z^3 + \frac{\sum_{n=1}^{\infty}\sum_{m=n+1}^{\infty}\frac{1}{n^2 m^2}}{\pi^4}z^5 + \cdots .$$

Writing the Taylor's series for $\sin z$ on the left-hand side of (2) we get

$$z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = z - \frac{\zeta(2)}{\pi^2}z^3 + \frac{\sum_{n=1}^{\infty}\sum_{m=n+1}^{\infty}\frac{1}{n^2 m^2}}{\pi^4}z^5 + \cdots .$$

Equating coefficients of z^3 we get $\zeta(2) = \pi^2/6$ at once. We can get $\zeta(4)$ from the coefficients of z^5, but we must first find the auxiliary relation $\zeta(2)^2 = \zeta(4) + 2\sum_{n=1}^{\infty}\sum_{m=n+1}^{\infty}\frac{1}{n^2 m^2}$. The auxiliary relations for $\zeta(6), \zeta(8), \zeta(10), \cdots$ become progressively more complex. Euler found the general nature of these auxiliary relations, and thereby found (1). The method we use avoids the need for auxiliary functions by replacing the function $\sin z$ by an appropriate function we call $g(z)$ to be described.

To motivate our method for finding $\zeta(4)$, consider the product

(3) $$z^2\prod_{n=1}^{\infty}\left(1-\frac{z^4}{\pi^4 n^4}\right) = z^2 - \frac{\zeta(4)}{\pi^4}z^6 + \cdots,$$

which displays $\zeta(4)$ directly as the coefficient of z^6 on the right hand side. Now the product on the left is

(4) $$z^2\prod_{n=1}^{\infty}\left(1-\frac{z^4}{\pi^4 n^4}\right) = z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{\pi^2 n^2}\right) z\prod_{n=1}^{\infty}\left(1+\frac{z^2}{\pi^2 n^2}\right).$$

We recognize this last product as $-i\sin z\sin(iz)$. Thus we have from (3) and (4)

(5) $$-i\sin z\sin(iz) = z^2 - \frac{\zeta(4)}{\pi^4}z^6 + \cdots.$$

Expanding the left side in Taylor's series and multiplying we get

(6) $$\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)\left(z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right) = z^2 + \left(\frac{2}{5!} - \frac{1}{(3!)^2}\right)z^6 + \cdots.$$

Equating coefficients of z^6 in (5) and (6) we get $\zeta(4) = \frac{\pi^4}{90}$.

Looking back, we saw that $\sin z = z - \frac{\zeta(2)}{\pi^2}z^3 + \cdots$, and that $-i\sin z\sin(iz) = z^2 - \frac{\zeta(4)}{\pi^4}z^6 + \cdots$. Now $\sin z$, (used to find $\zeta(2)$) has zeroes in the complex z plane on the two rays emerging from the origin making angles 0 and π. The function $-i\sin z\ \sin(iz)$, (used to find $\zeta(4)$), has zeros on the four rays making angles $0,\ \pi/2,\ \pi,$ and $3\pi/2$. It is now not difficult to anticipate the pattern emerging. To find $\zeta(6)$ we need a function which has zeroes on six rays making angles 0, $\pi/3$, $2\pi/3$, π, $4\pi/3$, and $5\pi/3$. A function having these zeroes is

$-\sin z\ \sin(\omega z)\ \sin\left(\omega^2 z\right) = z^3 - \frac{\zeta(6)}{\pi^6}z^6 + \cdots$, where $\omega = e^{\pi i/3}$. In general, to find $\zeta(2p)$ we need a function with zeroes on $2p$ rays. Such a function is

$g(z) = i^{1-p}\sin z\ \sin(\omega z)\sin(\omega^2 z)\sin(\omega^3 z)\cdots\sin(\omega^{p-1}z)$, where $\omega = \exp\left(\pi i/p\right)$.

Expanding each sine function as a product and multiplying we get

$g(z) = z^p\prod_{n=1}^{\infty}\left(1 - \frac{z^{2p}}{\pi^{2p}n^{2p}}\right) = \left(z^p + \frac{\zeta(2p)}{\pi^{2p}}z^{3p} - \cdots\right)$. Thus we have

$$(7)\qquad i^{1-p}\sin z\ \sin(\omega z)\sin(\omega^2 z)\sin(\omega^3 z)\cdots\sin(\omega^{p-1}z) = z^p + \frac{\zeta(2p)}{\pi^{2p}} z^{3p} - \cdots .$$

Since $\zeta(2p)$ occurs explicitly in (7) like $\zeta(2)$ does in (2)and $\zeta(4)$ does in (5), its calculation is now possible.

To find $\zeta(2p)$ from (7), we could replace each sine function on the left-hand side of (7) by its Taylor's series and then multiply and equate coefficients of z^{3p} as we did above to determine $\zeta(2)$ and $\zeta(4)$. However, to derive (1), it is easier if we first take the derivative of both sides of (7) with respect to *z*. Differentiating the product on the left we get

$$\begin{aligned} g'(z) = i^{1-p}\{\ &\cos z\ \sin(\omega z)\cdots\sin(\omega^{p-1}z) + \\ &\omega\sin z\ \cos(\omega z)\cdots\sin(\omega^{p-1}z) + \cdots + \\ &\omega^{p-1}\sin z\ \sin(\omega z)\cdots\cos(\omega^{p-1})\ \}\end{aligned} .$$

This simplifies to

$$g'(z) = g(z)\left(\cot z + \omega\cot(\omega z) + \omega^2\cot(\omega^2 z) + \cdots + \omega^{p-1}\cot(\omega^{p-1}z)\right).$$

Equating this last result to the derivative of the right-hand side of (7) we get

$$(8)\qquad \begin{aligned} &g(z)\left(\cot z + \omega\cot(\omega z) + \omega^2\cot(\omega^2 z) + \cdots + \omega^{p-1}\cot(\omega^{p-1}z)\right) = \\ &\qquad pz^{p-1} - \frac{3p\zeta(2p)}{\pi^{2p}} z^{3p-1} + \cdots . \end{aligned}$$

We know (see Knopp [6], p. 204) that $\cot z = \frac{1}{z} + \sum_{k=1}^{\infty} c_k z^{2k-1}$, where $c_k = (-1)^k \frac{2^{2k} B_{2k}}{(2k)!}$.

Next we rewrite (8) by substituting for $g(z)$ the Taylor's series on the right side of (7) and for each cotangent we substitute the series just described to get

$$\left(z^p - \frac{\zeta(2p)}{\pi^{2p}} z^{3p} + \cdots\right)\left(\frac{p}{z} + \sum_{k=1}^{\infty}\left(1+\omega^{2k}+\omega^{4k}+\cdots+\omega^{2(p-1)k}\right)c_k z^{2k-1}\right) =$$
$$pz^{p-1} - \frac{3p\zeta(2p)}{\pi^{2p}} z^{3p-1} + \cdots.$$

The expression $\left(1+\omega^{2k}+\omega^{4k}+\cdots+\omega^{2(p-1)k}\right) = p$ if k is a multiple of p, otherwise, it is zero. Thus we have

$$\left(z^p - \frac{\zeta(2p)}{\pi^{2p}} z^{3p} + \cdots\right)\left(\frac{p}{z} + pc_p z^{2p-1} + \cdots\right) = pz^{p-1} - \frac{3p\zeta(2p)}{\pi^{2p}} z^{3p-1} + \cdots.$$

Multiplying the two series on the left we get

$$pz^{p-1} + \left(pc_p - \frac{p\zeta(2p)}{\pi^{2p}}\right) z^{3p-1} + \cdots = pz^{p-1} - \frac{3p\zeta(2p)}{\pi^{2p}} z^{3p-1} + \cdots.$$

Equating the coefficients of z^{3p-1}, we get $\zeta(2p) = -\frac{c_p \pi^{2p}}{2}$. Since $c_p = (-1)^p \frac{2^{2p} B_{2p}}{(2p)!}$, we have derived (1).

REFERENCES

[1] Apostol, Tom M., *Another elementary proof of Euler's formula for* $\zeta(2n)$, American Mathematical Monthly, 80(1973), pp. 425-431.

[2] Ayoub, Raymond, *Euler and the Zeta function*, American Mathematical Monthly, 81(1974), pp. 1067-1086.

[3] Berndt, Bruce C., *Elementary evaluation of* $\zeta(z)$, Mathematics Magazine, 48(1975), pp. 148-154.

[4] Edwards, H. M., *Riemann's Zeta Function*, Academic Press, New York, 1974.

[5] Euler, Leonard *Introduction to Analysis of the Infinite, Book I,* (Translated by John D. Blanton) Springer-Verlag, New York, 1988, pp.137-153.

[6] Knopp, Konrad, *Theory and Application of Infinite Series*, Dover Publications, New York, 1990, pp. 236-240. (A translation by R. C. H. Young of the 4th German addition of 1947.)