The Spirit of Discovery

The Digital Roots of Integers

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In many mathematics classes, students are asked to learn via the discovery method, in the hope that the intrinsic beauty of mathematics becomes more accessible and that making conjectures, forming hypotheses, and analyzing patterns will help them compute fluently and solve problems creatively and resourcefully (NCTM 2000). The activity discussed in this article was conducted with a group of preservice and inservice teachers, and the objectives included examining patterns, making conjectures, and using data analysis to construct scatter plots and tables, all in the spirit of discovering mathematics. This activity is based on a concept called the multiplicative digital root of an integer (Sloane 1973). Here we take the term *integers*, unless otherwise qualified, to mean positive integers.

The activity focuses on generating a possible next term in the following sequence:

$$6788, 2688, 768, 336, 54$$

Analyzing this sequence leads us to see that successively multiplying the digits of each integer produces the succeeding integer, as shown below:

$$6 \cdot 7 \cdot 8 \cdot 8 = 2688$$
$$2 \cdot 6 \cdot 8 \cdot 8 = 768$$
$$7 \cdot 6 \cdot 8 = 336$$
$$3 \cdot 3 \cdot 6 = 54$$
$$5 \cdot 4 = 20$$

Table 1

Multiplicative Digital Roots and Persistence of the Integers 10–30

Integer	Multiplicative Digital Root	Multiplicative Persistence
10	0	1
11	1	1
12	2	1
13	3	1
14	4	1
15	5	1
16	6	1
17	7	1
18	8	1
19	9	1
20	0	1
21	2	1
22	4	1
23	6	1
24	8	1
25	0	2
26	2	2
27	4	2
28	6	2
29	8	2
30	0	1

Our students agreed that the sixth term in this sequence is 20 and that an additional iteration would yield $2 \cdot 0 = 0$. The students also observed that six iterations were required to arrive at the single digit 0, not including the initial integer. Analyzing the initial integer, 6788, we called 0 its *multiplicative digital root* and 6 its *multiplicative persistence.*

To investigate the multiplicative digital root and the multiplicative persistence of all two-digit integers, the students deployed their graphing calculators and created a table denoting the two-digit integers, their multiplicative digital roots, and their multiplicative persistence. Students' results for the integers 10 through 30 is shown in **table 1**.

To illustrate, observe that if the seed integer is 25, then $25 \rightarrow 10 \rightarrow 0$. Hence, the multiplicative digital root of 25 is 0 and its multiplicative persistence is 2. As a second example, observe that for the seed integer 27, we have $27 \rightarrow 14 \rightarrow 4$. Consequently, the multiplicative digital root of 27 is 4 and its multiplicative persistence is 2.

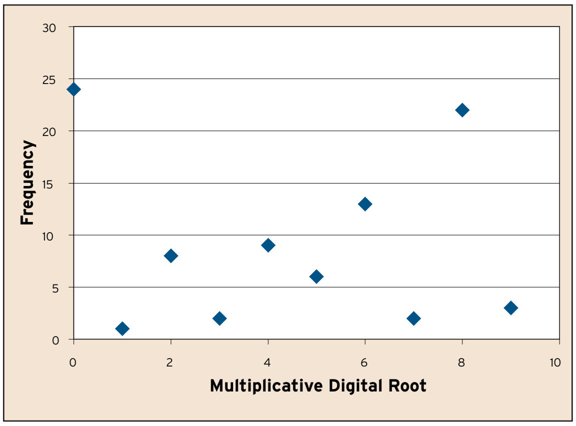

Fig. 1 Scatter plot of the multiplicative digital root of all two-digit integers

Table 2

Multiplicative Digital Roots and Their Frequencies for Two-Digit Integers

Multiplicative Digital Root	Frequency of the Multiplicative Digital Root
0	24
1	1
2	8
3	2
4	9
5	6
6	13
7	2
8	22
9	3

Table 3

Multiplicative Persistences and Their Frequencies for Two-Digit Integers

Multiplicative Persistence	Frequency of the Multiplicative Persistence
1	32
2	34
3	23
4	1

The summary of the collected data for the ninety two-digit integers is shown in **figure 1** and **tables 2** and **3**.

Figure 2 shows that there is only one two-digit integer having multiplicative persistence 4, namely, 77. Moreover, students noted that 0 was the mode for two-digit integers with a frequency of 24, followed closely by 8, which has a frequency of 22. They concluded that any integer that had 0 as one

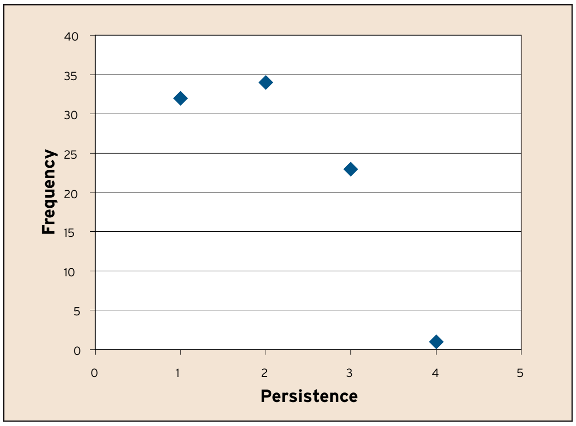

Fig. 2 Scatter plot of the multiplicative persistence of all two-digit integers

of its digits naturally would have multiplicative digital root 0. The digital root 1 occurred with only one integer (11), and the digital roots 3, 7, and 9 were rather scarce as well. Given a week to do so, students classified all nine hundred three-digit integers with respect to their multiplicative digital roots and their multiplicative persistence. See **tables 4** and **5** for these results.

Once again, the multiplicative digital root of 0 was the mode, occurring about 50 percent of the time. Meanwhile, 1 as a digital root appeared only once (with the integer 111), while 3, 7, and 9 as digital roots were quite infrequent. Students familiar with permutations realized that the integers 245, 254, 425, 452, 524, and 542 all had the same multiplicative digital root of 0. In the course of this activity, students posed numerous questions, such as "How many three-digit integers have zero as at least one digit?" To answer this question, students constructed the following argument:

In any set of one hundred consecutive integers, nine have 0 as a middle digit. In addition, for any one hundred consecutive three-digit integers, ten end in 0. Thus, there are nineteen integers with at least one digit that is 0 among the nine sets of three-digit integers: 100–199, 200–299, … up to 900–999. Therefore, there are 9 • 19, or 171, three-digit numbers having 0 as a digit.

At this juncture, students drew on a number of combinatorial ideas to count those three-digit numbers that did not have 0 as a digit. They observed that every three-digit integer having no digit of 0 has at most $3! = 3 \cdot 2 \cdot 1 = 6$ different permutations, or rearrangements, as previously illustrated by the integer 245. In addition, the integer 335 can be rearranged in only three different ways—as 335, 353, and 533. In this case, the two 3s are considered indistinguishable. These counting excursions aid in reducing the laborious task of determining the multiplicative digital roots and persistences for the nine hundred three-digit integers. Students were quickly able to place the integers in groups (for example, 245, 254, 425, 452, 524, and 542 belonged to one basic group) and thus create a reduced list of three-digit integers containing no digits of 0. Numbers were ordered as if they were page numbers in a textbook (called lexicographical ordering because it is akin to words being alphabetized in a dictionary). A systematic list was thus obtained. The following is an example of one group's solution:

111, 112, 113, 114, 115, 116, 117, 118, 119, 122, 123, 124, 125, 126, 127, 128, 129, 133, 134, 135, 136, 137, 138, 139, 144, 145, 146, 147, 148, 149, 155, 156, 157, 158, 159, 166, 167, 168, 169, 177, 178, 179, 188, 189, 199, 222, 223, 224, 225, 226, 227, 228, 229, 233, 234, 235, 236, 237, 238, 239, 244, 245, 246, 247, 248, 249, 255, 256, 257, 258, 259, 266, 267, 268, 269, 277, 278, 279, 288, 289, 299, 333, 334, 335, 336, 337, 338, 339, 344, 345, 346, 347, 348, 349, 355, 356, 357, 358, 359, 366, 367, 368, 369, 377, 378, 379, 388, 389, 399, 444, 445, 446, 447, 448, 449, 455, 456, 457, 458, 459, 466, 467, 468, 469, 477, 478, 479, 488, 489, 499, 555, 556, 557, 558, 559, 566, 567, 568, 569, 577, 578, 579, 588, 589, 599, 666, 667, 668, 669, 677, 678, 679, 688, 689, 699, 777, 778, 779, 788, 789, 799, 888, 889, 899, 999.

Table 4

Multiplicative Digital Roots and Their Frequencies for Three-Digit Integers

Multiplicative Digital Root	Frequency of the Multiplicative Digital Root
0	452
1	1
2	68
3	3
4	55
5	33
6	141
7	3
8	138
9	6

Table 5

Multiplicative Persistences and Their Frequencies for Three-Digit Integers

Multiplicative Persistence	Frequency of the Multiplicative Persistence
1	215
2	306
3	287
4	83
5	9

Table 6

Multiplicative Digital Roots and Their Frequencies for Four-Digit Integers

Multiplicative Digital Root	Frequency of the Multiplicative Digital Root
0	6263
1	1
2	466
3	4
4	214
5	132
6	1017
7	4
8	889
9	10

These 165 integers include 84 with all distinct digits. Students noted that there were nine choices for the hundreds digit, eight choices for the tens digit, and seven choices for the units digit. Applying the Fundamental Principle of Counting shows that the number of three-digit integers that can be formed with all digits different is 9 • 8 • 7 = 504. There are 3! = 6 different permutations of the 84 combinations of integers, and 504 ÷ 6 = 84. If two of the three digits are distinct, then the number of possible combinations of scenarios is 9 • 8 • 1 = 72, and the possible different permutations is 3 • 72 = 216 (the repeated digit may appear in both the hundreds and tens places, the hundreds and units places, or the tens and units places). Finally, there are clearly nine scenarios if all the digits are identical (as in the integer 111). If we add these results, we obtain 504 + 216 + 9 = 729 integers having no digits of 0. The students had previously found that 171 three-digit integers contain at least one digit of 0 and that 729 + 171 = 900; thus, they successfully completed the activity.

The enumeration above elicited additional patterns. For integers in the range 100–199 having no digits of 0, there were nine in the range 110–119, eight in the range 120–129, and seven in the range 130–139, followed by 6, 5, 4, 3, 2, and 1, respectively, for the next six groups of ten integers. Hence, the total is given by

$$9+8+7+6+5+4+3+2+1=\sum_{k=1}^{9} k = 45.$$

If one considers the next one hundred natural numbers (the range 200–299) having no digits of 0, the count by tens was reduced by one. Our count is

$$8+7+6+5+4+3+2+1=\sum_{k=1}^{8} k = 36.$$

Table 7

Multiplicative Persistences and Their Frequencies for Four-Digit Integers

Multiplicative Persistence	Frequency of Multiplicative Persistence
1	2512
2	2918
3	2280
4	1116
5	162
6	12

The total count for the next seven groups of one hundred natural numbers is 28 + 21 + 15 + 10 + 6 + 3 + 1 = 84. Adding 84 to the previous two sums yields 45 + 36 + 84 = 165, the number arrived at previously. In addition, students recognized that the terms in our sum (45, 36, … 6, 3, 1) were triangular numbers.

At this point, we sought to extend the activity further. We encouraged students to obtain a tally for both the multiplicative digital roots and the multiplicative persistences of all four-digit integers. **Tables 6** and **7** summarize their results.

An examination of **Table 6** indicates that more than two-thirds of the nine thousand four-digit integers have multiplicative digital root of 0. This finding is not that surprising, because any integer having 0 as a digit automatically has a multiplicative digital root of 0. The first one hundred integers in each group of one thousand have at least one digit of 0. It can be shown that 2439 four-digit integers have a digit of 0. In addition, integers containing 5 as a digit together with an even digit automatically will have a multiplicative digital root of 0 with a persistence of 2. Other combinations are possible.

By engaging in this activity, the students developed numerous conjectures, including the following:

Conjecture 1. The only integers having multiplicative digital root 1 are the repunits (repeated digits) 1, 11, 111, 1111, ….

Conjecture 2. The only integers containing at least two digits that have 3 as multiplicative digital root are those integers that have 3 as a single digit and 1 as all the remaining digits. Examples include 13, 31, 113, 131, 311, 1113, 1131, 1311, and 3111.

Conjecture 3. The only integers containing at least two digits that have 7 as multiplicative digital root are those integers that have 7 as a single digit and 1 as all the remaining digits. Examples include 17, 71, 117, 171, 711, 1117, 1171, 1711, and 7111.

Table 8	
The Initial Integer Having the Given Multiplicative Persistence	
Multiplicative Persistence	**Integer**
1	10
2	25
3	39
4	77
5	679
6	6788
7	68889
8	2677889
9	26888999
10	3778888999
11	277777788888899

Conjecture 4. The only integers containing at least two digits that have 9 as multiplicative digital root are those integers that contain 9 as a single digit and 1 as all the remaining digits or that contain 3 as two digits and 1 as all the remaining digits. Examples include 19, 33, 91, 119, 133, 191, 313, 331, 911, 1119, 1133, 1191, 1313, 1331, 1911, 3113, 3131, 3311, and 9111.

Conjecture 5. All integers having multiplicative digital root 1, 3, 7, or 9 have a multiplicative persistence of 1.

Conjecture 6. There are exactly n n-digit integers having multiplicative digital roots of either 3 or 7.

Justifications for conjectures 1–6 are relatively routine and certainly within the capabilities of middle school and high school students. Several other more involved conjectures were discussed by our group but not formally written. For example, with respect to the multiplicative digital root of 5, in addition to the two-digit integers 15 and 51, the following have a multiplicative digital root of 5: 35, 53, 57, and 75. In such instances, more than one iteration is required. Similarly, more than one iteration is required for integers having a multiplicative digital root of 0, 2, 4, 6, or 8. It is possible for such integers to have a multiplicative persistence greater than 1. An additional activity involves generating all integers that have up to ten digits and have an odd multiplicative digital root.

The curiosity of a few of our students involved finding the "first" integer having a given multiplicative persistence. To discover this, we turned to research and the Internet. **Table 8** presents the first integer having a multiplicative persistence of 1–11, as discovered by Sloane (1973). Sloane (1973) alludes to the fact that **Table 8** was generated by a computer program, although he does not name the program that he used. Naturally, one could appeal to the conjectures above as well as resort to rather labor-intensive brute force to generate the first integer having a multiplicative persistence of 1–6 inclusive. Without the use of an appropriate computer program, the initial integer having a multiplicative persistence of 7–11 would be extremely difficult to find.

In this article, we have explored an activity that can stimulate meaningful discovery by having participants collect data, harness technology, explore patterns, form conjectures based on the analysis of such patterns, use counting tools, and improve computational proficiency as part of their professional development. Through discovery-based learning, middle and high school teachers should be encouraged to achieve a deeper understanding of the mathematics they teach.

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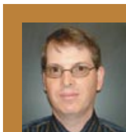


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