THE RIEMANN ZETA FUNCTION AND ITS APPLICATION TO NUMBER THEORY

ABDULKADIR HASSEN AND MARVIN I KNOPP

1. Introduction

The Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{1}$$

where $s = \sigma + it$. This notation for a complex number s is due to Riemann and now it is a standard notation in this context. In this article we will consider the basic properties of $\zeta(s)$ and prove some of its interesting properties. Our main goal will be to show an application of the zeta function in the proof of the Prime Number Theorem, henceforth abbreviated by PNT.

Legendre and Gauss independently conjectured the PNT as follows. Let $\pi(x)$ be the number of primes less than or equal to x, where x is a positive real number. Then

$$\lim_{x\to\infty} \frac{\pi(x)}{\frac{x}{\log x}} = 1 \tag{2}$$

In his only paper in Number Theory, in 1859, Riemann showed a deep relationship between the zeros of the zeta function and $\pi(x)$. This eight pages paper in fact gave rise to what is now known as Analytic Number Theory, a branch of Number Theory that uses complex analysis in tackling problems involving integers. To show pont of view is one of the intentions of this paper. We hope that the reader will be curios

This paper is based on the lecture notes given by Marvin Knopp of Temple University. The second author was a graduate students who took these notes. It was in these lectures that Marvin introduced me to the theory of the Riemann zeta function. For his lively lectures, everlasting encouragement, and becoming my thesis advisor, I say thank you! I hope that this paper will introduce young mathematicians to this beautiful theory and inspire them to go beyond these pages!

and interested enough to explore this rich and vibrant field of mathematics. For this we recommend the introductory texts in this area, among which we mention Apostol [1], Chandrasekharan [3], Ireland [5], and Patterson [10].

In section 2 will review the necessary background material needed to develop the theory of the Riemann zeta function as it pertains to the proof of PNT. In section 3 we will develop the properties of the zeta function and proof the functional equation it satisfies. In section 4 we will first give some elementary theorems involving $\pi(x)$ and conclude the section with Newman's proof of PNT.

2. Preliminaries

Clearly the series defining $\zeta(s)$ in (1) convergence for $\sigma > 1$. However the most interesting properties of zeta is in the region where $\sigma \leq 1$. The series representation given by (1) is invalid in this region and therefore we have to find a way to extend it to this region. The most fruitful *analytic continuation* of zeta is by way of (improper) integrals. Thus we will be defining functions using integrals and justify that such functions are analytic. In many instances we need to interchange the process of integration and limit and summation.

For the sake of minimizing our exposition and to focus on the important technical aspect of the application of the zeta function to PNT, we shall assume that the reader is familiar with the theory of the functions of one complex variable and theorems of convergence of real analysis. One of the theorems of real analysis we will be using often is the Weierstrass M-test for uniform convergence of series of functions. We will also be using consequence of uniform convergence. For readers with a graduate level real analysis, we point out that the integrals we deal with can be considered as Lebesgue integrals and thus we can easily appeal to the Dominated Lebesgue Convergence Theorem. For proofs of theorems related to these topics, we refer the reader to any standard text book of advanced real analysis but we mention Goldberg [4](Chapter 9), and Knopp [6](Chapter XII, Sections 56 to 58).

One of the most important theorems of complex analysis that we will be using frequently is the Identity Theorem. Here is the statement of the theorem. For the proof we refer the reader to Marsden [8](Page 397).

Theorem 1. Identity Theorem or The Principle of Analytic Continuation *Let f and g be analytic in a region R. Suppose that there is a sequence $\{z_n\}$ of distinct points of R converging to a point $z_0 \in R$ such that $f(z_n) = g(z_n)$ for all $n = 1,\ 2,\ 3,\ \cdots$. Then $f = g$ on all of R.*

Example: Let $g(z) = \sum_{n=0}^{\infty} z^n$ and $f(z) = \frac{1}{1-z}$. If $|z| < 1$, $g(z) = f(z)$.(The series is a geometric series.) $f(z)$ is analytic every where except at $z = 1$. Thus f is an analytic continuation of g in the sense that we define $g(z)$ to be $f(z)$ for $z \neq 1$.

To obtain a different representation of the Riemann zeta function it is essential to use the Gamma and Theta functions. We shall define this two function next and state the main properties that we shall need for our investigation of ζ.

Definition 1. *The Gamma function, denoted by $\Gamma(s)$ is defined by*

$$\Gamma(s) \ = \ \int_0^\infty x^{s-1} e^{-x}\, dx, \qquad \sigma > 0.$$

Integration by parts ($u = e^{-x}, dv = x^{s-1}dx$) yields

$$\Gamma(s) \ = \ \frac{1}{s}\Gamma(s+1).$$

Note then that if $s = n$ is a positive integer, then $\Gamma(n) \ = \ (n-1)!$. More importantly, we note that the integral defining $\Gamma(s+1)$ is convergent for $Re(s) > -1$ and hence $\frac{1}{s}\Gamma(s+1)$is the analytic continuation of $\Gamma(s)$ to the region $Re(s) > -1$. We repeat this process to extend $\Gamma(s)$ to the whole plane with simple poles at the nonpositive integers. One of the classical books on Special Functions, Lebedev [7](Chapter 1) treats many interesting properties and applications of the gamma function.

Next we introduce the theta function.

Definition 2. *The theta function* $\Theta(z)$ *is defined by*

$$\Theta(z) \;=\; \sum_{n=-\infty}^{\infty} e^{n^2\pi i z} \;=\; 1+2\sum_{n=1}^{\infty} e^{n^2\pi i z} \qquad\qquad Im(z)=y>0.$$

The importance of the theta function lies in its property that we state in

Theorem 2. (The Transformation Law of Theta)

(1) $\Theta(z+2) \;=\; \Theta(z)$.

(2) $\Theta\left(\frac{-1}{z}\right) \;=\; e^{\frac{-\pi i}{4}} z^{\frac{1}{2}}\Theta(z)$.

As we shall see later, the transformation law stated above plays an important role in the analytic continuation of the zeta function. In fact, the functional equation is a consequence of this transformation law. For the sake of simplicity we will proof a special case of Theorem 2 that we state in the following proposition. However, we note that the Identity Theorem can easily be used to deduce Theorem 2 from

Proposition 1. *If* $x>0$*, then*

$$\Theta\left(\frac{1}{x}\right) \;=\; x^{\frac{1}{2}}\Theta(x).$$

To prove this form of the transformation law of the theta function, we first need the following theorem from analysis.

Theorem 3. (Poisson Summation Formula) *If f is continuous and $\sum_{n=-\infty}^{\infty} f(n+t)$ converges uniformly on $0 \leq t \leq 1$ and if $\sum_{n=-\infty}^{\infty} f(n)e^{2\pi int}$ converges, then*

$$\sum_{n=-\infty}^{\infty} f(n+t) \; = \; \sum_{n=-\infty}^{\infty} \hat{f}(ne^{2\pi int},$$

where

$$\hat{f}(n) \; = \; \int_{-\infty}^{\infty} f(x)e^{-2\pi inx}\, dx.$$

Proof: Define $\phi(t) \; = \; \sum_{m=-\infty}^{\infty} f(t+m)$. By assumption ϕ is continuous and clearly $\phi(t+1) = \phi(t)$. Thus $\phi(t)$ has a Fourier series expansion given by

$$\phi(t) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi int},$$

where

$$a_n = \int_0^1 \phi(x)e^{-2\pi inx}\, dx.$$

Let us find a_n by substituting the summation for $\phi(x)$ in the integral.(We leave it to the reader to justify the permissibility of interchanging summation and integration.)

$$\begin{aligned} a_n = \int_0^1 \phi(x)e^{-2\pi inx}\, dx \; &= \; \int_0^1 \sum_{m=-\infty}^{\infty} f(x+m)e^{-2\pi inx}\, dx \\ = \sum_{m=-\infty}^{\infty} \int_0^1 f(x+m)e^{-2\pi inx}\, dx \; &= \; \sum_{m=-\infty}^{\infty} \int_m^{m+1} f(x)e^{-2\pi inx}\, dx \\ = \int_{-\infty}^{\infty} f(x)e^{-2\pi inx}\, dx = \hat{f}(n), \end{aligned}$$

as desired.

Proof of Proposition 1: In what follows x is a fixed positive real number. Again we leave it to the reader to justify interchanging summations and integration.

Define $f(u) \; = \; e^{-\pi u^2 x}$, Then $\sum_{n=-\infty}^{\infty} f(n) \; = \; \sum_{n=-\infty}^{\infty} e^{-\pi n^2 x}$ converges and by Poisson Summation Formula (with $t = 0$), we have

$$\Theta(x) = \sum_{n=-\infty}^{\infty} f(n) \; = \; \sum_{n=-\infty}^{\infty} \hat{f}(n) \; = \; \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(u) e^{-2\pi i n u}\, du. \tag{3}$$

But $f(u) \; = \; e^{-u^2 x}$ and hence by completing the square we have

$$e^{-\pi u^2 x - 2\pi i n u} = e^{-\pi x \left(u^2 + \frac{2in}{x} u\right)} = e^{-\pi x \left(u + \frac{in}{x}\right)^2 - \frac{\pi n^2}{x}}.$$

Change of variable $t = u + \frac{in}{x}$ then yields

$$\int_{-\infty}^{\infty} f(u) e^{-2\pi i n u}\, du = \int_{-\infty}^{\infty} e^{-\pi x \left(u + \frac{in}{x}\right)^2 - \frac{\pi n^2}{x}}\, du = e^{-\frac{\pi n^2}{x}} \int_{-\infty + \frac{in}{x}}^{\infty + \frac{in}{x}} e^{-\pi x t^2}\, dt.$$

It can be shown that

$$\int_{-\infty + \frac{in}{x}}^{\infty + \frac{in}{x}} e^{-\pi x t^2}\, dt = \int_{-\infty}^{\infty} e^{-\pi x t^2}\, dt. \tag{4}$$

Thus we have

$$\int_{-\infty}^{\infty} f(u) e^{-2\pi i n u}\, du = e^{-\frac{\pi n^2}{x}} \int_{-\infty}^{\infty} e^{-\pi x t^2}\, dt.$$

Finally to remove x form the integral, we let $t = \frac{y}{\sqrt{\pi x}}$. This yeilds

$$\int_{-\infty}^{\infty} f(u) e^{-2\pi i n u}\, du = \frac{e^{-\frac{\pi n^2}{x}}}{\sqrt{\pi x}} \int_{-\infty}^{\infty} e^{-y^2}\, dy = \gamma \frac{e^{-\frac{\pi n^2}{x}}}{\sqrt{x}}, \tag{5}$$

where

$$\gamma = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-y^2}\, dy.$$

Substituting (5) in (3) and noting that γ is a constant, we obtain

$$\Theta(x) = \frac{\gamma}{\sqrt{x}} \Theta\left(\frac{1}{x}\right). \tag{6}$$

To complete the proof wee need to show $\gamma = 1$. Since (6) holds for all $x > 0$, putting $x = 1$ in the equation yields $\gamma = 1$, thereby completing the proof of the Proposition.

As a consequence of Proposition 1, we have the following

Corollary 1. *For $t > 0$, let $\Psi(t) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$. Then*

$$\Psi\left(\frac{1}{t}\right) = -\frac{1}{2} + \frac{1}{2} t^{\frac{1}{t}} + t^{\frac{1}{2}} \Psi(t). \tag{7}$$

Proof: Follows from $\Psi(t) = \frac{\Theta(t)-1}{2}$.

3. The Riemann Zeta Function

Recall that the Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

where $s = \sigma + it$. Since $|n^{-s}| = n^{-\sigma}$, it follows from integral test for convergence of infinite series that the series convergence absolutely for $\sigma > 1$. Furthermore, if $a > 1$ and $\sigma \geq a$, then

$$\sum_{n=1}^{\infty} |\frac{1}{n^s}| = \sum_{n=1}^{\infty} |\frac{1}{n^{\sigma}}| \leq \sum_{n=1}^{\infty} |\frac{1}{n^a}| < \infty.$$

Thus convergence is uniform and therefore $\zeta(s)$ is analytic in the region $\sigma > 1$. For in depth analysis and detailed proofs of properties of the Riemann zeta function, we recommend Titchmarsh [12].

Theorem 4. *$\zeta(s)$ can be extended to the right half plane $\sigma > 0$ such that $\zeta(s) - \frac{1}{s-1}$ is analytic in $\sigma > 0$.*

Poof: For $\sigma > 0$, define

$$\phi_n(s) = \frac{1}{n^s} - \int_n^{n+1} \frac{1}{u^s}\, du = \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{u^s}\right) du.$$

Then

$$|\phi_n(s)| = |\int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{u^s}\right) du| \leq \max_{u \in [n, n+1]} \left(n^{-s} - u^{-s}\right).$$

But $n^{-s} - u^{-s} = \int_n^u sx^{-s-1}\, dx$. Thus we have

$$|n^{-s} - u^{-s}| \le |s| \int_n^{n+1} x^{-\sigma-1}\, dx \ \le \frac{|s|}{\sigma} \left(n^{-\sigma} - (n+1)^{-\sigma}\right).$$

Adding over n we get

$$\sum_{n=1}^{\infty} |\phi_n(s)| \ \le \ \frac{|s|}{\sigma} \sum_{n=1}^{\infty} \left(n^{-\sigma} - (n+1)^{-\sigma}\right) \ = \ \frac{|s|}{\sigma}.$$

Thus $\sum_{n=1}^{\infty} \phi_n(s)$ converges absolutely and since $\phi_n(s)$ is entire, it follows that the function $F(s)$ defined by

$$F(s) \ = \ \sum_{n=1}^{\infty} \phi_n(s)$$

is analytic function in $\sigma > 0$.

On the other hand, we have, for $\sigma > 1$,

$$\begin{aligned} F(s) \ &= \ \sum_{n=1}^{\infty} \phi(s) = \ \sum_{n=1}^{\infty} \left(\frac{1}{n^s} - \int_n^{n+1} \frac{1}{u^s}\, du \right) \ = \sum_{n=1}^{\infty} \frac{1}{n^s} - \sum_{n=1}^{\infty} \int_n^{n+1} \frac{1}{u^s}\, du \\ &= \ \sum_{n=1}^{\infty} \frac{1}{n^s} - \sum_{n=1}^{\infty} \left(\frac{(n+1)^{-s+1} - n^{-s+1}}{-s+1} \right) \\ &= \ \sum_{n=1}^{\infty} \frac{1}{n^s} - \frac{1}{s-1} \\ &= \ \zeta(s) - \frac{1}{s-1}. \end{aligned}$$

Thus $\zeta(s) \ = \ F(s) \ + \ \frac{1}{s-1}$ for $\sigma > 1$. Since $F(s)$ and $\frac{1}{s-1}$ are analytic for $\sigma > 0$, we see that $F(s) \ + \ \frac{1}{s-1}$ the analytic continuation of $\zeta(s)$ to the region $\sigma > 0$ with a simple pole at $s = 1$.

Next we extend this theorem by proving the functional equation of the Riemann zeta function.

Theorem 5. *Let*

$$\Phi(s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s).$$

Then $\Phi(s)$ can be continued analytically to the whole plane and it satisfies the functional equation

$$\Phi(s) = \Phi(1-s).$$

Proof: From the definition of $\zeta(s)$ and $\Gamma(s)$, we have

$$\Phi(s) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{s}{2}}\left(\int_0^\infty x^{\frac{s}{2}-1}e^{-x}\,dx\right)\left(\sum_{n=1}^\infty \frac{1}{n^s}\right) = \sum_{n=1}^\infty \int_0^\infty \left(\frac{x}{n^2\pi}\right)^{-\frac{s}{2}} e^{-x}\,dx.$$

Let $y = \frac{x}{n^2\pi}$. Then

$$\Phi(s) = \sum_{n=1}^\infty \int_0^\infty y^{\frac{s}{2}-1}e^{-n^2\pi y}\,dy \;=\; \int_0^\infty y^{\frac{s}{2}-1}\sum_{n=1}^\infty e^{-n^2\pi y}\,dy \;=\; \int_0^\infty y^{\frac{s}{2}-1}\Psi(y)\,dy$$

Now split the integral into two pieces to get

$$\Phi(s) = \int_0^1 y^{\frac{s}{2}-1}\Psi(y)\,dy \;+\; \int_1^\infty y^{\frac{s}{2}-1}\Psi(y)\,dy.$$

In the first integral, let us change the variable by letting $y = \frac{1}{u}$. Then

$$\Phi(s) = \int_1^\infty u^{-\frac{s}{2}-1}\Psi\left(\frac{1}{u}\right)\,du \;+\; \int_1^\infty y^{\frac{s}{2}-1}\Psi(y)\,dy. \tag{8}$$

Using (7), we can rewrite (8) as

$$\begin{aligned}
\Phi(s) &= \int_1^\infty u^{-\frac{s}{2}-1}\left(-\frac{1}{2}+\frac{1}{2}u^{\frac{1}{2}}+u^{\frac{1}{2}}\Psi(u)\right)\,du \;+\; \int_1^\infty y^{\frac{s}{2}-1}\Psi(y)\,dy \\
&= -\frac{1}{2}\int_0^\infty u^{-\frac{s}{2}-1}\,du \;+\; \frac{1}{2}\int_0^\infty u^{-\frac{s}{2}-\frac{1}{2}}\,du \;+\; \int_0^\infty u^{-\frac{s}{2}-1}\Psi(u)\,du \;+\; \int_1^\infty y^{\frac{s}{2}-1}\Psi(y)\,dy \\
&= -\frac{1}{s}+\frac{1}{s-1} \;+\; \int_0^\infty u^{-\frac{s}{2}-1}\Psi(u)\,du \;+\; \int_1^\infty y^{\frac{s}{2}-1}\Psi(y)\,dy.
\end{aligned}$$

Replacing u by y in the first integral on the last line of the above equation, we get

$$\Phi(s) = -\frac{1}{s} \;+\; \frac{1}{s-1} \;+\; G(s), \tag{9}$$

where

$$G(s) \;=\; \int_0^\infty y^{-\frac{s}{2}-1}\Psi(y)\,du \;+\; \int_1^\infty y^{\frac{s}{2}-1}\Psi(y)\,dy \tag{10}$$

Note then that $G(s)$ is an entire function, since clearly $\Psi(s) \le Ae^{-\alpha y}$, as $y \to \infty$ for some constants A and α. Also $\frac{1}{s}$ and $\frac{1}{s-1}$ are analytic except at 0 and 1, respectively. Therefore $-\frac{1}{s} \;+\; \frac{1}{s-1} \;+\; G(s)$ is the analytic continuation of $\Phi(s)$ to the whole plane.

To see that $\Phi(s)$ satisfies the functional equation $\Phi(s) = \Phi(1-s)$, we need only to observe that under the transformation $s \to 1-s$ the first integral in $G(s)$ goes to the second and vice versa. Clearly $\frac{1}{s} - \frac{1}{s-1}$ goes back to itself when s is replaced by $1-s$. This completes the proof of the theorem.

Remark 1. (1) *From the fact that* $\Gamma(s) = \frac{1}{s}\Gamma(s+1)$, *we deduce that* $\lim_{s\to 0} s\Gamma(s) = \lim_{s\to 0}\Gamma(s+1) = 1$. *It follows that* $\lim_{s\to 0} s\Gamma(\frac{s}{2} = 2$. *Since* $G(s)$ *is entire,* $G(0)$ *is finite and hence* $\lim_{s\to 0} sG(s) = 0$. *But then*

$$\lim_{s\to 0}(s\Phi(s)) = \lim_{s\to 0}\left(-1 + \frac{s}{s-1} + sG(s)\right) = -1.$$

From these facts and the definition of $\Phi(s)$, *we conclude that* $\zeta(0) = \frac{-1}{2}$

(2) *The function* $G(s)$ *defined by* (10) *is entire. Thus* $\Phi(s)$ *is analytic everywhere except at* $s = 0$ *and* $= 1$, *where it has simple poles. Solving for* $\zeta(s)$ *from* $\Phi(s) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$, *we get*

$$\zeta(s) = \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)}\Phi(s).$$

Since the only poles of $\Gamma(s)$ *are* $0,\ -1,\ -2,\ -3,\ \cdot$, *it follows that* $\frac{1}{\Gamma(\frac{s}{2})}$ *is entire with zeros at* $s = 0, -2, -4, -6, \cdots$. *The functional equation* $\Phi(s) = \Phi(1-s)$ *and the fact that* $\zeta(n)$ *and* $\Gamma(n)$ *are nonzero for positive integers* n, *implies that* $s = -2, -4, -6, \cdots$ *are zeroes of* $\zeta(s)$. *These zeroes are called the trivial zeroes.*

(3) *We will show shortly that if* $Re(s) = \sigma > 1$, *then* $\zeta(s) \neq 0$. *This fact and the functional equation imply that all other zeroes are in the vertical strip* $0 < \sigma < 1$. *This is known as the* **critical strip**.

The Riemann Hypothesis, *one of the most famous open problems of the last 14 decades, states that* **all the nontrivial zeroes of $\zeta(s)$ lie on the line $\sigma = \frac{1}{2}$**

Next we derive the Euler Product Formula for $\zeta(s)$. For the remainder of our discussion p will be used exclusively to denote a prime number.

Theorem 6. *If $Re(s) = \sigma > 1$, then $\zeta(s)$ has following infinite product expansion*

$$\zeta(s) = \prod_p \left(1 - p^{-s}\right)^{-1} \tag{11}$$

where the product is taken over all primes p. This assumption will be used throughout our discussion.

Proof: Let $X > 0$ be a positive integer. Consider the product $\prod_{p \le X} \left(1 - p^{-s}\right)^{-1}$. We expand each term of this product in to power series to get

$$\left(1 - p^{-s}\right)^{-1} = 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} \frac{1}{p^{3s}} + \cdots .$$

Substituting in the above product and multiplying out the terms(note the we have an absolutely uniform convergent series), we get

$$\prod_{p \le X} \left(1 - p^{-s}\right)^{-1} = \prod_{p \le X} \sum_{k=0}^{\infty} \frac{1}{p^{ks}} = \sum_{k=1}^{\infty} \frac{1}{n_k^s}$$

where the n_k are those integers for which their maximum prime divisor But all integers less than or equal to X are of the form n_k for some k,it follows that

$$\left|\zeta(s) - \prod_{p \le X} \left(1 - p^{-s}\right)^{-1}\right| \le \frac{1}{(X+1)^\sigma} + \frac{1}{(X+2)^\sigma + \cdots}.$$

We now let $X \to \infty$ and observe that

$$\frac{1}{(X+1)^\sigma} + \frac{1}{(X+2)^\sigma + \cdots}$$

is the tail end of a convergent series for $\sigma > 1$ and hence goes to 0. This proves the theorem.

One of the many consequences of Euler's product formula (11) is that it gives us a proof for the infinitude of primes. For if there were a finite number of primes then the product in (11) would be finite for $s = 1$ which in term would imply that $\lim_{s\to 1}\zeta(s)$ is finite. Since $\lim_{s\to 1}\zeta(s)$ is the harmonic series, we have a contradiction. Here is another consequence of (11)

Corollary 2. *$\zeta(s) \neq 0$ for $Re(s) = \sigma > 1$*

Proof: Follows from Euler Product formula (11) and the fact that for $\sigma > 1$, $1 - p^{-s} \neq 0$ for all primes p.

The following theorem is critical in the proof of the Prime Number Theorem that will be considered in the next section.

Theorem 7. *$|\zeta(1 + it|) \neq 0$ for $Re(s) = \sigma \geq 1$.*

We first prove

Lemma 1. *For $s = \sigma + it$, $\sigma > 1$,and $t \neq 0$, we have*

$$\left|\zeta^3(\sigma)\zeta^4(s)\zeta(s + it)\right| \geq 1$$

Proof: Let $\rho = e^{i\phi}$, ϕ real. Then $\rho^{1/2} + \rho^{-1/2} = 2\cos(\phi/2)$ and hence

$$0 \leq \left(\rho^{1/2} + \rho^{-1/2}\right)^4 = \rho^2 + \rho^{-2} + 4(\rho + \rho^{-1}) + 6 = 2Re(\rho^2) + 8Re(\rho) + 6.$$

Thus

$$Re(\rho^2) + 4Re(\rho) + 3 \geq 0 \tag{12}$$

Assume for now that $Re(s) = \sigma > 1$ and take the logarithm of both sides of (11) to get

$$\log\zeta(s) = -\sum_p \log\left(1 - p^{-s}\right) = \sum_p \sum_{m=1}^{\infty} \frac{1}{m}\frac{1}{p^{ms}} = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

where

$$a_n = \begin{cases} \frac{1}{m}, & n = p^m \\ 0, & \text{otherwise} \end{cases}$$

and we have applied the power series expansion of $\log(1-x)$ and rearranged the double sum. From this we conclude that

$$\log\zeta(\sigma) = \sum_{n=1}^{\infty} a_n n^{-\sigma} \qquad \log\zeta(s) = \sum_{n=1}^{\infty} \left(a_n n^{-\sigma}\right) n^{-it} \qquad \log\zeta(s+it) = \sum_{n=1}^{\infty} \left(a_n n^{-\sigma}\right) n^{-2it}.$$

Note that $a_n n^{-\sigma} > 0$. Put $\rho = n^{-it} = e^{-it\log n}$. Then $|\rho| = 1$ and by (12) we have

$$Re(n^{-2it}) + 4Re(n^{-it}) + 3 \geq 0.$$

Multiplying this inequality by $a_n n^{-\sigma} > 0$, noting the fact that $Re\big(a_n n^{-\sigma} n^{-it}\big) = (a_n n^{-\sigma}) Re\big(n^{-it}\big)$, and adding the resulting inequality over n, we get

$$\sum_{n=1}^{\infty} \left\{ Re\big(a_n n^{-\sigma} n^{-2it}\big) + 4Re\big(a_n n^{-\sigma} n^{-it}\big) + 3a_n n^{-\sigma} \right\} \geq 0$$

Thus we have

$$Re\Big(\log\zeta(\sigma+2it) + 4\log\zeta(s) + 3\log\zeta(\sigma)\Big) \geq 0$$

and exponentiation yields

$$e^{Re\Big(\log\zeta(\sigma+2it)+4\log\zeta(s)+3\log\zeta(\sigma)\Big)} \geq 1$$

This implies that

$$e^{Re(\log\zeta(\sigma+2it)} e^{4\log\zeta(s)} e^{3\log\zeta(\sigma))} \geq 1$$

Since $|z| = e^{Re\big(\log(z)\big)}$ holds for all z, we conclude that

$$\left|\zeta^3(\sigma)\zeta^4(s)\zeta(s+it)\right| \geq 1$$

as desired.

Proof of Theorem 7. From the above lemma we have, for $\sigma > 1$,

$$\Big|(\sigma-1)\zeta(s+it)\Big|\Big|\left(\frac{\zeta(s)}{\sigma-1}\right)\Big|^4\Big|\big((\sigma-1)\zeta(\sigma)\big)\Big|^3 \geq 1 \tag{13}$$

Now suppose $|\zeta(1+it)| = 0$ for $t \neq 0$. Since $\zeta(s)$ has a simple pole at $s = 1$, we have

$$\lim_{\sigma\to 1}(\sigma-1)\zeta(\sigma) = 1. \tag{14}$$

Note also that

$$\zeta'(1+it) = \lim_{\sigma\to 1}\frac{\zeta(\sigma+it)-\zeta(1+it)}{(\sigma+it)-(1+it)} = \lim_{\sigma\to 1}\frac{\zeta(\sigma+it)}{\sigma-1} \tag{15}$$

exists and is finite. Since $\zeta(s+it) = \zeta(1+2it)$, we see that

$$\lim_{\sigma\to 1}(\sigma-1)\zeta(s+it) = 0. \tag{16}$$

Taking the limit as $\sigma \to 1^+$ for the expression on the left side of (13) and using (14), (15) and (16), we conclude that the limit of the left hand side of (13) is 0 which is a contradiction to the inequality stated there.

4. The Prime Number Theorem

Let $\pi(x)$ denote the number of primes less than or equal to x. Euclid proved that there are infinitely many prime numbers. (Euclid's proof of the infinitude of primes can be found in any introductory level Number Theory books such as [11].) Thus clearly $\lim_{x\to\infty}\pi(x) = \infty$. The question then becomes how does $\pi(x)$ behave at infinity? In other words, how does it go to infinity? To answer this question we first define the following notations.

Definition 3. *Let $f,\ g\ : R \to R$ be functions such that $g(x) \geq 0$. Then we say*

(1) $f(x) = \bigcirc(g(x))$ *if and only if there exists $R > 0$ and $M > 0$ such that $|f(x)| \leq Mg(x)$ for all $x > R$.*

(2) $f(x) = \circ\left(g(x)\right)$ *if and only if* $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$.

(3) $f(x) \sim g(x)$ *if and only if* $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$.

If f and g satisfy definition 3, we say that they are asymptotic. The following facts can easily be proven and will be used freely.

(1) $\bigcirc\left(\bigcirc\left(g(x)\right)\right) = \bigcirc\left(g(x)\right)$

(2) $\bigcirc\left(g(x)\right) \pm \bigcirc\left(g(x)\right) = \bigcirc\left(g(x)\right)$

(3) $\bigcirc\left(g(x)\right) \pm \circ\left(g(x)\right) = \bigcirc\left(g(x)\right)$

(4) $\left(\bigcirc\left(g(x)\right)\right)^2 = \bigcirc\left((g(x))^2\right)$

Theorem 8. *(The Prime Number Theorem - PNT)*

$$\lim_{x\to\infty} \frac{\pi(x)}{\left(\frac{x}{\log x}\right)} = 1. \tag{17}$$

The proof of PNT is one of the crown achievements of modern mathematics. The effort made in proving it has tremendous impact in the development of complex analysis in the 19th and 20th centuries. Among the principal contributers to the proof of the PNT were Legendre, Gauss, Tchebychev, Riemann, Dirichlet, Hadamard, and De la Valle Poussin. Each of these mathematicians used the methods of analysis. In the 1929 Erdos and Selberg gave an *elementary proof* in the sense that their proof did not use the methods of analysis. For a brief summary of the history of the theorem and its impact see the excellent and readable paper of Bateman and Diamond of 1996(Bateman[1]). The first proof the theorem appeared in 1896.It was proved independently by Hadamard and De la Valle Poussin. In this section we present the proof of Newman [6](Chapter 7). We begin with a brief historical development of this theorem.

The first asymptotic result we will look at is that of Legendre. This states:

$$\pi(x) \sim \frac{x}{\log x - 1.08\cdots} \tag{18}$$

We pose to the reader: Is it possible to find such a constants (or functions) A and B such that

$$\pi(x) \sim \frac{x}{A\log x + B} \tag{19}$$

gives a better asymptotic formula for $\pi(x)$ than (18)? For readers with access to the computer algebra system Mathematica, the following module is helpful for experimental purposes and to check the accuracy of (18).

$$\textbf{Table[\{ n, PrimePi[n],Floor[}\frac{n}{\textbf{Log}[n-1.08]}\textbf{\},\{n, a, b, c\}]//TableForm}$$

Here **Floor[t]** is the greatest integer less than or equal to t, the variable a is the first value of n in the table, b is the last value and c is the increment. These should be specified when evaluating the command.

The next improvement on this is given by the following integral, due to Gauss.

Theorem 9. *Define*

$$Li(x) = \int_3^x \frac{dt}{\log t} \tag{20}$$

Then

$$\pi(x) \sim Li(x) \tag{21}$$

It is a good exercise to show that Li(x) has the following expansion.

$$Li(x) = \frac{x}{\log x} + \frac{x}{(\log x)^2} + \frac{2!x}{(\log x)^3} + \frac{3!x}{(\log x)^4} + \cdots + + \frac{n!x}{(\log x)^{n+1}}\left[1 + \epsilon(x)\right], \tag{22}$$

where $\epsilon(x) \to 0$ as $x \to \infty$. Again using Mathematica or other computer programs, one can compute $Li(x)$ and $\pi(x)$ and compare the validity of the asymptotic formula.

The third improvement in the proof of PNT is due to Tchebychev. Tchebychev proved the following two statements about distribution of primes.

Theorem 10. *(Tchebychev) For $Li(x)$ as in (20) we have*

$$\varliminf_{x\to\infty} \frac{\pi(x)}{Li(x)} \le 1 \le \varlimsup_{x\to\infty} \frac{\pi(x)}{Li(x)} \tag{23}$$

$$c_1 \frac{x}{\log x} \le \pi(x) \le c_2 \frac{x}{\log x} \tag{24}$$

where $c_1 = 0.92\cdots$ and $c_2 = 1.105\cdots$.

We will prove (24) for $c_1 = \log 2$ and $c_2 = 4\log 2$. To this end , we need to define the following functions.

Definition 4. (1) $\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^\alpha \\ 0, & \text{else} \end{cases}$

(2) $\vartheta(x) = \sum_{p\le x} \log p$

(3) $\Psi(x) = \sum_{p^m \le x} \log p$

Remark 2. *The reader should note that*

$$\Psi(x) = \sum_{n\le x} \Lambda(n) = \sum_{p\le x} \left[\frac{\log x}{\log p}\right] \log p$$

and that

$$\Psi(x) = \vartheta(x) + \vartheta(x^{\frac{1}{2}}) + \vartheta(x^{\frac{1}{3}}) + \cdots .$$

The following theorem gives the connection between the above function and the Prime Number Theorem

Theorem 11. *Let*

$$L_1 = \overline{\lim}_{x\to\infty} \frac{\pi(x)}{\frac{x}{\log x}} \qquad L_2 = \overline{\lim}_{x\to\infty} \frac{\vartheta(x)}{x} \qquad L_3 = \overline{\lim}_{x\to\infty} \frac{\Psi(x)}{x} \tag{25}$$

$$l_1 = \underline{\lim}_{x\to\infty} \frac{\pi(x)}{\frac{x}{\log x}} \qquad l_2 = \underline{\lim}_{x\to\infty} \frac{\vartheta(x)}{x} \qquad l_3 = \underline{\lim}_{x\to\infty} \frac{\Psi(x)}{x} \tag{26}$$

Then

$$l_1 = l_2 = l_3 \qquad \textit{and} \qquad L_1 = L_2 = L_3.$$

Proof: As pointed out in the preceeding remark, we have $\Psi(x) = \vartheta(x) + \vartheta(x^{\frac{1}{2}}) + \vartheta(x^{\frac{1}{3}}) + \cdots$. Thus $\vartheta(x) \le \Psi(x)$. Also $\Psi(x) = \sum_{p\le x} \left[\frac{\log x}{\log p}\right] \log p \le \sum_{p\le x} \log x = \pi(x)\log x$. Thus we have

$$\frac{\vartheta(x)}{x} \le \frac{\Psi(x)}{x} \le \frac{\pi(x)\log x}{x}.$$

Taking lim sup we get $L_2 \le L_3 \le L_1$. To complete the proof of $L_1 = L_2 = L_3$, it suffices to show $L_2 \ge L_1$. To this end, let $0 < \alpha < 1$ and $x > 1$. Then

$$\vartheta(x) \ge \sum_{x^\alpha < p \ge x} \log p \ge \alpha \log x \sum_{x^\alpha < p \le x} 1 = \alpha \log x\,(\pi(x) - \pi(x^\alpha)) \ge \alpha \log x\,(\pi(x) - x^\alpha)$$

since $\pi(x^\alpha) \le x^\alpha$. Thus we have

$$\vartheta(x) > \alpha \frac{\pi(x)\log x}{x} - \alpha \frac{\log x}{x^{1-\alpha}}$$

Since $\lim_{x\to\infty} \frac{\log x}{x^{1-\alpha}} = 0$, we conclude that

$$\overline{\lim}_{x\to\infty} \frac{\vartheta(x)}{x} \ge \alpha \overline{\lim}_{x\to\infty} \frac{\pi(x)}{\frac{x}{\log x}}$$

Thus $L_2 \geq \alpha L_1$ and taking the limit as $\alpha \to 1$, we conclude that $L_2 \geq L_1$.

The same argument can be used to show $l_1 = l_2 = l_3$.

Remark 3. *In view of the above theorem, note that PNT follows if we can show that $l_2 = L_2 = 1$. The main goal of the remainder of this paper is to prove this fact.*

We are now in a position to prove (24). If we show that $L_2 = \overline{\lim}_{x\to\infty} \frac{\vartheta(x)}{x} \leq c_2 = 1.0204$ and$L_3 = \overline{\lim}_{x\to\infty} \frac{\Psi(x)}{x} \geq c_1 = .6932$, we can then appeal to Theorem 10 to conclude that (24) holds.

For any positive integer n, let

$$N = \begin{pmatrix} 2n \\ \\ n \end{pmatrix} = \frac{(2n)!}{n!n!} = \frac{(n+1)(n+2)\cdots 2n}{1\cdot 2\cdots n}.$$

Then it is easy to show that

$$\prod_{n<p\leq 2n} p \leq N < 2^{2n} \leq N(2n+1) \tag{27}$$

Taking log of (27), we get

$$\log\left(\prod_{n<p\leq 2n} p\right) \leq \log N < 2n\log 2$$

which yields

$$\vartheta(2n) - \vartheta(n) < 2n\log 2.$$

Put $n = 2^m$and add the result for $m = 0,\ 1, \cdots k$, to obtain

$$\sum_{m=0}^{k} \vartheta(2^{m+1}) - \vartheta(2^m) < \sum_{m=0}^{k} \left(2^{m+1}\log 2\right) < 2^{k+2}\log 2.$$

From this we conclude that

$$\frac{\vartheta(2^{k+1})}{2^k} < 4\log 2.$$

Finally for any $x > 2$, choose k so that $2^{k-1} \leq x \leq 2^k$ and apply the last inequality to obtain

$$\frac{\vartheta(x)}{x} \leq 4\log 2 = c_2.$$

This implies that $L_2 \leq 4\log 2 = 1.0204\cdot = c_2$.

To prove $L_3 = \overline{\lim}_{x\to\infty} \frac{\Psi(x)}{x} \geq c_1 = .6932$, we define $M_p = \left[\frac{\log 2n}{\log p}\right]$. Then if $N = \prod_{p\leq 2n} p^{\nu_p}$, then $\nu_p < M_p$.(The reader should proof this easily.)Consequently

$$\Psi(2n) = \sum_{p\leq 2n} \left[\frac{\log 2n}{\log p}\right] \log p = \sum_{p\leq 2n} M_p \log p = \log\left(\prod_{p\leq 2n} p^{M_p}\right) \geq \log N.$$

On the other hand $N = \begin{pmatrix} 2n \\ n \end{pmatrix} < 2^{2n} \leq (2n+1)N$ implies $N > \frac{2^{2n}}{2n+1}$ and hence $\log N \geq 2n\log 2 - \log(2n+1)$. Therefore we have

$$\Psi(2n) \geq 2n\log 2 - \log(2n+1).$$

For $x > 2$, let $n = \left[\frac{x}{2}\right]$. Then $n \geq 1$, $n > \frac{x}{2} - 1$, and $2n \leq x$. Hence

$$\Psi(x) \geq \Psi(2n) \geq 2n\log 2 - \log(2n+1) \geq 2(\frac{x}{2} - 1)\log 2 - \log x = (x-2)\log 2 - \log x$$

and we obtain

$$\frac{\Psi(x)}{x} \geq \frac{(x-2)\log 2}{x} - \frac{\log x}{x}.$$

Taking lim sup and noting that $\lim_{x\to\infty} \frac{\log x}{x} = 0$, we conclude that

$$L_3 \geq log 2 = .069\cdots$$

as desired.

We now return to the proof of $\lim_{x\to\infty} \frac{\vartheta(x)}{x} = 1$. We first prove the following theorem.

Theorem 12. *Let*

$$\rho(x) = \sum_{p \leq x} \frac{\log p}{p} \tag{28}$$

Then the following statements are equivalent.

(1) $\lim_{x\to\infty} \frac{\vartheta(x)}{x} = 1$.

(2) $\lim_{x\to\infty} (\rho(x) - \log x)$ *exists.*

Proof: We will show (2) $\Rightarrow$ (1) and leave it to the reader to prove (1) $\Rightarrow$ (2).

Let $E = \lim_{x\to\infty} (\rho(x) - \log x)$. We observe that

$$\rho(n) - \rho(n-1) = \begin{cases} \log p, & \text{if } \ n = p \\ 0, & \text{else} \end{cases}$$

and hence we have

$$\vartheta(x) = \sum_{p \leq x} \log p = \sum_{2 \leq n \leq x} (\rho(n) - \rho(n-1))\, n = \sum_{2 \leq n \leq x} \{[\rho(n) - \log n] - [\rho(n-1) - \log(n-1)]\}\, n + \sum_{2 \leq n \leq x} n \log\left(\frac{n}{n-1}\right).$$

Let

$$F_1(x) = \sum_{2 \leq n \leq x} \{[\rho(n) - \log n] - [\rho(n-1) - \log(n-1)]\}\, n, \qquad F_2(x) = \sum_{2 \leq n \leq x} n \log\left(\frac{n}{n-1}\right).$$

Claim

$$F_1(x) = \circ(x) \qquad \text{and} \qquad F_2(x) = x + \bigcirc(\log x).$$

Note then that

$$\vartheta(x) = F_1(x) + F_2(x) = \circ(x) + x + \bigcirc(\log x) = x + \circ(x)$$

and hence

$$\lim_{x\to\infty} \frac{\vartheta(x)}{x} = 1.$$

To prove the claim, we rewrite F_1 as

$$\begin{aligned}
F_1(x) &= \sum_{2\le n\le x} n\,[\rho(n)-\log n] - \sum_{2\le n\le x} n\,[\rho(n-1)-\log(n-1)] \\
&= \sum_{2\le n\le x} n\,[\rho(n)-\log n] - \sum_{1\le n\le x-1} (n+1)\,[\rho(n)-\log(n)] \\
&= (\rho([x])-\log([x]))\,[x] + \sum_{2\le n\le x-1} n\,[\rho(n)-\log n] - \sum_{1\le n\le x-1} n\,[\rho(n)-\log(n)] - \sum_{1\le n\le x-1} [\rho(n)-\log(n)] \\
&= (\rho([x])-\log([x]))\,([x]+1) - \sum_{1\le n\le x} [\rho(n)-\log(n)]
\end{aligned}$$

Let $\delta(x)=\rho(x)-\log x-E$. Then $\lim_{x\to\infty}\delta(x)=0$. That is $\delta(x)=\circ(1)$ and we have

$$\begin{aligned}
F_1(x) &= (\delta([x])+E)\,([x]+1) - \sum_{1\le n\le x} [\delta(n)+E] \\
&= (\delta([x])+E)\,([x]+1) - ([x]+1)\,E - \sum_{1\le n\le x} \delta(n) \\
&= 2E + \delta([x])\,([x]+1) - \sum_{1\le n\le x} \delta(n)
\end{aligned}$$

But $\sum_{1\le n\le x}\delta(n)=\circ(x)$ and $\delta([x])\,([x]+1)=\circ(x)$ Therefore

$$F_1(x) = 2E + \circ(x).$$

Also

$$\begin{aligned}
F_2(x) &= \sum_{2\le n\le x} n\log\left(\frac{n}{n-1}\right) = \sum_{2\le n\le x} n\log\left(1+\frac{1}{n-1}\right) \\
&= 2\log 2 + \sum_{3\le n\le x} n\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\left(\frac{1}{n-1}\right)^k \\
&= 2\log 2 + \sum_{3\le n\le x}\left\{\frac{n}{n-1} + \sum_{k=2}^{\infty} n\frac{(-1)^{k+1}}{k(n-1)^k}\right\}
\end{aligned}$$

Since

$$\left|\sum_{k=2}^{\infty} n\frac{(-1)^{k+1}}{k(n-1)^k}\right| \le \frac{n}{2}\sum_{k=2}^{\infty}\left(\frac{1}{n-1}\right)^2 = \frac{n}{2(n-1)(n-2)} = \bigcirc\left(\frac{1}{n}\right),$$

we have

$$\begin{aligned} F_2(x) &= 2\log 2 + \sum_{3\le n\le x}\left\{\frac{n}{n-1} + \bigcirc\left(\frac{1}{n}\right)\right\} \\ &= 2\log 2 + \sum_{3\le n\le x}\left(1+\frac{1}{n-1}\right) + \bigcirc\left(\sum_{3\le n\le x}\frac{1}{n}\right) \\ &= 2\log 2 + ([x]-2) + \bigcirc(\log x) \\ &= = [x] + \bigcirc(\log x) \end{aligned}$$

as claimed and we have completed the proof of the Theorem.

To complete the proof of PNT we must prove

Theorem 13. $\lim_{x\to\infty}(\rho(x)-\log x)$ *exists.*

We shall present the beautiful proof of D. Newman. His proof depends on the following convergence theorem. The theorem is due to Ingham and dates back to 1929. The proof by Ingham uses Fourier analysis while that of Newman's proof of the convergence theorem uses only theory of complex analysis.

Theorem 14. (Convergence Theorem) *Let* $\{a_n\}$ *be a bounded sequence of complex numbers. For* $Re(s) = \sigma > 1$*, assume that*

$$F(s) = \sum_{n=1}^{\infty}\frac{a_n}{n^s}$$

is analytic in an open set containing the region $Re(s) \ge 1$*. The series* $\sum_{n=1}^{\infty}\frac{a_n}{n^s}$ *converges for* $\Re(s) \ge 1$*.*

We shall return to the proof of the convergence theorem later. Let us assume its validity for now and use to proof Theorem 13.

Proof of Theorem 11: Define

$$f(s) = \sum_{n=1}^{\infty}\frac{\rho(n)}{n^s} = \sum_{n=1}^{\infty}\left(\sum_{p\le n}\frac{\log p}{p}\right)\frac{1}{n^s}$$

We rewrite f as $f)(s) = \sum_p \frac{\log p}{p}\left(\sum_{n\geq p}^{\infty}\frac{1}{n^s}\right)$ and note that

$$\sum_{n\geq p}^{\infty}\frac{1}{n^s} = \frac{1}{(s-1)p^{s-1}} + s\int_p^{\infty}\frac{1-\{t\}}{t^{s+1}}\,dt$$

where $\{t\} = t - [t]$ is the fractional part of the real number t. Define

$$A_p(s) = \frac{1}{p^s} - \frac{1}{p^s-1} + \frac{s(s-1)}{p}\int_p^{\infty}\frac{1-\{t\}}{t^{s+1}}\,dt$$

so that

$$\sum_{n\geq p}^{\infty}\frac{1}{n^s} = \frac{p}{s-1}\left(\frac{1}{p^s-1} + A_p(s)\right).$$

Clearly $A_p(s)$ is analytic in $Re(s) = \sigma > 0$(one can appeal to the Dominated Convergence Theorem to justify differentiating inside the integral to see this) and is bounded there by

$$\frac{1}{p^{\sigma}(p^{\sigma}-1)} + \frac{|s(s-1)|}{\sigma p^{\sigma+1}}.$$

By the Weierstrass M-test we conclude that the series defined by

$$A(s) = \sum_p A_p(s)\frac{p}{s-1}$$

is analytic in $Re(s) > \frac{1}{2}$. We now conclude that

$$f(s) = \frac{1}{s-1}\left(\sum_p \frac{\log p}{p^s-1} \ + \ A(s)\right) \tag{29}$$

is analytic in $\sigma > \frac{1}{2}$.

On the other hand from the Euler Product Formula (11) for $\zeta(s)$, (yes! finally $\zeta(s)$ is coming to the scene) we have $\zeta(s) = \prod_p (1-p^{-s})^{-1}$, we obtain, upon logarithmic differentiation,

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_p \frac{\log p}{p^s-1} \tag{30}$$

Using (30) in (29) we see that

$$f(s) = \frac{1}{s-1}\left(-\frac{\zeta'(s)}{\zeta(s)} \ + \ A(s)\right) \tag{31}$$

Since $|\zeta(1+it)| \neq 0$ and $\zeta(s)$ has a simple pole at $s=1$, we see that $(s-1)\zeta(s) \neq 0$ in $\sigma > \frac{1}{2}$. In fact we can conclude that $f(s)$ is analytic in an open interval containing $\Re(s) \geq 1$ with the principal part

$$\frac{1}{(s-1)^2} + \frac{c}{s-1},$$

where c is a complex number. This can be obtained by observing that $-\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{s-1} + g(s)$ where $g(s)$ is analytic at $s=1$. We have proved that the function

$$F(s) = f(s) + \zeta'(s) - c\zeta(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

, where

$$a_n = \sum_{p \leq n} \frac{\log p}{p} - \log n \ - \ c = \rho(n) - \log n \ - \ c$$

is analytic in an open set containing $\sigma \geq 1$. By the Convergence Theorem we conclude that

$$\sum_{n=1} \frac{a_n}{n}$$

converges. Our theorem now follows if we prove

Claim:

$$\lim_{n\to\infty} a_n = 0$$

To prove the Claim we use the Cauchy criterion. First note that $\log n + a_n + \sum_{p\leq n} \frac{\log p}{p} \ - E$ is non-decreasing. Now let $\epsilon > 0$ be given. Then there exists $N_0 > 0$ such that for $K \geq N \geq M > N_0$, we have

$$\sum_{n=N}^{K} \frac{a_n}{n} \ \leq \ \epsilon^2 \tag{32}$$

and

$$\sum_{n=M}^{N} \frac{a_n}{n} \ \geq \ -\epsilon^2 \tag{33}$$

But then for $N \le n \le K$ we have

$$a_n - a_N = \sum_{N<p\le n} \log\left(\frac{N}{n}\right) + \log\left(\frac{N}{n}\right) \ge \log\left(\frac{N}{n}\right) \ge \log\left(\frac{N}{K}\right)$$

We now choose $K \ge N(1+\epsilon)$ so that $\log\left(\frac{N}{K}\right) > -\epsilon$. This yields $a_n \ge a_N - \epsilon$. It then follows that $\sum_{n=N}^{K} \frac{a_n}{n} \ge (a_M - \epsilon)\sum_{n=N}^{K} \frac{1}{n}$ and therefore,

$$a_N \le \frac{\epsilon^2}{\sum_{n=M}^{K} \frac{1}{n}} + \epsilon. \tag{34}$$

But $\sum_{n=M}^{K} \frac{1}{n} \ge \frac{1}{K}(K - N + 1) \ge \frac{\epsilon}{1+\epsilon}$. Using this in (34) and simplifying, we get

$$a_N \le 2\epsilon + \epsilon^2 \tag{35}$$

On the other hand for $M < n < N$, we have $a_N - a_n = \sum_{n<p\le N} \log\left(\frac{n}{N}\right) + \log\left(\frac{n}{N}\right) \ge \log\left(\frac{n}{N}\right) \ge \log\left(\frac{M}{N}\right)$. We now choose $M \le N(1-\epsilon)$ and note that $\log\left(\frac{M}{N}\right) \ge \epsilon$. This gives $a_n \le a_N + \frac{\epsilon}{1-\epsilon}$ and hence

$$\sum_{n=M}^{N} \frac{a_n}{n} \le \left(a_N + \frac{\epsilon}{1-\epsilon}\right) \sum_{n=M}^{N} \frac{1}{n}.$$

Thus

$$a_N \ge \frac{\sum_{n=M}^{N} \frac{a_n}{n}}{\sum_{n=M}^{N} \frac{1}{n}} - \frac{\epsilon}{1-\epsilon} \ge \frac{-\epsilon^2}{\sum_{n=M}^{N} \frac{1}{n}} - \frac{\epsilon}{1-\epsilon} \tag{36}$$

As before we estimate $\sum_{n=M}^{N} \frac{1}{n} \ge \frac{1}{N}(N - M + 1) \ge \frac{1}{N}(N - N(1-\epsilon)) = \epsilon$. Using this in (36) and simplifying, we get

$$a_N \ge -\frac{\epsilon(2-\epsilon)}{1-\epsilon} \tag{37}$$

Combining (35) and (37), we have verified the definition of

$$\lim_{n\to\infty} a_n = 0.$$

The proof of PNT is now complete except for the proof of the Convergence Theorem.

5. Proof of The Convergence Theorem

In this section we give an outline of the proof of the Convergence Theorem. We hope that the reader will take a moment to verify the inequalities we state here.

Theorem 15. *Suppose $\{a_n\}$ is a bounded sequence and $\sum_{n=0}^{\infty} \frac{a_n}{n^s}$ converges to an analytic function $F(s)$ for $\Re(s) > 1$. If $F(s)$ is analytic throughout $\Re(s) \geq 1$, then the series $\sum_{n=0}^{\infty} \frac{a_n}{n^s}$ converges to $F(s)$ for $\Re(s) \geq 1$.*

Proof: Suppose $|a_n| \leq K$. Since we can replace a_n by $\frac{a_n}{K}$, we may as well assume $K = 1$. Fix w with $\Re(w) \geq 1$ and let $\epsilon > 0$ be given. Let $R = \max\{\frac{3}{\epsilon}, 1\}$. Then $F(s+w)$ is analytic in $\Re(s) \geq 0$. Hence there exist positive numbers δ and M, depending on R, with $0 < \delta < \frac{1}{2}$ such that $F(s+w)$ is analytic and

$$|F(s+w)| \leq M \text{ in } -\delta \leq \Re(s), \;\; |s| \leq R \tag{38}$$

Let Γ be the curve, with counter clockwise orientation, be given by

$$\Gamma = \{s \in \mathcal{C} || \int | = \mathcal{R}, \; \Re(\int) \geq -\delta\} \bigcup \{s \in \mathcal{C} | \Re(\int) = -\delta, |\int| = \mathcal{R}\}$$

Let Γ_r be the portion of Γ for which $\Re(s) = \sigma > 0$ and let Γ_l be the portion for which $\sigma \leq 0$. By Cauchy Residue Theorem, we have

$$2\pi i F(w) = \int_{\Gamma} F(s+w) N^s \left(\frac{1}{s} + \frac{s}{R^2}\right) ds \tag{39}$$

On Γ_r, $F(s+w)$ is given by its series and we may write it as $F(s+w) = S_N(s+w) + r_N(s+w)$, where

$$S_N(s+w) = \sum_{n=0}^{N} \frac{a_n}{n^{s+w}} \qquad \text{and} \qquad r_N(s+w) = \sum_{n=N+1}^{\infty} \frac{a_n}{n^{s+w}}.$$

Note then that $S_N(s+w)$ is entire and hence

$$2\pi i S_N(w) = \int_{|S|=R} S_N(s+w) N^s \left(\frac{1}{s} + \frac{s}{R^2}\right) ds$$

On the other hand,

$$\int_{|S|=R} S_N(s+w) N^s \left(\frac{1}{s} + \frac{s}{R^2}\right) ds = \int_{\Gamma_r} S_N(s+w) N^s \left(\frac{1}{s} + \frac{s}{R^2}\right) ds \ + \ \int_{-\Gamma_r} S_N(s+w) N^s \left(\frac{1}{s} + \frac{s}{R^2}\right) ds.$$

Therefore, we have

$$\int_{\Gamma_r} S_N(s+w) N^s \left(\frac{1}{s} + \frac{s}{R^2}\right) ds = 2\pi i S_N(w) \ - \ \int_{-\Gamma_r} S_N(s+w) N^s \left(\frac{1}{s} + \frac{s}{R^2}\right) ds,$$

which we express as

$$\int_{\Gamma_r} S_N(s+w) N^s \left(\frac{1}{s} + \frac{s}{R^2}\right) ds = 2\pi i S_N(w) \ - \ \int_{\Gamma_r} S_N(w-s) N^{-s} \left(\frac{1}{s} + \frac{s}{R^2}\right) ds \tag{40}$$

Combining (39) and (40) we get

$$2\pi i \left(F(w) - S_N(w)\right) = \int_{\Gamma_r} \left[r_N(s+w) N^s - \frac{S_N(w-s)}{N^s}\right] \left(\frac{1}{s} + \frac{s}{R^2}\right) ds + \int_{\Gamma_l} F(s+w) N^s \left(\frac{1}{s} + \frac{s}{R^2}\right) ds \tag{41}$$

To estimate the integrals in (41), we note the following whose details are left to the reader.

On $|s| = R$ we have

$$\left|\frac{1}{s} + \frac{s}{R^2}\right| \leq \frac{2\sigma}{R^2}. \tag{42}$$

On $\sigma = -\delta, \ \ |s| = \leq R$, we have

$$\left|\frac{1}{s}+\frac{s}{R^2}\right| \leq \frac{1}{\delta}\left(1+\frac{|s|^2}{R^2}\right). \tag{43}$$

We also have

$$|r_N(s+w)| \leq \sum_{n=N+1}^{\infty} \frac{1}{n^{\sigma+1}} \leq \int_N^{\infty} \frac{du}{u^{\sigma+1}} du = \frac{1}{\sigma N^{\sigma}} \tag{44}$$

and

$$|S_N(w-s)| \leq \sum_{n=0}^{N} n^{\sigma-1} \leq N^{\sigma-1} + \int_0^N u^{\sigma-1} du N^{\sigma} \left(\frac{1}{N} + \frac{1}{\sigma}\right) \tag{45}$$

Combining (42), (44), (45) we obtain

$$\left[r_N(s+w)N^s - \frac{S_N(w-s)}{N^s}\right]\left(\frac{1}{s}+\frac{s}{R^2}\right) \leq \left(\frac{1}{\sigma}+\frac{1}{\sigma}+\frac{1}{N}\right)\frac{2\sigma}{R^2} \leq \frac{4}{R^2}+\frac{2}{RN}$$

and hence

$$\left|\int_{\Gamma_r}\left[r_N(s+w)N^s - \frac{S_N(w-s)}{N^s}\right]\left(\frac{1}{s}+\frac{s}{R^2}\right) ds\right| \leq \frac{4\pi}{R}+\frac{2\pi}{N}. \tag{46}$$

On the other hand, by (38), (42), (45) we have

$$\left|\int_{\Gamma_l} F(s+w)N^s\left(\frac{1}{s}+\frac{s}{R^2}\right) ds\right| \leq \int_{-R}^{R} MN^{-\sigma}\frac{2}{\sigma} dt + 2M\int_{-\delta}^{0} n^{\sigma}\frac{2|\sigma|}{R^2}\frac{3}{2} d\sigma \leq \frac{4MR}{\delta N^{\sigma}} + \frac{6M}{R^2\log^2 N}. \tag{47}$$

Using (46) and (47) in (39) we get

$$|F(w) - S_N(w)| \leq \frac{2}{R} + \frac{1}{N} + \frac{MR}{\delta N^{\delta}} + \frac{M}{R^2 \log^2 N}.$$

Since $R = \frac{3}{\epsilon}$, we can take N large enough to conclude that $|F(w) - S_N(w)| < \epsilon$ thereby proving the fact that the infinite series $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ converges for $\Re(s) \geq 1$. This completes the proof of the Convergence Theorem.

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