Garden State Undergraduate Mathematics Competition
Individual Component Solutions

1. $\boxed{\frac{1}{5} = 20\%}$ There are 5 possible rolls of the dice whose sum is 8: $(2,6),(3,5),(4,4),(5,3),(6,2)$. Of these 5, one is the possibility $(4,4)$. So the probability is $1/5$.

2. $\boxed{120}$. There are 2 possibilities for the way the first two letters can be vowels. The remaining 5 letters can be rearranged in $5!/2 = 60$ possibilities (the doubled T means each of 5! possibilities are counted twice). Total, we have $2 \times 60 = 120$ rearrangements.

3. $\boxed{\frac{8}{3\sqrt{3}} = \frac{8\sqrt{3}}{9}}$. Let s be the length of an edge of the cube. Then the diagonal of a side has length $\sqrt{2}s$ and the diagonal of the cube is $\sqrt{3}s$. Now since the cube is inscribed in the sphere, $\sqrt{3}s = 2$ and $s = 2/\sqrt{3}$, so the volume is $s^3 = 8/(3\sqrt{3})$.

4. $\boxed{14}$. We want $2^{-n} < 10^{-4}$, so $2^n > 10^4 = 10000$. Now $2^8 = 256, 2^{10} = 1024, 2^{13} = 8192$, so $n = 14$ is the smallest solution.

5. $\boxed{6\sqrt{6}}$. Let $\triangle ABC$ be the triangle and $AB = 5$, $BC = 6$, $AC = 7$. Let the altitude from B to AC intersect AC at D. Let x be the length AD. Then $7 - x$ is the length of DC. Now by the Pythagorean theorem, $(BD)^2 + x^2 = 5^2$, and $(BD)^2 + (7-x)^2 = 6^2$. Then $25 - x^2 = 36 - (7-x)^2 = -13 + 14x - x^2$. So $14x = 38$ and $x = \frac{19}{7}$. Then the length of BD is $y\sqrt{25 - (19/7)^2} = 12\sqrt{6}/7$. So the area of the triangle is $7y/2 = 6\sqrt{6}$.

6. $\boxed{12}$. The fourth derivative of $\cos(x)$ is $\cos(x)$, so n needs to be multiple of 4. Correspondingly, we need n to be a multiple of 3 from the hypothesis on $f(x)$. $n = 12$ is the smallest number that satisfies both conditions.

7. $\boxed{4}$. The remainder of $0! + 1! + 2! + \cdots + 1000!$ when divided by 5 is $1 + 1 + 2 + 1 + 4 + 0 + 0 + \cdots + 0 = 9 = 4$. Cubed, the remainder is 4.

8. $\boxed{\sqrt{a}}$. From the limit, we have $A = \lim_{x \to 0} x + \frac{a}{A}$. So $A = a/A$ and $A^2 = a$. So $A = \sqrt{a}$.

9. $\boxed{\text{(a) 23(b) 12}}$ To maximize the sum, you want the smallest numbers $1,2$ to be a_1, a_4 and be on the edges of the sum. To minimize the sum, you want the largest numbers on the edges. Trial and error shows that $2(4)+(4)(3)+(3)(1) = 23$ and $4(1)+(1)(2)+(2)(3) = 12$.

10. $\boxed{57}$. Let r, w, b be the number of red, white, and blue marbles. Then $b \geq w/2$, $b \leq r/3$, and $w + b \geq 55$. Since $2b \geq w$, $3b \geq 55$, $b \geq 19$. Minimizing w, r, we get $w = 36$, and $r = 57$.

11. $\boxed{162}$. We have $a_3 = a_1 + a_2$, $a_4 = a_1 + 2a_2$, $a_5 = 2a_1 + 3a_2$, $a_6 = 3a_1 + 5a_2$, $a_7 = 5a_1 + 8a_2$. Now $5a_1 + 8a_2 = 100$ has only one integral solution with $a_2 > a_1 > 0$. It is $a_1 = 4$, $a_2 = 10$. Then the sequence is $4, 10, 14, 24, 38, 62, 100, 162, \ldots$.

12. $\boxed{(500,502),(164,170)}$ We have $2004 = n^2 - m^2 = (n-m)(n+m)$. Since $2004 = 2^2(3)(167)$, and $n-m$, $n+m$ have the same parity, we must have that $n-m$, $n+m$ are even. Since $n + m > n - m$, there are only two possibilities: $(n+m, n-m) = (1002, 2), (334, 6)$. Solving for (m, n), we obtain $(500,502),(164,170)$ as solutions.

13. $\boxed{n = 1, 2, 5, 125}$ If $1/n$ has a terminating decimal expansion, then $n = 2^k 5^l$. So its last digit is $0, 1, 2, 4, 5, or 8$. Now 1 can only occur when $n = 1$. So the last digits of $(n, n+3)$ could be $(2, 5)$, $(5, 8)$, or $(1, 4)$. Since $n < 1000$, the first case can only happen when $(n, n+3) = (2, 5)$, the second case can only happen when $(n, n+3) = (5, 8), (125, 128)$, and the third case can only happen when $(n, n+3) = (1, 4)$. So $n = 1, 2, 5, 125$.

14. $\boxed{\text{radius} = \sqrt[3]{\frac{V}{2\pi}},\ \text{height} = \sqrt[3]{\frac{4V}{\pi}}}$ Let r, h, be the radius and height of the cylinder. Then $V = \pi r^2 h$. Now the surface area is $S = 2\pi rh + 2\pi r^2$. Substituting $h = V/(\pi r^2)$, we have $S(r) = 2V/r + 2\pi r^2$. Differentiating, $S'(r) = -2V/r^2 + 4\pi r$. $S'(r) = 0$ gives $r^3 = V/(2\pi)$, so $r = \sqrt[3]{V/(2\pi)}$. So $h = \sqrt[3]{4V/\pi}$.

Garden State Undergraduate Mathematics Competition
Team Component Solutions

1. $\boxed{\text{False}}$. Consider the following situation. Suppose Player A gets one hit in her one at-bat in the first half of the season, and gets 1 hits in 199 at-bats in the second half of the season. Then Player A bats 1.000 in the first half of the season, and .005 in the second half of the season, and .0100 for the entire season.

Now if Player B gets 100 hits in 199 at-bats in the first half of the season and no hits in 1 at-bat the second half of the season. Then Player B bats .505 in the first half of the season, .000 in the second half of the season, and .500 for the entire season.

2. $\boxed{A = 20}$. We want to solve the following equations:

$$\begin{aligned} 30 + B + C &= 4A \\ 60 + A + D &= 4B \\ 70 + B + C &= 4D \\ 40 + A + D &= 4C \end{aligned}$$

Solving this equations gives $A = 20$, $B = 35/2$, $C = 45/2$, $D = 30$.

3. $\boxed{\det(A) = (a-b)^{n-1}[a + (n-1)b]}$. Let A_n denote the $n \times n$ matrix. We can prove this by induction. When $n = 1$, we have $\det(A_1) = a$. We assume that the formula is proved for a $n \times n$ matrix. Given A_{n+1} subtract the top row from the bottom row. Then the bottom row has $b - a$ in the first column, and $a - b$ in the last column. Expanding along the last row, we have

$$|(-1)^{2+n}(b-a)\begin{vmatrix} b & b & \dots & b & b \\ a & b & \dots & b & b \\ b & a & \dots & b & b \\ \vdots & \vdots & \ddots & b & b \\ b & b & \dots & a & b \end{vmatrix} + (-1)^{2n+2}(a-b)|A_n|.$$

Now

$$\begin{vmatrix} b & b & \dots & b & b \\ a & b & \dots & b & b \\ b & a & \dots & b & b \\ \vdots & \vdots & \ddots & b & b \\ b & b & \dots & a & b \end{vmatrix} = \begin{vmatrix} 0 & 0 & \dots & 0 & b \\ a-b & 0 & \dots & 0 & b \\ 0 & a-b & \dots & 0 & b \\ \vdots & \vdots & \ddots & 0 & b \\ 0 & 0 & \dots & a-b & b \end{vmatrix} = (-1)^{n+1}b(a-b)^{n-1}$$

And

$$\begin{aligned} |A_{n+1}| &= b(a-b)(a-b)^{n-1} + (a-b)[(a-b)^{n-1}[a + (n-1)b] \\ &= (a-b)^n[b + a + (n-1)b] = (a-b)^n[a + nb]. \end{aligned}$$

4. (a) The set $S = \{0, 1, 2, 3, \dots, 9\}$ satisfies the hypothesis.
(b) $\boxed{\text{The statement is true.}}$ Proof: Let n be the number of elements of A. The hypothesis on A implies $n > \frac{2}{3}|S|$. Let m be the number of elements of S not in A. Then $m < |S|/3$ and $2m < n$. Assume that there is an element $x \in S$ that is not the sum or difference of two elements of A. Then if $y \in A$ and $y < x$, then $x = y + (x - y)$, so $x - y \in S - A$ (otherwise x is the sum of two elements of S). Similarly if $y \in A$ and $y \geq x$, then $x = y - (y - x)$, so $y - x \in S - A$ (otherwise x is the difference of two elements of S). Thus if $y_1, y_2, \dots, y_n$ are the elements of A, then $|x - y_1|, |x - y_2|, \dots, |x - y_n|$, is a list of elements of $S - A$. As $|x - y_i|$ may equal $|y_j - x|$, the second list may list some elements of $S - A$ twice. But

there are at least $[(n-1)/2]+1$ distinct elements. Hence $m \geq [(n-1)/2]+1 \geq n/2$ and $2m \geq n$. But this contradicts the inequality $2m < n$ derived above. Hence our assumption is incorrect and every element of S must be the sum or difference of two elements of A.

5. $\boxed{\text{Both associativity and commutativity hold.}}$
(b) Commutativity:

$$\begin{aligned} a \circ b &= (e \circ a) \circ (e \circ b) \\ &= (e \circ b) \circ (e \circ a) \text{ by the given} \\ &= b \circ a. \end{aligned}$$

(a) Associativity:

$$\begin{aligned} a \circ (b \circ c) &= a \circ (c \circ b) \text{ by (b)} \\ &= (a \circ e) \circ (c \circ b) \\ &= (a \circ b) \circ (c \circ e) \\ &= (a \circ b) \circ c \end{aligned}$$

6. If $n = 1$, $\phi(1) = \gamma(1) = 1$. Now $\phi(2) \neq \gamma(2)$, and for $n > 2$, $\phi(n)$ is even. Now if $\gamma(n)$ is even, n is even and $\gamma(n) = 2^2 m$, with m odd. Rewriting the formula for $\phi(n)$ as $\prod_i p_i^{k_i-1}(p_i-1)$, where $n = \prod_p p_i^{k_i}$ is the prime factorization of n, we see that there can only be two odd primes p_i in the factorization. For if there are more than 2 odd primes p_i, $\phi(n)$ is divisible by 8 and the equality $\phi(n) = \gamma(n)^2$ cannot hold. Hence there are three cases:

$$1)\ n = 2^k, \qquad 2)\ n = 2^k p^l, \qquad 3)\ n = 2^k p^l q^m,$$

for which $\phi(n) = \gamma(n)^2$ holds. In (1), $2^{k-1} = 4$, so $k = 3$, and $n = 8$. In (2), $2^{k-1}p^{l-1}(p-1) = 4p^2$. So $l = 3$, and $2^{k-1}(p-1) = 4$. If $p-1 = 2$, $p = 3$ and $k = 2$, so $n = 2^2 3^3 = 108$. If $p-1 = 4$, $p = 5$ and $k = 1$ so $n = 2^1 5^3 = 250$. In case (3), $2^{k-1}(p-1)p^{l-1}(q-1)q^{m-1} = 4p^2q^2$. We can assume q is the largest prime. Then $m = 3$ and we have $2^{k-1}(p-1)p^{l-1}(q-1) = 4p^2$. Since $p-1$, $q-1$ are even $k = 1$. Since $(p-1, p) = 1$, $p-1$ divides 4. Now $p-1 = 1$ forces $p = 2$ and $p-1 = 4$ forces $q = 2$. So the only choice is $p-1 = 2$. Then $p = 3$ and $3^{l-1}(q-1) = 18$. Then $(l, q) = (1, 18), (2, 7)$ are the only two solutions. These lead to $n = 2(3)(19)^3 = 41154$ and $n = 2(3)^2(7)^3 = 6174$. So the six values of n are

$$\boxed{\begin{aligned} n =& 1, 8 = 2^3, 108 = 2^2 3^3, 250 = 2(5)^3, \\ & 6174 = 2(3)^2(7)^3, 41154 = 2(3)(19)^3 \end{aligned}}$$

7. For $n = 2, 3$, we have $\cos(\pi/2n) = \sqrt{n}/2$. Then using the half-angle identity,

$$\cos(\theta) = \sqrt{\frac{\cos(2\theta)+1}{2}}, \text{ for } 0 \leq \theta < \pi/2,$$

we have $\cos(\pi/(4n)) = \frac{1}{2}\sqrt{2+\sqrt{n}}$. In general,

$$\cos(\pi/(2^k n)) = \frac{1}{2}\sqrt{2+\sqrt{2+\cdots+\sqrt{2+\sqrt{n}}}},$$

where there are k square roots. We can prove this by induction. We have already established it for $k = 2$. Then assuming the identity is true for k, we have

$$\begin{aligned}
\cos(\pi/(2^{k+1}n)) &= \sqrt{\frac{1+\cos(\pi/(2^k n))}{2}} \\
&= \sqrt{\frac{1}{4}(2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2+\sqrt{n}}}})} \\
&= \frac{1}{2}\sqrt{2+\sqrt{2+\cdots+\sqrt{2+\sqrt{n}}}},
\end{aligned}$$

where the last equation has $k+1$ square roots. Now $\sin(\theta) = \sqrt{\frac{1-\cos(2\theta)}{2}}$, so

$$R_k(n) = \sqrt{2-2\cos(\pi/(2^k n))} = 2\sin(\pi/(2^{k+1}n)), \text{ for } n = 2, 3.$$

Then

$$\begin{aligned}
\lim_{k\to\infty} \frac{R_k(2)}{R_k(3)} = \lim_{k\to\infty} \frac{2\sin(\pi/(2^{k+1}2))}{2\sin(\pi/(2^{k+1}3))} &= \lim_{x\to 0} \frac{\sin(x/2)}{\sin(x/3)} \\
&= \lim_{x\to 0} \left(\frac{\sin(x/2)}{x/2} \frac{x/3}{\sin(x/3)} \frac{3}{2} \right) \\
&= \lim_{x\to 0} \frac{\sin(x/2)}{x/2} \lim_{x\to 0} \frac{x/3}{\sin(x/3)} \lim_{x\to 0} \frac{3}{2} \\
&= 1 \cdot 1 \cdot \frac{3}{2} = \boxed{\frac{3}{2}}.
\end{aligned}$$