Odd Abundant Numbers

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The natural numbers can be divided into three types, the *abundant*, *deficient*, and *perfect* numbers. If $\sigma(n)$ denotes the sum of all the positive divisors of n (including 1 and n), then n is classified as *deficient* if $\sigma(n) < 2n$, *perfect* if $\sigma(n) = 2n$, and *abundant* if $\sigma(n) > 2n$. For example, 5 is deficient ($\sigma(5) = 1 + 5 = 6 < 10 = 2 \times 5$), 28 is perfect ($\sigma(28) = 1+2+4+7+14+28 = 56 = 2 \times 28$), and 12 is abundant ($\sigma(12) = 1+2+3+4+6+12 = 28 > 24 = 2 \times 12$). Empirical evidence might erroneously lead us to believe that all the positive odd integers are deficient, particularly if the search did not exceed 500. Indeed, odd abundant numbers exist with 945 as the first member. The purpose of this article is to explore the sequence $u_n = 945 + 630n$ for the initial 52 values of n $(0 \leq n \leq 51)$. What transpires is rather curious and remarkable; our formula always generates an odd abundant number. In a manner similar to E. B. Escott's prime generating formula, $f(n) = n^2 - 79n + 1601$, which produces prime outputs for the initial 80 whole numbers (Escott's formula fails when $n = 80$ which yields the composite output $1681 = 41^2$), our formula fails when $n = 52$. Such investigations are nonetheless useful for two reasons, firstly, never abandon a problem after only the observation of a few cases and, secondly, the formulation of conjectures is an essential ingredient in mathematical discovery. We initiate our discussion with the following results, whose proofs are available in most elementary number theory references, including reference 1.

1. Every prime number as well as every power of a prime is deficient.

2. Every divisor of a perfect number or a deficient number is deficient.

3. Every multiple of a perfect number or an abundant number is abundant.

4. Using the formula for the sum of a finite geometric progression, we have $\sigma(p^n) = (p^{n+1} - 1)/(p - 1)$, where p is any prime.

5. The function σ is a multiplicative number–theoretic function in the sense that

$$\sigma(p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_r^{\alpha_r}) = \sigma(p_1^{\alpha_1})\sigma(p_2^{\alpha_2})\sigma(p_3^{\alpha_3}) \cdots \sigma(p_r^{\alpha_r})$$

   for distinct primes $p_1, \ldots, p_r$ and distinct whole numbers $\alpha_1, \ldots, \alpha_r$.

We are now in a position to prove that 945 is abundant. To see this, observe $\sigma(945) = \sigma(3^3 \times 5 \times 7) = \sigma(3^3)\sigma(5)\sigma(7) = ((3^{3+1} - 1)/(3 - 1))(5 + 1)(7 + 1) = 40 \times 6 \times 8 = 1920 > 1890 = 2 \times 945$.

The following theorem is an immediate consequence of the third result above.

Theorem 1 *There exist infinitely many odd abundant numbers.*

Table 1 shows all the odd abundant numbers less than 50 000. Outcomes of the form $945 + 630n$ are denoted by an asterisk in the first column. Not all odd abundant numbers are of this form.

Table 1

Odd abundant number n	Prime factorization of n	$\sigma(n)$	$2n$
945*	$3^3 \times 5 \times 7$	1 920	1 890
1 575*	$3^2 \times 5^2 \times 7$	3 224	3 150
2 205*	$3^2 \times 5 \times 7^2$	4 446	4 410
2 835*	$3^4 \times 5 \times 7$	5 808	5 670
3 465*	$3^2 \times 5 \times 7 \times 11$	7 488	6 930
4 095*	$3^2 \times 5 \times 7 \times 13$	8 736	8 190
4 725*	$3^3 \times 5^2 \times 7$	9 920	9 450
5 355*	$3^2 \times 5 \times 7 \times 17$	11 232	10 710
5 775	$3 \times 5^2 \times 7 \times 11$	11 904	11 550
5 985*	$3^2 \times 5 \times 7 \times 19$	12 480	11 970
6 435	$3^2 \times 5 \times 11 \times 13$	13 104	12 870
6 615*	$3^3 \times 5 \times 7^2$	13 680	13 230
6 825	$3 \times 5^2 \times 7 \times 13$	13 888	13 650
7 245*	$3^2 \times 5 \times 7 \times 23$	14 976	14 490
7 425	$3^3 \times 5^2 \times 11$	14 880	14 850
7 875*	$3^2 \times 5^3 \times 7$	16 224	15 750
8 085	$3 \times 5 \times 7^2 \times 11$	16 416	16 170
8 415	$3^2 \times 5 \times 11 \times 17$	16 848	16 830
8 505*	$3^5 \times 5 \times 7$	17 472	17 010
8 925	$3 \times 5^2 \times 7 \times 17$	17 856	17 850
9 135*	$3^2 \times 5 \times 7 \times 29$	18 720	18 270
9 555	$3 \times 5 \times 7^2 \times 13$	19 152	19 110
9 765*	$3^2 \times 5 \times 7 \times 31$	19 968	19 530
10 395*	$3^3 \times 5 \times 7 \times 11$	23 040	20 790
11 025*	$3^2 \times 5^2 \times 7^2$	22 971	22 050
11 655*	$3^2 \times 5 \times 7 \times 37$	23 712	23 310
12 285*	$3^3 \times 5 \times 7 \times 13$	26 880	24 570
12 705	$3 \times 5 \times 7 \times 11^2$	25 536	25 410
12 915*	$3^2 \times 5 \times 7 \times 41$	26 208	25 830
13 545*	$3^2 \times 5 \times 7 \times 43$	27 456	27 090
14 175*	$3^4 \times 5^2 \times 7$	30 008	28 350
14 805*	$3^2 \times 5 \times 7 \times 47$	29 952	29 610
15 015	$3 \times 5 \times 7 \times 11 \times 13$	32 256	30 030
15 435*	$3^2 \times 5 \times 7^3$	31 200	30 870
16 065*	$3^3 \times 5 \times 7 \times 17$	34 560	32 120
16 695*	$3^2 \times 5 \times 7 \times 53$	33 696	33 390
17 325*	$3^2 \times 5^2 \times 7 \times 11$	38 688	34 650
17 955*	$3^3 \times 5 \times 7 \times 19$	38 400	35 910
18 585*	$3^2 \times 5 \times 7 \times 59$	37 440	37 170
19 215*	$3^2 \times 5 \times 7 \times 61$	38 688	38 430
19 305	$3^3 \times 5 \times 11 \times 13$	40 320	38 610
19 635	$3 \times 5 \times 7 \times 11 \times 17$	41 472	39 270
19 845*	$3^4 \times 5 \times 7^2$	41 382	39 690
20 475*	$3^2 \times 5^2 \times 7 \times 13$	45 136	40 950
21 105*	$3^2 \times 5 \times 7 \times 67$	42 432	42 210
21 735*	$3^3 \times 5 \times 7 \times 23$	46 080	43 470
21 945	$3 \times 5 \times 7 \times 11 \times 19$	46 080	43 890
22 275	$3^4 \times 5^2 \times 11$	45 012	44 550
22 365*	$3^2 \times 5 \times 7 \times 71$	44 928	44 730
22 995*	$3^2 \times 5 \times 7 \times 73$	46 176	45 990
23 205	$3 \times 5 \times 7 \times 13 \times 17$	48 384	46 410
23 625*	$3^3 \times 5^3 \times 7$	49 920	47 250
24 255*	$3^2 \times 5 \times 7^2 \times 11$	53 352	48 510
24 885*	$3^2 \times 5 \times 7 \times 79$	49 920	49 770
25 245	$3^3 \times 5 \times 11 \times 17$	51 840	50 490
25 515*	$3^6 \times 5 \times 7$	52 464	51 030
25 935	$3 \times 5 \times 7 \times 13 \times 19$	53 760	51 870
26 145*	$3^2 \times 5 \times 7 \times 83$	52 416	52 290
26 565	$3 \times 5 \times 7 \times 11 \times 23$	55 296	53 130
26 775*	$3^2 \times 5^2 \times 7 \times 17$	58 032	53 550
27 405*	$3^3 \times 5 \times 7 \times 29$	57 600	54 810
28 035*	$3^2 \times 5 \times 7 \times 89$	56 160	56 070
28 215	$3^3 \times 5 \times 11 \times 19$	57 600	56 430
28 665*	$3^2 \times 5 \times 7^2 \times 13$	62 244	57 330
28 875	$3 \times 5^3 \times 7 \times 11$	59 904	57 750
29 295*	$3^3 \times 5 \times 7 \times 31$	61 440	58 590
29 835	$3^3 \times 5 \times 13 \times 17$	60 480	59 670
29 925*	$3^2 \times 5^2 \times 7 \times 19$	64 480	59 850
30 555*	$3^2 \times 5 \times 7 \times 97$	61 152	61 110
31 185*	$3^4 \times 5 \times 7 \times 11$	69 696	62 370
31 395	$3 \times 5 \times 7 \times 13 \times 23$	64 512	62 790
31 815*	$3^2 \times 5 \times 7 \times 101$	63 648	63 630
32 175	$3^2 \times 5^2 \times 11 \times 13$	67 704	64 350
32 445*	$3^2 \times 5 \times 7 \times 103$	64 896	64 890
33 075*	$3^3 \times 5^2 \times 7^2$	70 680	66 150
33 345	$3^3 \times 5 \times 13 \times 19$	67 200	66 690
33 495	$3 \times 5 \times 7 \times 11 \times 29$	69 120	66 990
33 915	$3 \times 5 \times 7 \times 17 \times 19$	69 120	67 830
34 125	$3 \times 5^3 \times 7 \times 13$	69 888	68 250
34 155	$3^3 \times 5 \times 11 \times 23$	69 120	68 310
34 965*	$3^3 \times 5 \times 7 \times 37$	72 960	69 930
35 805	$3 \times 5 \times 7 \times 11 \times 31$	73 728	71 610
36 225*	$3^2 \times 5^2 \times 7 \times 23$	77 376	72 450
36 855*	$3^4 \times 5 \times 7 \times 13$	81 312	73 710
37 125	$3^3 \times 5^3 \times 11$	74 880	74 250
37 485*	$3^2 \times 5 \times 7^2 \times 17$	80 028	74 970
38 115*	$3^2 \times 5 \times 7 \times 11^2$	82 992	76 230
38 745*	$3^3 \times 5 \times 7 \times 41$	80 640	77 490
39 375*	$3^2 \times 5^4 \times 7$	81 224	78 750
39 585	$3 \times 5 \times 7 \times 13 \times 29$	80 640	79 170
40 425	$3 \times 5^2 \times 7^2 \times 11$	84 816	80 850
40 635*	$3^3 \times 5 \times 7 \times 43$	84 480	81 270
41 055	$3 \times 5 \times 7 \times 17 \times 23$	82 944	82 110
41 895*	$3^2 \times 5 \times 7^2 \times 19$	88 920	83 790
42 075	$3^2 \times 5^2 \times 11 \times 17$	87 048	84 150
42 315	$3 \times 5 \times 7 \times 13 \times 31$	86 016	84 630
42 525*	$3^5 \times 5^2 \times 7$	90 272	85 050
42 735	$3 \times 5 \times 7 \times 11 \times 37$	87 552	85 470
43 065	$3^3 \times 5 \times 11 \times 29$	86 400	86 130
44 415*	$3^3 \times 5 \times 7 \times 47$	92 160	88 830
44 625	$3 \times 5^3 \times 7 \times 17$	89 856	89 250
45 045*	$3^2 \times 5 \times 7 \times 11 \times 13$	104 832	90 090
45 675*	$3^2 \times 5^2 \times 7 \times 29$	96 720	91 350
45 885	$3 \times 5 \times 7 \times 19 \times 23$	92 160	91 770
46 035	$3^3 \times 5 \times 11 \times 31$	92 160	92 070
46 305*	$3^3 \times 5 \times 7^3$	96 000	92 610
47 025	$3^2 \times 5^2 \times 11 \times 19$	96 720	94 050
47 355	$3 \times 5 \times 7 \times 11 \times 41$	96 768	94 710
47 775	$3 \times 5^2 \times 7^2 \times 13$	98 952	95 550
48 195*	$3^4 \times 5 \times 7 \times 17$	104 544	96 390
48 825*	$3^2 \times 5^2 \times 7 \times 31$	103 168	97 650
49 665	$3 \times 5 \times 7 \times 11 \times 43$	101 376	99 330
49 725	$3^2 \times 5^2 \times 13 \times 17$	101 556	99 450
49 875	$3 \times 5^3 \times 7 \times 19$	99 840	99 750

The number 33 705 corresponding to the value of $945 + 630n$ for $n = 52$ is missing from Table 1, so this formula fails to always produce an odd abundant number. In fact, $\sigma(33\,705) = \sigma(3^2 \times 5 \times 7 \times 107) = \sigma(3^2)\,\sigma(5)\,\sigma(7)\,\sigma(107) = 13 \times 6 \times 8 \times 108 = 67\,392 < 67\,410 = 2 \times 33\,705$, so 33 705 is deficient.

A second observation from Table 1 is that none of our integers of the form $945 + 630n$ are square-free. (An integer is *square-free* if it is not divisible by the square of any prime.) It is clear that all are divisible by 9.

We might erroneously believe from looking at Table 1 that all odd abundant numbers are divisible by 3 and end in the digit 5. The following counterexamples serve to refute these notions.

Counterexample 1 The odd abundant number 81 081 does not terminate in the digit 5, and is the first such number. In fact, $\sigma(81\,081) = \sigma(3^4 \times 7 \times 11 \times 13) = \sigma(3^4)\,\sigma(7)\,\sigma(11)\,\sigma(13) = 121 \times 8 \times 12 \times 14 = 162\,624 > 162\,162 = 2 \times 81\,081$.

Counterexample 2 The odd abundant number $1\,382\,511\,906\,801\,025 = 5^2 \times 7^2 \times 11^2 \times 13^2 \times 17^2 \times 19^2 \times 23^2$ is not a multiple of 3. It can be verified that $\sigma(1\,382\,511\,906\,801\,025 = 31 \times 57 \times 133 \times 183 \times 307 \times 381 \times 553 = 2\,781\,811\,913\,132\,763 > 2\,765\,023\,813\,602\,050 = 2 \times 1\,382\,511\,906\,801\,025$.

Theorem 2 *There are infinitely many odd abundant numbers terminating in any odd digit.*

Proof Since 81 081 is abundant, any odd integer multiple of this integer will again be an odd abundant number. The first few odd abundant integers terminating in digits other than 5 are 81 081, 153 153, 171 171, 189 189, 207 207, 243 243 and 297 297.

In conclusion, the set of odd abundant numbers lends itself to interesting investigations in number theory. The next step, for example, might be to explore square-free odd abundant numbers that have a maximum of five prime factors. With technology such as modern graphical calculators and MATHEMATICA®, further investigations are possible.

Reference

1 R. W. Prielipp, Perfect numbers, abundant numbers, and deficient numbers, *Math. Teacher* **63** (1970), pp. 690–692.

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+ equals ×

Find all pairs of real numbers whose sum is equal to their product.

Tienen, Belgium **Guido Lasters**