

Real Analysis I: Problems for Chapter 3.

1. Prove or find a counterexample to the following statements. In all cases we assume that f and g are functions defined on the indicated intervals.

- (a) If f is bounded on $[a, b]$, then f is continuous on $[a, b]$.
- (b) If f is continuous on $[a, b]$, then f is bounded on $[a, b]$.
- (c) If f is continuous on (a, b) , then f is bounded on (a, b) .
- (d) If $(f(x))^2$ is continuous on (a, b) , then f is continuous on (a, b) .
- (e) If f and g are not continuous on (a, b) , then fg is not continuous on (a, b) .
- (f) If f and g are not continuous on (a, b) , then $f + g$ is not continuous on (a, b) .
- (g) If f and g are not continuous on (a, b) , then $f \circ g$ is not continuous on (a, b) .
- (h) If $f + g$ and g are continuous on (a, b) , then f is continuous on (a, b) .
- (i) If fg and g are continuous on (a, b) , then f is continuous on (a, b) .
- (j) If $|f(x)|$ is continuous on (a, b) , then f is continuous on (a, b) .

2. Determine whether or not the given functions are uniformly continuous.

- (a) $f(x) = \frac{1}{x}$ with $x \in (0, 1)$
- (b) $f(x) = x^3$ with $x \in [0, 2)$.
- (c) $f(x) = \frac{x}{x+4}$ with $x \in [0, 2)$.

3. Let $f(x) = (x^2 + x - 6)/(x - 2)$. Can f be defined at $x = 2$ so that f is continuous everywhere?

4. Let $f(x) = \begin{cases} 2x, & \text{if } x \in \mathbf{Q} \\ x + 3, & \text{if } x \notin \mathbf{Q} \end{cases}$

Find all points at which f is continuous.

5. Let f be a continuous function on \mathbf{R} and let $S = \{x \in \mathbf{R} \mid f(x) = 0\}$. If $\{x_n\}$ is a sequence of points in S and if $x_n \rightarrow x$, then show that $x \in S$.

6. Let f and g be continuous functions on \mathbf{R} and let $S = \{x \in \mathbf{R} \mid f(x) \geq g(x)\}$. If $\{x_n\}$ is a sequence of points in S and if $x_n \rightarrow x$, then show that $x \in S$.

7. Let f and g be continuous functions on \mathbf{R} . If $f(x) = g(x)$ for all rational numbers x , is it true that $f(x) = g(x)$ for all real numbers x ? Explain.

8. Use Intermediate Value Theorem to show that the polynomial $p(x) = x^4 + 7x^3 - 9$ has two real roots. Then use your calculator to find the roots.

9. Let $f(x) = x^2$ with $x \in [0, \infty)$. Find a positive number ϵ and two sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim_{n \rightarrow \infty} (x_n - y_n) = 0$ but $|f(x_n) - f(y_n)| \geq \epsilon$. Then conclude that $f(x) = x^2$ is not uniformly continuous on $[0, \infty)$.

10. Let $f(x) = \sqrt{x}$ for $x \in [0, \infty)$. Show that f is uniformly continuous but f is not Lipschitz continuous on $[0, \infty)$.

11. For each of the following functions, find $L_P(f)$ and $U_P(f)$, where $P = \{x_0, x_1, x_2, \dots, x_N\}$ is the partition of $[a, b]$ such that each subinterval has length equal to $h = \frac{b-a}{N}$.

- a) $f(x) = x^2 + 2x + 1$ on $[0, 1]$ with $N = 4$, $N = 8$, and N is any integer.
b) $f(x) = x^2 - 2x + 1$ on $[0, 2]$ with $N = 4$, $N = 8$, and N is any integer.

12. Let $f(x) = \begin{cases} 1, & \text{if } 0 \leq x \leq 1 \\ 2, & \text{if } 1 < x \leq 2. \end{cases}$

- a) Find a partition P of $[0, 2]$ such that $U_P(f) - L_P(f) \leq \frac{1}{10}$.
b) For any $\epsilon > 0$, show that there exists a partition P such that $U_P(f) - L_P(f) \leq \epsilon$.
c) Use the definition to find $\int_0^2 f(x) dx$.

13. Use upper and lower sums to evaluate the following integrals.

a) $\int_0^1 3x^2 dx$ b) $\int_0^2 (x^2 - 2x + 1) dx$

14. Suppose $f(x) \geq 0$ for all $x \in [a, b]$. If f is bounded, show that $L_P(f) \geq 0$ for all partitions P of $[a, b]$.

15. Prove or give a counter example: If $f(x) \leq g(x) \leq h(x)$ for all x and if f and h are Riemann integrable on $[a, b]$, then so is g .

16. Give an example of a function with domain $[0, 1]$, such that

- a) f is bounded but not Riemann integrable.
b) f is Riemann integrable but not monotone.
c) f is Riemann integrable but not continuous.

17. Suppose f is bounded on $[a, b]$ and $f(x) = 0$ for all x except at finitely many values in $[a, b]$, prove that $\int_a^b f(x) dx = 0$.

18. If f is Riemann integrable on $[a, b]$, then prove that $|f|$ is also Riemann integrable on $[a, b]$. Is the converse of this statement true? Prove or give a counter example.

19. (Refer to problems #3 and #7 page 102.) Define

$$T_N = \frac{h}{2} (f(x_0) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_{N-1}) + f(x_N))$$

and for even N

$$S_N = \frac{h}{6} \sum_{i=1}^N (f(x_{i-1}) + 4f(\bar{x}_i) + f(x_i)),$$

where \bar{x}_i is the midpoint of $[x_{i-1}, x_i]$ for each i .

- a) For $f(x) = x^2$ on $[0, 1]$, find M_N , T_N , and S_N for $N = 4$ and $N = 8$.
b) For $f(x) = 1/x$ on $[1, 2]$, find M_N , T_N , and S_N for $N = 4$ and $N = 8$.

20. DO problem #9 on page 103.