THE INTEGERS OF JAMES BOOTH

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1. Introduction

In 1854, the Reverend James Booth published a little note [1], less than one page long in which he proved that six digit integers of the form *abcabc* like 376376 or 459459 are all divisible by the numbers 7, 11 and 13. In this note we will repeat Booth's original demonstration and give a second proof that shows how to generalize Booth's observation.

Let $N(p,n)$ denote the set of all positive integers with a sequence of *p* digits repeated *n* times. Thus the number 321432143214 in which the sequence 3214 of length 4 is repeated 3 times is an element of the set $N(4,3)$. The numbers described by Booth are from the set $N(3,2)$.

2. Booth's theorem and a generalization

We begin by giving Booth's original proof in the following theorem.

Theorem 1: All numbers from the set $N(3,2)$ are divisible by 7, 11 and 13.

Proof: Consider dividing the number

$$abcabc = 100,000a + 10,000b + 1,000c + 100a + 10b + c$$

by 7. Since 100,000 = 7x14285 + 5, 10,000 = 7x1428 + 4, 1,000 = 7x142 + 6, 100 = 7x14 + 2, 10 = 7x1 + 3 and 1 = 7x0 + 1, the division of *abcabc* by 7 has the remainder

$$5a + 4b + 6c + 2a + 3b + c = 7a + 7b + 7c.$$

Thus *abcabc* is divisible by 7. In the same way we can show that this number is divisible by 11 and 13 and the theorem is proved. □

A more instructive proof of this theorem begins by observing that 1001 = 7 x 11 x 13. Since our number can be written as

$$\begin{aligned} abcabc &= a00a00 + 0b00b0 + 00c00c \\ &= 100100a + 10010b + 1001c \\ &= 1001(100a) + 1001(10b) + 1001c \end{aligned}$$

it is clear that it is divisible by 7, 11, and 13. This second proof now allows us to easily generalize Theorem 1. Since the divisors of $1001 = 10^3 + 1$ divide any number in $N(3,2)$ we see that the proof of the following theorem is an easy extension of the above argument.

Theorem 2: Any divisor of $10^p + 1$ also divides all numbers in $N(p,2)$.

In the following table we list the divisors of $10^p + 1$ for use with Theorem 2.

p	**Prime Divisors of** $10^p + 1$	***p***	**Prime Divisors of** $10^p + 1$	***p***	**Prime Divisors of** $10^p + 1$
1	11	11	11, 23, 4093, 8779	21	7, 11, 13, 127, 2689, 459691, 909091
2	101	12	73, 137, 99990001	22	89, 101, 1052788969, 1056689261
3	7, 11, 13	13	11, 859, 1058313049	23	11, 47, 139, 2531, 549797184491917
4	73, 137	14	29, 101, 281, 121499449	24	17, 5882353, 9999999900000001
5	11, 9091	15	7, 11, 13, 211, 241, 2161, 9091	25	11, 251, 5051, 9091, 78875943472201
6	101, 9901	16	353, 449, 641, 1409, 69857	26	101, 521, 1900381976777332243781
7	11, 909091	17	11, 103, 4013, 21993833369	27	7, 11, 13, 19, 52579, 70541929, 14175966169
8	17, 5882353	18	101, 9901, 999999000001	28	73, 137, 7841, 127522001020150503761
9	7, 11, 13, 19, 52579	19	11, 909090909090909091	29	11, 59, 154083204930662557781201849
10	101, 3541, 27961	20	73, 137, 1676321, 5964848081	30	61, 101, 3541, 9901, 27961, 4188901, 39526741

From this table we see that 11 and 9091 divide 100001, and thus from Theorem 2 we know that 11 and 9091 divide every number in $N(5,2)$ such as 1234512345 and 5402154021. (Check these on your calculator.)

It is also easy to see that you can concatenate two or more members of $N(3,2)$ and the new number will also be divisible by 7, 11 and 13. This is because $abcabcdefdef = 1000000abcabc + defdef$, and the divisibility is now clear. For example, 123123456456789789 is divisible by 7, 11 and 13. In general we have the following theorem.

Theorem 3: Let two or more members of $N(p,2)$ be concatenated together to form a new number *z*. Then any divisor of $10^p + 1$ also divides *z*.

3. A useful algebraic identity

In the above table we see that 11 divides $10^p + 1$ whenever *p* is odd. Also notice that 101 divides $10^p + 1$ for *p* = 2, 6, 10, 14, 18, …. These are special cases of the following important theorem.

Theorem 4: Let *x* be an integer (not equal to –1). Then $x+1$ divides $x^p + 1$ if p is odd.

Proof: By direct multiplication we see that

(1) $$(x+1)\left(x^{p-1} - x^{p-2} + x^{p-3} - \cdots - x + 1\right)$$

$$= x\left(x^{p-1} - x^{p-2} + x^{p-3} - \cdots - x + 1\right) + \left(x^{p-1} - x^{p-2} + x^{p-3} - \cdots - x + 1\right)$$

$$= x^p - x^{p-1} + x^{p-2} - \cdots - x^2 + x + \left(x^{p-1} - x^{p-2} + x^{p-3} - \cdots - x + 1\right)$$

$$= x^p + 1.$$

and thus the theorem is proved. □

With $x = 10$ in (1) we see why 11 divides $10^p + 1$ for odd p, and with $x = 100$ we see why 101 divides $10^{2p} + 1$ for odd p.

The following corollary follows at once.

Corollary: No integer of the form $10^p + 1$, where p is an odd integer greater than 1, is prime.

Theorem 5: If y divides $10^q + 1$, then it divides every number in $N(pq, 2)$ where p is odd.

Proof: Let $x = 10^q$. By Theorem 4, $10^q + 1$ divides $10^{pq} + 1$, where p is an odd integer greater than 1), and thus y divides $10^{pq} + 1$ from Theorem 2. □

We can check Theorem 5 using the table. Notice that 37 divides $10^4 + 1$, and by our theorem 37 should divide $10^{4p} + 1$ where p is odd. Thus with the exponents of ten 4x1 = 4, 4x3 = 12, 4x5 = 20 and 4x7 = 28 we that 37 is a divisor listed in the table.

Just as the number 1001 was fundamental in the examination of the divisibility properties of numbers in the set $N(3, 2)$, the reader will have no trouble showing that the number 1001001 governs such questions for numbers in the set $N(3, 3)$. With the help of mathematical software, we can construct a table of the prime divisors of $10^{2p} + 10^p + 1$ for p =1, 2, 3, … so that the divisibility of numbers in $N(p, 3)$ can be explained. Further extensions to $N(p, 4)$, $N(p, 5)$, … can be studied. This completes the presentation of our extension of the numbers introduced by James Booth.

References

[1] Booth, James, *On a property of numbers*, Proceedings of the Royal Society of London, 7(1854-1855), pp. 42-43.