

Garden State Undergraduate Mathematics Competition
Individual Component – March 27, 2004

1. Suppose that a pair of fair dice, each numbered 1-6, is rolled. Determine the conditional probability that both dice bear the number 4, given that the sum of the numbers is 8.
2. How many distinguishable rearrangements of the letters in CONTEST have both the vowels first? For example, OECNTST is one such rearrangement, but OCNTSTE is not.
3. Suppose a cube is inscribed in a sphere, and that the sphere has radius 1. Determine the volume of the cube.
4. Suppose $f(x) = 2^{-x}$. Find the least positive integer n such that $f(n) < 0.0001$.
5. Determine the area of a triangle with side lengths $a = 5$, $b = 6$, and $c = 7$.
6. Suppose $f(x)$ is a differentiable function and $\frac{d^n}{dx^n}f(x)$ is defined for all $n \geq 1$. Suppose $\frac{d^3}{dx^3}f(x) = f(x)$ and let $g(x) = \cos(x) + f(x)$. Find the smallest positive integer n such that the following equality must hold: $\frac{d^n}{dx^n}g(x) = g(x)$.
7. What is the remainder of $(0! + 1! + 2! + 3! + \dots + 1000!)^3$ when divided by 5?
8. Let a be a positive integer. In terms of a , determine the value of

$$A = \lim_{x \rightarrow 0} x + \frac{a}{x + \frac{a}{x + \frac{a}{x + \dots}}}$$

9. Suppose (a_1, a_2, a_3, a_4) is a permutation of $(1, 2, 3, 4)$. Let $S = a_1a_2 + a_2a_3 + a_3a_4$.
 - (a) Determine the maximum possible value for S .
 - (b) Determine the minimum possible value for S .
10. A box contains marbles, each of which is red, white, or blue. The number of blue marbles is at least half the number of white marbles and at most one-third the number of red marbles. The number which are white or blue is at least 55. Find the minimum possible number of red marbles.
11. Suppose that $a_1, a_2, a_3, \dots, a_n, \dots$, is an increasing sequence of positive integers such that $a_{n+1} = a_n + a_{n-1}$ for $n \geq 2$ and $a_7 = 100$. Determine the value of a_8 .
12. Determine all positive integral solutions (m, n) to the equation $m^2 + 2004 = n^2$.
13. Determine the set of all positive integers $n \leq 1000$ such that both of the fractions $\frac{1}{n}$ and $\frac{1}{n+3}$ have terminating decimal expansions.
14. Determine the dimensions of the right circular cylinder of volume V which has the least surface area. We note that the top and bottom of the cylinder are part of the surface.

Garden State Undergraduate Mathematics Competition
Team Component – March 27, 2004

1. In a certain game a player's batting average is defined to be her score divided by her number of at-bats. Assume that the number of at-bats is always a positive integer n and the score is a non-negative integer s with $s \leq n$. Suppose that player A has a higher batting average than player B in both the first half and the second half of the season. Prove or disprove the following statement:

“For the full season, player A has a higher batting average than player B.”

2. In the following diagram, the temperature at nodes A, B, C, D is the equal to the average of all adjacent nodes (a node is *adjacent* if it is to the immediate left, right, top, or bottom of a given node). Find the temperature at node A .

	20	20	
10	A	B	40
10	C	D	40
	30	30	

3. Let a, b be variables. Find and prove a formula for $\det(A)$, where A is the $n \times n$ matrix defined by

$$A = \begin{pmatrix} a & b & b & \dots & b \\ b & a & b & \dots & b \\ b & b & a & \dots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \dots & a \end{pmatrix}.$$

4. Let S be a finite set of distinct non-negative integers such that $|x - y| \in S$ for all $x, y \in S$.

(a) Give an example of such a set S with 10 elements.

(b) If A is a subset of S containing more than two-thirds of the elements of S , prove or disprove that *every* element of S is the sum or difference of two elements of A .

5. Let S be a set which is closed under the binary operation \circ with the following properties:

(1) There is an element $e \in S$ such that $a \circ e = e \circ a = a$ for each $a \in S$.

(2) $(a \circ b) \circ (c \circ d) = (a \circ c) \circ (b \circ d)$ for all $a, b, c, d \in S$.

Prove or disprove the following statements:

(a) \circ is associative on S .

(b) \circ is commutative on S .

6. Let ϕ denote the Euler phi function, which is defined for positive integers n by the formula

$$\phi(n) = n \prod_{\substack{p|n \\ p \text{ prime}}} \left(1 - \frac{1}{p}\right),$$

for $n > 1$, and $\phi(1) = 1$. Define $\gamma(n)$, for a positive integer n , by

$$\gamma(n) = \begin{cases} \prod_{\substack{p|n \\ p \text{ prime}}} p, & \text{for } n > 1 \\ 1, & \text{if } n = 1. \end{cases}$$

Find the six positive integers n which satisfy $\gamma(n)^2 = \phi(n)$. Please leave your answers in their prime factorization form.

7. Evaluate $\lim_{k \rightarrow \infty} \frac{R_k(2)}{R_k(3)}$, where

$$R_k(n) = \sqrt{2 - \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2 + \sqrt{n}}}}},$$

is defined using k square-roots. Hint: Trigonometry.