Direct Products

Let A and B be sets. The Cartesian cross product of A with B, denoted $A \times B$, is the set of all ordered pairs (x, y) where $x \in A$ and $y \in B$. If G_1 and G_2 are groups, the external direct product of G_1 with G_2 is also denoted in the same manner: $G_1 \times G_2$. The set of elements that constitute $G_1 \times G_2$ is precisely the same set as the Cartesian cross product of G_1 with G_2. However, the external direct product of G_1 with G_2 is endowed with an operator. If x_1 and x_2 are in G_1 and y_1 and y_2 are in G_2, $(x_1, y_1)\bigstar(x_2, y_2)$ is defined to be $(x_1 \cdot x_2, y_1 \cdot y_2)$ where the " $\cdot$ " in the first coordinate is the G_1 operator and the " $\cdot$ " in the second coordinate is the G_2 operator. There should be no ambiguity in using " $\times$ " in these two different but related ways in this text.

Is $G_1 \times G_2$ a group? That $\bigstar$ is closed, associative and well defined is inherited from the G_1 and G_2 operators. (e_1, e_2) acts as an identity for $G_1 \times G_2$ where e_1 is the identity of G_1 and e_2 is the identity of G_2. Finally, $(x, y)^{-1}$ is (x^{-1}, y^{-1}) where the inverses in the last ordered pair take place in G_1 and G_2 respectively. $G_1 \times G_2$ is a group.

External direct products can be used to create new groups from ones we've already studied. If either G_1 or G_2 are non-abelian, $G_1 \times G_2$ will be non-abelian. Sometimes $G_1 \times G_2$ will be an isomorphic copy of a group we have already studied. As an easy example, consider $\mathbb{Z}_2 \times \mathbb{Z}_2$. The elements are ([0], [0]), ([0], [1]), ([1], [0]) and ([1], [1]). An operating table is:

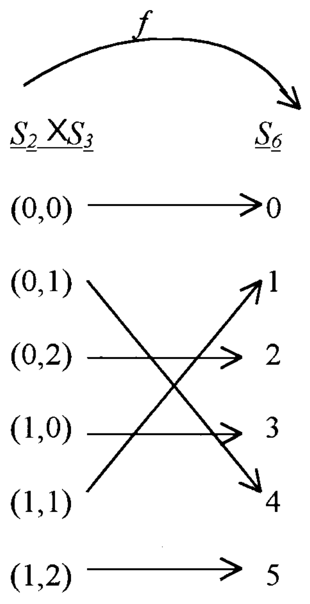
$\mathbb{Z}_2 \times \mathbb{Z}_2$	([0], [0])	([0], [1])	([1], [0])	([1], [1])
([0], [0])	([0], [0])	([0], [1])	([1], [0])	([1], [1])
([0], [1])	([0], [1])	([0], [0])	([1], [1])	([1], [0])
([1], [0])	([1], [0])	([1], [1])	([0], [0])	([0], [1])
([1], [1])	([1], [1])	([1], [0])	([0], [1])	([0], [0])

$\mathbb{Z}_2 \times \mathbb{Z}_2$ is an isomorphic copy of K_4. We will use the symbol " $\simeq$ " to mean "is isomorphic to". Therefore, $\mathbb{Z}_2 \times \mathbb{Z}_2 \simeq K_4$.

Let's construct an operating table for $\mathbb{Z}_2 \times \mathbb{Z}_3$. We will drop the brackets that we have been using to denoted equivalence classes in order to shorten our notation.

$\mathbb{Z}_2 \times \mathbb{Z}_3$	$(0,0)$	$(0,1)$	$(0,2)$	$(1,0)$	$(1,1)$	$(1,2)$
$(0,0)$	$(0,0)$	$(0,1)$	$(0,2)$	$(1,0)$	$(1,1)$	$(1,2)$
$(0,1)$	$(0,1)$	$(0,2)$	$(0,0)$	$(1,1)$	$(1,2)$	$(1,0)$
$(0,2)$	$(0,2)$	$(0,0)$	$(0,1)$	$(1,2)$	$(1,0)$	$(1,1)$
$(1,0)$	$(1,0)$	$(1,1)$	$(1,2)$	$(0,0)$	$(0,1)$	$(0,2)$
$(1,1)$	$(1,1)$	$(1,2)$	$(1,0)$	$(0,1)$	$(0,2)$	$(0,0)$
$(1,2)$	$(1,2)$	$(1,0)$	$(1,1)$	$(0,2)$	$(0,0)$	$(0,1)$

Consider the injective function:



You should verify that this is a homomorphism (36 equations must be confirmed) and therefore, since it is injective, an isomorphism. Once this is accomplished, we have established that $\mathbb{Z}_2 \times \mathbb{Z}_3 \simeq \mathbb{Z}_6$. As we will see later in this chapter, it is no accident that $\mathbb{Z}_2 \times \mathbb{Z}_3 \simeq \mathbb{Z}_6$ but $\mathbb{Z}_2 \times \mathbb{Z}_2 \not\simeq \mathbb{Z}_4$.

A fascinating use of external direct products is to create new groups from the groups we already know. For example, if we cross Q_8 with the integers under addition we create an infinite non-abelian group.

For any two groups G_1 and G_2, it is easy to show that $G_1 \times G_2 \simeq G_2 \times G$. The function $f(a,b) = (b,a)$ can be used to establish this isomorphic relationship. As a result, order will be of little consequence when writing external direct products. Also, we can extend the definition of external direct products to as many groups as we wish. For example, $G_1 \times G_2 \times G_3$ will be a group of ordered triples.

Let G be a group. Let N_1 and N_2 be normal subgroups of G. We say that G is the internal direct product of N_1 and N_2 (written $G = N_2 \otimes N_2$) iff:

1. $N_1 \cdot N_2$ spans G. This means that if x is any element of G then there exists an element y in N_1 and an element z in $N_2 \ni x = y \cdot z$

2. $N_1 \cap N_2 = \{e\}$

As an example, let $G_1 = \mathbb{Z}_6$. Let $N_1 = \{[0], [3]\}$ and $N_2 = \{[0], [2], [4]\}$. If you compute $N_1 \cdot N_2$ you obtain:

$$\begin{gathered}[0]+[0]=[0]\\ [0]+[2]=[2]\\ [0]+[4]=[4]\\ [3]+[0]=[3]\\ [3]+[2]=[5]\\ [3]+[4]=[1]\end{gathered}$$

Also, $N_1 \cap N_2 = [0]$. $\therefore \mathbb{Z}_6 = \{[0],[3]\} \otimes \{[0],[2],[4]\}$

Let's look at some non-examples. $D_3 \neq \{e,a\} \otimes \{e,b,b^2\}$. While the product of these sets spans D_3 and their intersection is $\{e\}$, only the second set is a normal subgroup of D_3. $Q_8 \neq \{1,-1,i,-i\} \otimes \{1,-1,j,-j\}$. Spanning is not the problem here nor is normalicy. The intersection of these sets has two elements. Let G be the positive reals under multiplication. Let N_1 be the set of all integral powers of 2 i.e. $N_1 = \{\ldots, 2^{-3}, 2^{-2}, 2^{-1}, 2^0, 2^1, 2^2, 2^3, \ldots\}$. Let N_2 be the set of all integral power of 3. $G \neq N_1 \otimes N_2$. Here the problem is spanning. 7, for example, cannot be written as the product of a power of 2 times a power of 3.

Theorem: Let G be a group. Let $N_1 \lhd G$ and $N_2 \lhd G_2$. If $G = N_1 \otimes N_2$ then for every x in N_1 and y in N_2, $x \cdot y = y \cdot x$.

Proof: Consider the element $xyx^{-1}y^{-1}$ (This product is called a commutator). Let's first group the product in the form $(xyx^{-1})y^{-1}$. We know that N_2 is normal. Thus, $xyx^{-1} \in N_2$. Since $y \in N_2$, we know $y^{-1} \in N_2$. Finally, the product of xyx^{-1} and y^{-1} must be in N_2 because of closure. $\therefore xyx^{-1}y^{-1} \in N_2$.

Let's associate the same product in the equivalent form $x(yx^{-1}y^{-1}) = x(yxy^{-1})^{-1}$. We know that N_1 is normal and $x \in N_1$. $\therefore yxy^{-1} \in N_1$ $\therefore (yxy^{-1})^{-1} \in N_1$. Hence $xyy^{-1}y^{-1} \in N_1$. Combining this fact with our previous result, we obtain:

$$xyx^{-1}y^{-1} \in N_1 \cap N_2$$

$$\begin{aligned} \text{However, } N_1 \cap N_2 &= \{e\} \\ \therefore\ xyx^{-1}y^{-1} &= e \\ \Rightarrow xyx^{-1} &= y \\ \Rightarrow xy &= yx \end{aligned}$$

QED

Care must be taken in interpreting the result of the last theorem. The theorem does not say that if $G = N_1 \otimes N_2$ then G must be abelian. It does say that any element of N_1 will commute with any element of N_2. However, if two elements are both from N_1 (or N_2), they might not commute with each other. For example, $D_6 = \{e, b^3\} \otimes \{e, a, b^2, b^4, ab^2, ab^4\}$. Note that e and b^3 commute with each of the elements in the second subgroup. However, if we select a and b^2 which are both in the second subgroup, $a \cdot b^2 \neq b^2 \cdot a$.

Theorem: If $G = N_1 \otimes N_2$ and x is any element in G then x has a unique representation of the form $h \cdot k$ where $h \in N_1$ and k $\in N_2$.

Proof: Since $N_1 \cdot N_2$ spans G, each x in G has at least one representation of the form $h \cdot k$ where $h \in N_1$ and $k \in N_2$. Could some x in G have two or more such representations? Suppose so. Suppose $x \in G$ and equals both h_1k_1 and h_2k_2 where $h_1, h_2 \in N_1$ and $k_1, k_2 \in N_2$ and either $h_1 \neq h_2$ or $k_1 \neq k_2$. We can deduce that :

$$\begin{aligned} h_1k_1 &= h_2k_2 \\ \Rightarrow h_1k_1k_2^{-1} &= h_2 \\ \Rightarrow k_1k_2^{-1} &= h_1^{-1}h_2 \end{aligned}$$

Because of the definition of a subgroup, $k_1k_2^{-1} \in N_2$. Also $h_1^{-1}h_2 \in N_1$. Our last equation says that an element of N_2 equals an element of N_1. Since $N_1 \cap N_2 = \{e\}$, both sides of the last equation equal e.

$$\begin{aligned} &\therefore k_1 \cdot k_2^{-1} = e \quad \text{and} \quad h_1^{-1}h_2 = e \\ &\Rightarrow k_1 = k_2 \qquad \text{and} \quad h_2 = h_1 \end{aligned}$$

$$\longrightarrow\!\!\longleftarrow$$

QED

The next two theorems establish that there is only a cosmetic difference between internal and external direct products.

Theorem: If $G = N_1 \otimes N_2$ then $N_1 \times N_2$ is isomorphic to G.

Proof: Consider what we obtain when we construct $N_1 \times N_2$. We obtain the set of all ordered pairs of the form (x, y) where $x \in N_1$ and $y \in N_2$. Define $f : N_1 \times N_2 \to G$ as $f\left[(x, y)\right] = x \cdot y$ where $x \in N_1$ and $y \in N_2$. Note that $N_2 \cdot N_2$ must span G so that this function must be surjective. Consider:
$f\left[(x_1, y_1) \cdot (x_2, y_2)\right] = f\left[(x_1 x_2, y_1 y_2)\right] = x_1 x_2 \cdot y_1 y_2$. By a previous theorem, every element of N_1 commutes with every element of N_2. Therefore, the previous product can be written $x_1 y_1 \cdot x_2 y_2 = f\left[(x_1, y_1)\right] \cdot f\left[(x_2, y_2)\right]$. $\therefore$ f is a homomorphism.

Is f injective?
Suppose $f\left[(x_1, y_1)\right] = f\left[(x_2, y_2)\right]$
$\Rightarrow x_1 y_1 = x_2 y_2$

By a previous theorem, each element in G has a unique representation as an element of N_1 times an element of N_2
$\therefore x_1 = x_2$ and $y_1 = y_2$
$\therefore$ f is injective.

QED

Theorem: If $G = G_1 \times G_2$ then there exist normal subgroups N_1 and N_2 of $G \ni$

1. N_1 is isomorphic to G_1
2. N_2 is isomorphic to G_2
3. $G = N_1 \otimes N_2$.

Proof: Let $N_1 = \{(x, e_2) \mid x \in G_1\}$. It is easy to show that N_1 is a normal subgroup of G. It is also easy to show that $f : N_1 \to G_1$ defined by $f\left[(x, e_2)\right] = x$ is an isomorphism. Similarly, the set $N_2 = \{(e_1, y) \mid y \in G_2\}$ can be shown to be a normal subgroup of G that is isomorphic to G_2.

Every element (x, y) in G can be written as $(x, e_2) \cdot (e_1, y)$. Therefore $N_1 \cdot N_2$ spans G. Suppose $(z, w) \in N_1 \cap N_2$. Because $(z, w) \in N_1$, $w = e_2$. Because $(z, w) \in N_2$, $z = e_1$. $\therefore (z, w) = (e_1, e_2)$ which is the identity of G.
$\therefore G = N_1 \otimes N_2$

QED

Example 1

When we examined external direct products, we discovered that $\mathbb{Z}_2 \times \mathbb{Z}_2 \simeq K_4$. The sets $\{e, p)$ and $\{e, q\}$ are both normal subgroups of K_4. Further, $\{e, p\} \otimes \{e, q\} = K_4$.

The following charts:

	e	p
e	e	p
p	p	e

	e	q
e	e	q
q	q	e

are both isomorphic copies of $\mathbb{Z}_2$:

	[0]	[1]
[0]	[0]	[1]
[1]	[1]	[0]

In a sense, K_4 equals $\mathbb{Z}_2 \times \mathbb{Z}_2$ both internally and externally. This is essentially the thrust of our theory.

Example 2

Earlier, we found that $\mathbb{Z}_2 \times 3$ is isomorphic to $\mathbb{Z}_6$. We also found that $\mathbb{Z}_6 = \{[0],[3]\} \times \{[0],[2],[4]\}$. Since $\{[0],[3]\}$ is isomorphic to $\mathbb{Z}_2$ and $\{[0],[2],[4]\}$ is isomorphic to $\mathbb{Z}_3$, we again confirm the internal/external equivalence of direct products.

Example 3

We pointed out in this chapter that

$$D_6 = \{e, b^3\} \otimes \{e, a, b^2, b^4, ab^2, ab^4\}.$$

The second normal subgroup in this product is an isomorphic copy of D_3 (you should verify this). The first normal subgroup is an isomorphic copy of $\mathbb{Z}_2$. We can now deduce that $D_6 \simeq \mathbb{Z}_2 \times D_3$.