ACTIVITIES for students

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Unit Fractions and Their "Basimal" Representations: Exploring Patterns

Major foci of secondary mathematics include understanding numbers, ways of representing numbers, and relationships among numbers (NCTM 2000). This article considers different representations of rational numbers and leads students through activities that explore patterns in base ten, as well as in other bases. These activities encourage students to solve problems and investigate situations designed to foster flexible thinking about rational numbers. Preservice teachers in a college-level mathematics course carried out these activities. Their conjectures and ideas are incorporated throughout this article.

BASE-TEN INVESTIGATIONS

The set of rational numbers is often formally defined as follows:

$$Q = \{ a/b: a \text{ and } b \text{ are integers and } b \neq 0 \}.$$

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This definition uses a fractional representation of rational numbers. However, a rational number can also be represented as a terminating decimal or as a repeating decimal. When they study rational numbers, students are often asked to convert fractional representations to decimal representations, and vice versa. The connections that they find can suggest interesting problem-solving situations. This article focuses on investigations into the reasons that some rational numbers can be represented as terminating decimals while others repeat, as well as various properties of repeating decimals.

For simplicity's sake, only rational numbers whose fractional representations have a numerator of 1 are investigated in this article. Such a number is called a *unit fraction*, which is defined as a fraction of the form $1/n$ where n is a natural number greater than 1. Studying numbers whose fractional representations have numerators other than 1 is not necessary, since such numbers can be expressed as multiples of unit fractions (for example, $5/6 = 5 \bullet (1/6)$); hence, patterns that hold for unit fractions also hold for other fractions.

Consider the first ten unit fractions. As a beginning activity, students can find the decimal representations by doing simple division on a calculator, as shown in **table 1.** Separating the findings into terminating and repeating sets yields

Terminating: 1/2, 1/4, 1/5, 1/8, 1/10
Repeating: 1/3, 1/6, 1/7, 1/9, 1/11

ABLE 1

First Set of Ten Unit Fractions and Their Decimal Representations

Unit Fraction	Decimal Representation	Terminating or Repeating
1/2	0.5	Terminating decimal
1/3	$0.333\ldots = 0.\overline{3}$	Repeating decimal
1/4	0.25	Terminating decimal
1/5	0.2	Terminating decimal
1/6	$0.1666\ldots = 0.1\overline{6}$	Repeating decimal
1/7	$0.142857142857\ldots = 0.\overline{142857}$	Repeating decimal
1/8	0.125	Terminating decimal
1/9	$0.111\ldots = 0.\overline{1}$	Repeating decimal
1/10	0.1	Terminating decimal
1/11	$0.090909\ldots = 0.\overline{09}$	Repeating decimal

ABLE 2

Second Set of Ten Unit Fractions and Their Decimal Representations

Unit Fraction	Decimal Representation	Terminating or Repeating
1/12	$0.08333\ldots = 0.08\overline{3}$	Repeating decimal
1/13	$0.076923076923\ldots = 0.\overline{076923}$	Repeating decimal
1/14	$0.0714285714285\ldots = 0.0\overline{714285}$	Repeating decimal
1/15	$0.0666\ldots = 0.0\overline{6}$	Repeating decimal
1/16	0.0625	Terminating decimal
1/17	$0.\overline{0588235294117647}$	Repeating decimal
1/18	$0.0555\ldots = 0.0\overline{5}$	Repeating decimal
1/19	$0.\overline{052631578947368421}$	Repeating decimal
1/20	0.05	Terminating decimal
1/21	$0.047619047619\ldots = 0.\overline{047619}$	Repeating decimal

Interestingly, the first ten unit fractions are split jually between terminating and repeating decimals. udents can conjecture which unit fractions termi- ite and which ones repeat. The preservice teachers ho completed this article's activities, for example, iickly saw that all the unit fractions that terminate, cept for 1/5, have even denominators, whereas all e unit fractions that repeat, except for 1/6, have ld denominators. Their insights prompted them to ok at the next ten unit fractions, which are shown **table 2.** Some of those decimal representations n be found by doing simple division on a calcula- r; however, others extend farther than the number digits that a calculator can display before revealing erminating or repeating pattern. Decimal repre- ntations of those unit fractions can be found by ing such computer software as Mathematica. For is exploration, a teacher could simply provide the cimal representations of 1/17 and 1/19 to stu- nts. Summarizing to this point, terminating frac- ns are 1/2, 1/4, 1/5, 1/8, 1/10, 1/16, 1/20; and re- ating fractions are 1/3, 1/6, 1/7, 1/9, 1/11, 1/12, 13, 1/14, 1/15, 1/17, 1/18, 1/19, 1/21.

The first twenty unit fractions are not split ually between terminating and repeating decimals. The preservice teachers noted that many more unit fractions seemed to repeat than to terminate, yielding many more rational numbers that can be represented as repeating decimals. "Many more" was the students' term; in fact, both sets have an infinite number of elements.

After students consider the first twenty unit fractions, they can test their first conjectures and conjecture which unit fractions terminate and which ones repeat. The preservice teachers still thought that for a unit fraction to terminate, it must have an even denominator (except for 1/5), whereas they thought that all the unit fractions that repeat have odd denominators, except for 1/6, 1/12, and 1/18. Some wondered whether all unit fractions with a denominator that is a multiple of 6 repeat. Others built from this conjecture, adding the thought that any unit fraction with a denominator that is a multiple of 3 or 7 repeats.

When students were prompted to look also at possible patterns with the terminating decimals, some of them noticed that unit fractions with denominators that are powers of 2 or powers of 5 terminate. This idea placed 1/2, 1/4, 1/5, 1/8, and 1/16 (but not 1/10 or 1/20) in the list of unit fractions

TABLE 3

Lengths of Periods of Decimal Representations of Unit Fractions with Prime Denominators

n	Unit Fraction	Decimal Representation	Length of Period
2	1/2	0.5	N/A
3	1/3	$0.0333\ldots = 0.\overline{3}$	1
5	1/5	0.2	N/A
7	1/7	$0.142857142857\ldots = 0.\overline{142857}$	6
11	1/11	$0.090909\ldots = 0.\overline{09}$	2
13	1/13	$0.076923076923\ldots = 0.\overline{076923}$	6
17	1/17	$0.\overline{0588235294117647}$	16
19	1/19	$0.\overline{052631578947368421}$	18
23	1/23	$0.\overline{0434782608695652173913}$	22

that correspond to terminating decimals. A few preservice teachers noticed, however, that the prime factorizations of 10 and 20 contain only 2s and 5s. In other words, they found an intriguing pattern in the unit fractions that terminate: the denominators can always be factored into powers of 2, 5, or both. All unit fractions with denominators that cannot be factored into only powers of 2, 5, or both seemed to repeat. This pattern accounted for everything that the preservice teachers had noticed. For example, multiples of 3 (including multiples of 6) and 7 contain powers of prime numbers other than 2 and 5. Thus, unit fractions with denominators that are multiples of 3 or 7 do indeed repeat.

The preservice teachers had, in fact, discovered a correct conjecture. Formally stated, a rational number a/b in simplest form can be written as a terminating decimal if and only if the prime factorization of the denominator contains no primes other than 2 and 5. Using the conjecture on the next four unit fractions gives 1/22 as a repeating decimal, 1/23 as a repeating decimal, 1/24 as a repeating decimal, and 1/25 as a terminating decimal. These results can be verified by division using a calculator or computer software.

Many preservice teachers became interested in the unit fractions 1/17 and 1/19 because of the large number of digits before the decimal repeats. The number of places before a decimal repeats is known as the *period*. When the preservice teachers noticed that 17 and 19 are prime numbers, that 1/17 has a period of 16, and that 1/19 has a period of 18, they wondered whether the next prime number, 23, would have a period of 22. Surely enough, it does. They conjectured that a unit fraction with denominator p, where p is any prime other than 2 or 5, repeats with a period of length $(p - 1)$.

To test this conjecture, students can build a chart like **table 3** of unit fractions with prime denominators and look at the period of each. The preservice teachers quickly noticed that counterexamples to the conjecture arise, since, for example, 1/13 has a period of 6 rather than 12. However, they were determined not to give up on finding a pattern. They were then very interested in securing a conjecture with respect to a unit fraction's denominator and the length of its period. Thus, they modified the conjecture to state that a unit fraction with denominator p, where p is any prime other than 2 or 5, repeats with a period of at most $(p - 1)$. Further, they noticed that if p is prime and if its period is not of length $(p - 1)$, then the length of the period is a factor of $(p - 1)$. Returning to the example 1/13, although the period is not of length 12, the period's length of 6 is a factor of 12.

Several preservice teachers were still interested in investigating other properties of the repeating decimals. They noticed that some decimal expansions have repeating portions that begin immediately to the right of the decimal point, whereas others have repeating portions that do not. The *purely periodic* decimals, whose decimal expansions have repeating portions that begin immediately to the right of the decimal point, can be seen in the decimal representations of 1/3, 1/7, 1/9, 1/11, 1/13, 1/17, 1/19, 1/21, and 1/23. The *delayed periodic* decimals, whose decimal expansions have repeating portions that begin after a delay to the right of the decimal point, include 1/6, 1/12, 1/14, 1/15, 1/18, 1/22, and 1/24. For example, $1/6 = 0.1\overline{6}$ has an infinitely repeating part that begins one place to the right of the decimal point, yielding a delay of length 1. Similarly, $1/12 = 0.08\overline{3}$ has an infinitely repeating part that begins two places to the right of the decimal point, yielding a delay of length 2. **Table 4** summarizes the lengths of delay of the unit fractions considered thus far.

The preservice teachers remembered that they could use the prime factorization of the denominator of a unit fraction to determine whether its decimal representation terminates or repeats; they suspected that the length of delay of a repeating decimal can be determined by appealing to prime factorization. They decided to consider the repeating decimals with delay lengths that are greater than zero (that is,

TABLE 4

Length of Delay of Unit Fractions Having Decimal Representations That Repeat

Unit Fraction	Decimal Representation	Length of Delay
1/3	$0.333\ldots = 0.\overline{3}$	0 (purely periodic)
1/6	$0.1666\ldots = 0.1\overline{6}$	1
1/7	$0.142857142857\ldots = 0.\overline{142857}$	0 (purely periodic)
1/9	$0.111\ldots = 0.\overline{1}$	0 (purely periodic)
1/11	$0.090909\ldots = 0.\overline{09}$	0 (purely periodic)
1/12	$0.08333\ldots = 0.08\overline{3}$	2
1/13	$0.076923076923\ldots = 0.\overline{076923}$	0 (purely periodic)
1/14	$0.0714285714285\ldots = 0.0\overline{714285}$	1
1/15	$0.0666\ldots = 0.0\overline{6}$	1
1/17	$0.\overline{0588235294117647}$	0 (purely periodic)
1/18	$0.0555\ldots = 0.0\overline{5}$	1
1/19	$0.\overline{052631578947368421}$	0 (purely periodic)
1/21	$0.047619047619\ldots = 0.\overline{047619}$	0 (purely periodic)
1/22	$0.0454545\ldots = 0.0\overline{45}$	1
1/23	$0.\overline{0434782608695652173913}$	0 (purely periodic)
1/24	$0.041666\ldots = .041\overline{6}$	3

TABLE 5

Prime Factorization of Denominators of Unit Fractions Having Delayed Periodic Decimal Representations

Unit Fraction	Decimal Representation	Length of Delay	Prime Factorization of Denominator
1/6	$0.1666\ldots = 0.1\overline{6}$	1	$2 \bullet 3$
1/12	$0.08333\ldots = 0.08\overline{3}$	2	$2^2 \bullet 3$
1/14	$0.0714285714285\ldots = 0.0\overline{714285}$	1	$2 \bullet 7$
1/15	$0.0666\ldots = 0.0\overline{6}$	1	$3 \bullet 5$
1/18	$0.0555\ldots = 0.0\overline{5}$	1	$2 \bullet 3^2$
1/22	$0.0454545\ldots = 0.0\overline{45}$	1	$2 \bullet 11$
1/24	$0.041666\ldots = 0.041\overline{6}$	3	$2^3 \bullet 3$

the delayed periodic decimals) and determine whether any patterns involved 2s and 5s again. The work for this investigation is shown in **table 5.**

One preservice teacher noted immediately that when the prime factorization involved 2s, the length of the delay matched the power of 2. Others noticed that when no 2s were in the prime factorization (such as with 1/15), they could look at 5s; the length of the delay matched the power of the 5. Most wanted to test these ideas with other cases and particularly wondered what would happen if both 2s and 5s appeared within a prime factorization. Clearly, they remembered that the denominator of the unit fraction must have at least one prime factor other than 2 or 5 for the decimal to repeat.

Cases such as the ones shown in **table 6** can be tested.

After testing some other cases, most preservice teachers were comfortable with the conjectures about factors involving 2s and 5s. They simply added that if both 2s and 5s appeared within a prime factorization, the length of delay is the highest power of either the 2 or the 5. Again, the preservice teachers had arrived at a correct hypothesis. That is, the length of delay of the decimal representation of a given rational number corresponds to the highest exponent on either the 2 or the 5 in the prime factorization of the denominator of the fractional representation of the rational number. It follows that a rational number is purely periodic if and only if the denominator has no factors of 2 and 5 (that is, the denominator is relatively prime to 10). It also follows that any rational number with a prime denominator greater than 5 must be purely periodic.

EXTENSION: BASE-FOUR INVESTIGATIONS

Learning whether the patterns found in base ten also work in other base systems is an interesting activity.

TABLE 6

More Cases of Prime Factorization of Denominators of Unit Fractions Having Delayed Periodic Decimal Representations

Unit Fraction	Decimal Representation	Length of Delay	Prime Factorization of Denominator
1/30	$0.0\overline{3}$	1	$2 \bullet 3 \bullet 5$
1/48	$0.0208\overline{3}$	4	$2^4 \bullet 3$
1/56	$0.017\overline{857142}$	3	$2^3 \bullet 7$
1/60	$0.01\overline{6}$	2	$2^2 \bullet 3 \bullet 5$
1/75	$0.01\overline{3}$	2	$3 \bullet 5^2$
1/150	$0.00\overline{6}$	2	$2 \bullet 3 \bullet 5^2$
1/1200	$0.0008\overline{3}$	4	$2^4 \bullet 3 \bullet 5^2$
1/30,000	$0.0000\overline{3}$	4	$2^4 \bullet 3 \bullet 5^4$

For example, one might wonder how to tell whether a unit fraction in base four has a "basimal" representation that terminates or one that repeats. (Note that the word *basimal* replaces the word *decimal*, since *deci* refers specifically to base ten only, whereas *basi* refers to any base. We chose the term *basimal representation* because some readers may not be familiar with the term *radix representation*. Radix refers to the number base of a numeral system. An Internet search on the word *basimal* led to results that may be more applicable for use in the high school classroom than the word *radix*, if a reader wants to further pursue the ideas presented in the article.)

Before we consider base-four basimals, we briefly review operations in base four. The base-ten system uses ten digits (0, 1, 2, 3, 4, 5, 6, 7, 8, and 9) and place value to denote sets of ten; the base-four system uses four digits (0, 1, 2, and 3) and place value to denote sets of four. Counting in base ten involves 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9, followed by 10 to represent one set of ten. As the numbers increase, every new set of ten creates a new place-value position. For example, 100 represents ten sets of ten. Similarly, base-four counting begins with 0_{four}, 1_{four}, 2_{four}, and 3_{four}, and then continues with 10_{four} to represent one set of four. As the numbers increase, every new set of four creates a new place-value position. In this manner, the numbers up to 100_{four} are 0_{four}, 1_{four}, 2_{four}, 3_{four}, 10_{four}, 11_{four}, 12_{four}, 13_{four}, 20_{four}, 21_{four}, 22_{four}, 23_{four}, 30_{four}, 31_{four}, 32_{four}, 33_{four}, and 100_{four}. Here, 100_{four} represents four sets of four, which is equivalent to 16 in base ten.

Addition, subtraction, multiplication, and division can be performed in base four with the same algorithms used in base ten. Addition can be completed by adding the digits in given place-value positions and carrying sets of four. Adding 213_{four} and 1220_{four}, for example, yields 2033_{four}. The 3 in the rightmost place-value position comes from adding 3_{four} and 0_{four}, and the 3 in the second place-value position (moving left) comes from adding 1_{four} and 2_{four}. The 0 in the third place-value position (continuing to move left) comes from adding 2_{four} and 2_{four}, getting 10_{four}, recording the 0, and carrying the 1 into the next place-value position (moving left). This carried 1_{four} is added to the 1_{four} from 1220_{four} to yield 2_{four} in the fourth position.

Much like addition, subtraction can be completed by subtracting the digits in given place-value positions and borrowing sets of four. Standard algorithms for multiplication and division incorporate addition and subtraction facts. The preservice teachers who completed the activities in this article practiced many base-four operations before doing a basimal investigation. While working on practice problems, they built a table, shown as **table 7,** of base-four multiplication facts to enable them to complete their work more quickly. Some completed the table using repeated addition, for example,

$$\begin{aligned} 31_{four} \times 3_{four} &= 31_{four} + 31_{four} + 31_{four} \\ &= 122_{four} + 31_{four} \\ &= 213_{four} \end{aligned}$$

TABLE 7

Base-Four Product Chart

×	0	1	2	3	10	11	12	13	20	21	22	23	30	31	32	33	100
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	10	11	12	13	20	21	22	23	30	31	32	33	100
2	0	2	10	12	20	22	30	32	100	102	110	112	120	122	130	132	200
3	0	3	12	21	30	33	102	111	120	123	132	201	210	213	222	231	300

whereas others noticed that quicker base-ten methods worked in base four (for example, $31_{four} \times 3_{four} = 213_{four}$, since $1_{four} \times 3_{four} = 3_{four}$ in the ones place-value column and since $3_{four} \times 3_{four} = 21_{four}$ in the next column, moving left).

A slightly more complicated multiplication example is $31_{four} \times 23_{four}$. When the standard algorithm for multiplication is used, the problem can be set up vertically and completed as in the steps in **figure 1.**

The multiplication facts in **table 7** are useful not only for more complicated multiplication problems, but also for long division. **Figure 2** shows the steps for completing $130_{four} \div 2_{four}$ by using multiplication and subtraction facts.

As **table 8** shows, students can use long division to complete work for a set of unit fractions. The first seventeen numbers in base four are 0_{four}, 1_{four}, 2_{four}, 3_{four}, 10_{four}, 11_{four}, 12_{four}, 13_{four}, 20_{four}, 21_{four}, 22_{four}, 23_{four}, 30_{four}, 31_{four}, 32_{four}, 33_{four}, and 100_{four}, so the first fifteen unit fractions can be compiled with these values (except for 0_{four} and 1_{four}) in denominators. Each long-division problem involves replacing the numerator of 1 with 1.0000 . . . , using as many 0's for place-holders as needed when carrying out the division process and looking for repeating patterns in the result. For example, the unit fraction $1/31_{four}$ requires an extension of seven 0's, the first six of which are needed to find the repeating portion of the basimal and the seventh of which is needed to discover when the repeating portion begins a new cycle. See **figure 3.**

The preservice teachers who completed **table 8** found it useful to see some long-division examples in base four and then completed the remaining problems. They then set out to find patterns using the denominators of the unit fractions that may determine whether the basimal representation terminates or repeats. Recalling that terminating decimals in base ten correspond to a prime factorization of the denominator containing no primes other than 2 and 5, the students guessed that a similar pattern may hold true in base four. Since the numeral 5 does not exist in base four, the preservice teachers knew that the exact pattern from base-ten computations could not hold true in base-four work. One student thought that the 5 could just be "thrown away" or ignored, leaving a pattern involving prime factorization of denominators containing no primes other than 2_{four}. Others thought that the 5 should be converted into base-four notation (that is, 11_{four}), yielding a pattern involving prime factorization of denominators containing no primes other than 2_{four} and 11_{four}.

To test these conjectures, the preservice teachers decided to stick with the idea of prime factorizations, and they used base-four multiplication facts to factor the denominators of the unit fractions in question.

$3\ 1_{four}$ $\times\ \ 2\ 3_{four}$ 2 1 3 1 2 2	Set up problem vertically. Multiply digits using standard algorithm. Place results in stair-step pattern ($31_{four} \times 3_{four} = 213_{four}$ written in the first row and $31_{four} \times 2_{four} = 122_{four}$ written in the second row one space from the rightmost place-value position).
$3\ 1_{four}$ $\times\ \ 2\ 3_{four}$ 2 1 3 1 2 2 $2\ 0\ 3\ 3_{four}$	Add digits in given place-value columns, carrying sets of four. Note: The 0 in the third place-value position (from the right) comes from adding 2_{four} and 2_{four}, getting 10_{four}, recording the 0, and carrying the 1 into the next place-value position (moving left). This 1 then gets added in the next (fourth) column.

Fig. 1 Steps for solving $31_{four} \times 23_{four}$

$2_{four}\overline{)130_{four}}$	Set up long division. Consider how many times 2_{four} goes into 1_{four}. Since 2_{four} does not go into 1_{four}, consider how many times 2_{four} goes into 13_{four}. Using multiplication facts, $2_{four} \times 1_{four} = 2_{four}$, $2_{four} \times 2_{four} = 10_{four}$, $2_{four} \times 3_{four} = 12_{four}$, and $2_{four} \times 10_{four} = 20_{four}$. Since $2_{four} \times 3_{four}$ is less than 13_{four} and since $2_{four} \times 10_{four}$ is greater than 13_{four}, 2_{four} goes into 13_{four} at most 3_{four} times.
3 $2_{four}\overline{)130_{four}}$ 12 10	Record the 3_{four} above the problem setup and the product 12_{four} below the problem setup. Subtract 12_{four} from 13_{four}, and bring down the next digit. This work yields a remainder of 10_{four}.
32 $2_{four}\overline{)130_{four}}$ 12 10 10 0	Consider how many times 2_{four} goes into 10_{four}. Since $2_{four} \times 2_{four} = 10_{four}$ 2_{four} goes into 10_{four} exactly 2_{four} times. Record the 2_{four} above the problem setup and record the product 10_{four} below the problem setup. Subtract 10_{four} from 10_{four}. This work yields a remainder of 0_{four}. Since no other digits can be brought down, the problem is finished.
$130_{four} \div 2_{four} = 32_{four}$	The result can be checked with multiplication: $32_{four} \times 2_{four} = 130_{four}$

Fig. 2 Steps for solving $130_{four} \div 2_{four}$

TABLE 8

Base-Four Unit Fractions and Their Basimal Representations

Unit Fraction	Basimal Representation	Terminating or Repeating
1/2	0.2	Terminating basimal
1/3	$0.111\ldots = 0.\overline{1}$	Repeating basimal
1/10	0.1	Terminating basimal
1/11	$0.0303\ldots = 0.\overline{03}$	Repeating basimal
1/12	$0.0222\ldots = 0.0\overline{2}$	Repeating basimal
1/13	$0.021021\ldots = 0.\overline{021}$	Repeating basimal
1/20	0.02	Terminating basimal
1/21	$0.013013\ldots = 0.\overline{013}$	Repeating basimal
1/22	$0.01212\ldots = 0.0\overline{12}$	Repeating basimal
1/23	$0.0113101131\ldots = 0.\overline{01131}$	Repeating basimal
1/30	$0.0111\ldots = 0.0\overline{1}$	Repeating basimal
1/31	$0.010323010323\ldots = 0.\overline{010323}$	Repeating basimal
1/32	$0.0102102\ldots = 0.0\overline{102}$	Repeating basimal
1/33	$0.0101\ldots = 0.\overline{01}$	Repeating basimal
1/100	0.01	Terminating basimal

```
              0.0103230 . . .four
31four  )1.0000000. . .four
           31
            300
            213
             210
             122
              220
              213
               100
                31 . . (problem repeats from beginning)
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Fig. 3 With long division, the unit fraction $1/31_{four}$ requires an extension of seven zeros.

TABLE 9

Base-Four Prime Factorization

Unit Fraction	Prime Factorization of Denominator
1/2	2
1/3	3
1/10	2^2
1/11	11
1/12	2 • 3
1/13	13
1/20	2^3
1/21	3^2
1/22	2 • 11
1/23	23
1/30	2^2 • 3
1/31	31
1/32	2 • 13
1/33	3 • 11
1/100	2^{10}

Table 9 shows this work. Since counterexamples arise, comparing the results in **table 8** and **table 9** rules out the conjecture about a pattern involving prime factorization of denominators containing no primes other than 2_{four} and 11_{four}. However, the conjecture about a pattern involving prime factorization of denominators containing no primes other than 2_{four} seems to work. On further reflection, the preservice teachers were content with this idea, thinking that the 2 and 5 are "special" in base ten because they are the prime factors of ten (10_{ten}) and likewise that the 2_{four} is "special" in base four because it is the only prime factor of four (10_{four}).

Thus, the conjecture that a rational number *a*/*b* in simplest form in base four can be written as a terminating decimal if and only if the prime factorization of the denominator contains no primes other than 2_{four} was generally accepted. Two more cases are checked in **table 10.**

SUMMARY OF INVESTIGATIONS

In working on all the preceding activities, students are encouraged to look for patterns, make conjectures, test conjectures, discover number-theory facts, and investigate mathematics. Exploring serious mathematics challenges both students and teachers beyond the usual mundane pursuits of ordinary arithmetic. The conjectures made by the preservice teachers who worked through the activities brought out many fascinating ideas and led them to delve further into mathematics. The preservice teachers developed theorems about rational numbers through discovery methods, rather than by memorizing them from a textbook. Many of the students were genuinely excited about their pursuits

TABLE 10

Base-Four Unit Fractions, Basimal Representations, and Prime Factorization			
Unit Fraction	Basimal Representation	Terminating or Repeating	Prime Factorization of Denominator
1/133	0.0020100201 . . . = $0.\overline{00201}$	Repeating basimal	133
1/200	0.002	Terminating basimal	2^n

and accomplishments and gained a sense of pride from developing their own ideas.

Teachers and students can use **sheets 1** and **2** to complete the activities that the preservice teachers did in this article. Solutions to these activity sheets can be found throughout the article. Readers are further encouraged to explore methods of determining the length of the period and the length of the delay of basimal representations that repeat in base four. In fact, readers can repeat the above activities in various other bases. Building on the ideas presented in the article, readers can develop general theorems about the basimal representations of unit fractions, given any base.

REFERENCE

National Council of Teachers of Mathematics (NCTM). *Principles and Standards for School Mathematics*. Reston, Va.: NCTM, 2000. ∞

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