A SIMPLE GEOMETRIC METHOD OF ESTIMATING THE ERROR IN USING VIETA'S PRODUCT FOR π

Thomas J. Osler
Mathematics Department
Rowan University
Glassboro, NJ 08028

Osler@rowan.edu

1. Introduction

There are many expressions in the mathematical literature for the number π. The beautiful infinite product of radicals, (see [1] and [2]),

(1) $$\frac{2}{\pi}=\sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdots$$

due to Vieta in 1592, is the oldest noniterative analytical expressions for π. Reciprocating (1) we obtain after a little manipulation

(2) $$\frac{\pi}{2}=\frac{2}{\sqrt{2}}\frac{2}{\sqrt{2+\sqrt{2}}}\frac{2}{\sqrt{2+\sqrt{2+\sqrt{2}}}}\cdots.$$

It is the purpose of this paper to find, using a simple geometric arguments, error bounds on the calculation of π when using (2). Let $f_n=\frac{2}{\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}_{n\ radicals}}$ be the n^{th} factor in the product (2), and let $p_n=f_1f_2f_3\cdots f_n$. Clearly p_n is the approximation to $\pi/2$ obtained by using n factors in the product (2). We will show that the error in this approximation satisfies

$$\frac{p_n^2\pi}{12\cdot 4^n}<\left|\pi/2-p_n\right|<0.8046\,\theta_n^2,\text{ with }\theta_n=\pi/2^{n+1}.$$

2. Geometric construction approximating $\pi/2$

We begin by describing a geometric construction of the partial products p_n of (2).

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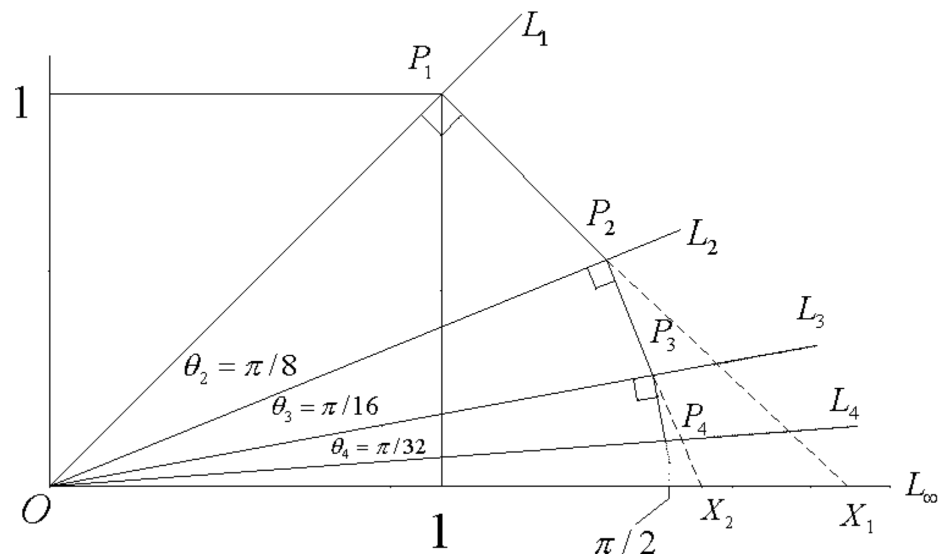

Figure 1: Constructing approximations to π/2

In Figure 1 we see the unit square. Angle L_1OL_∞ is $\pi/4$. Line OL_2 bisects angle L_1OL_∞ so that $\theta_2 = \pi/8$. Construct line OL_3 so that it bisects angle L_2OL_∞ making angle $\theta_3 = \pi/16$. Continuing we construct line OL_4 so that it bisects angle L_3OL_∞ making angle $\theta_4 = \pi/32$, etc.

From the corner of our unit square P_1 we construct a line perpendicular to line OL_1 meeting line OL_2 at P_2 and extended meets L_∞ at X_1. From P_2 we construct a line perpendicular to line OL_2 meeting line OL_3 at P_3 and extended meets L_∞ at X_2. We continue in this way forming additional points P_4, P_5, $\cdots$, and X_3, X_4, $\cdots$.We will show that the lengths of the line segments $\overline{OP_1}$, $\overline{OP_2}$, $\overline{OP_3}$, … converge to $\pi/2$. In fact we will show that

$$p_1 = \overline{OP_1} = \frac{2}{\sqrt{2}}, \quad p_2 = \overline{OP_2} = \frac{2^2}{\sqrt{2}\sqrt{2+\sqrt{2}}}, \quad p_3 = \overline{OP_3} = \frac{2^3}{\sqrt{2}\sqrt{2+\sqrt{2}}\sqrt{2+\sqrt{2+\sqrt{2}}}}, \ldots .$$

Comparing these with (2) we see that the length of $\overline{OP_n}$ is the reciprocal of the first n factors of Vieta's product.

3. Derivation of Vieta's product and verification of the construction

We now review a derivation of Vieta's product (1). Repeated use of a familiar trigonometric identity gives us

$$\sin x = 2\cos\frac{x}{2}\sin\frac{x}{2}$$

$$\sin x = 2^2\cos\frac{x}{2}\cos\frac{x}{2^2}\sin\frac{x}{2^2}$$

$$\sin x = 2^3\cos\frac{x}{2}\cos\frac{x}{2^2}\cos\frac{x}{2^3}\sin\frac{x}{2^3}$$

Continuing this way, and dividing by x we get

(3) $$\frac{\sin x}{x} = \frac{\sin\dfrac{x}{2^N}}{x/2^N}\prod_{k=1}^{N}\cos\frac{x}{2^k},$$

or taking the reciprocal we get

(4) $$\frac{x}{\sin x} = \frac{x/2^N}{\sin\dfrac{x}{2^N}}\prod_{k=1}^{N}\sec\frac{x}{2^k}.$$

Next we use a half angle formula to replace $\cos\dfrac{x}{2^k}$.

$$\cos\frac{x}{2}=\sqrt{\frac{1}{2}+\frac{1}{2}\cos x}$$

$$\cos\frac{x}{2^2}=\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\cos x}}$$

$$\cos\frac{x}{2^3}=\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\cos x}}}}\ .$$

Now (3) becomes

$$\frac{\sin x}{x}=\frac{\sin\frac{x}{2^N}}{x/2^N}\prod_{k=1}^{N}\underbrace{\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\cdots\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\cos x}}}}}_{k\ radicals}\ .$$

Finally we set $x=\pi/2$ and let N pass to infinity to get Vieta's product (1). Notice that (2) now takes the form

(5) $$\frac{\pi}{2}=\prod_{k=1}^{\infty}\sec\frac{\pi}{2^{k+1}}\ .$$

From Figure 1 we see that

$$p_1=\overline{OP_1}=\sec(\pi/4)$$

$$p_2=\overline{OP_2}=\overline{OP_1}\sec(\pi/8)=\sec(\pi/4)\sec(\pi/8)$$

$$p_3=\overline{OP_3}=\overline{OP_2}\sec(\pi/16)=\sec(\pi/4)\sec(\pi/8)\sec(\pi/16),$$

etc. It is now clear from (5) that our constructions converge to $\pi/2$.

4. An upper bound for the error

Call x_n the length of the line segment $\overline{OX_n}$. It is clear that the sequence $p_1, p_2, p_3, \cdots$ is increasing and approaches $\pi / 2$ from bellow. From Figure 1, we see that the sequence $x_1, x_2, x_3, \cdots$, is decreasing and approaches $\pi / 2$ from above. Also from Figure 1 we see that

$$x_n = \overline{OX_n} = \overline{OP_n} \sec\theta_n = p_n \sec\theta_n .$$

Thus we have

(6) $$p_n < \pi / 2 < p_n \sec\theta_n .$$

From Taylor's theorem with the remainder we have $f(\theta) = f(0) + f'(0)\theta + f''(c)\theta^2 / 2$, where $0 < c < \theta$. Let $f(\theta_n) = \sec\theta_n$ and get

$$\sec\theta_n = 1 + \frac{1+\sin^2 c}{2\cos^3 c}\theta_n^2 ,$$

where $0 < c < \theta_n$. From (6) we now have $p_n < \pi / 2 < p_n + p_n \frac{1+\sin^2 c}{2\cos^3 c}\theta_n^2$. Thus the error is

(7) $$\left|\pi / 2 - p_n\right| < p_n \frac{1+\sin^2 c}{2\cos^3 c}\theta_n^2 .$$

It is easy to see that $0 < \frac{1+\sin^2 c}{2\cos^3 c} < \frac{1+\sin^2\theta_n}{2\cos^3\theta_n}$, and that $\lim_{n\to\infty} \frac{1+\sin^2\theta_n}{2\cos^3\theta_n} = \frac{1}{2}$. Since $\theta_n = \pi / 2^{n+1}$, for $n \ge 4$ we have $\frac{1+\sin^2 c}{2\cos^3 c} < \frac{1+\sin^2\theta_4}{2\cos^3\theta_4} < 0.5122$. Using these numbers in (7) we find the error is less than

(8) $$\left|\pi / 2 - p_n\right| < 0.8046\,\theta_n^2, \text{ for } n \ge 4 .$$

This last relation is our desired upper bound for the error.

5. A lower bound for the error

In Figure 1, call $\overline{P_{n-1}P_n} = t_n$. We have, from the Pythagorean theorem

$$p_n^2 - p_{n-1}^2 = t_n^2$$

$$p_{n-1}^2 - p_{n-2}^2 = t_{n-1}^2$$

$$p_{n-2}^2 - p_{n-3}^2 = t_{n-2}^2$$

$$\cdots$$

$$p_{r+1}^2 - p_r^2 = t_{r+1}^2.$$

Adding the above we get

$$p_n^2 - p_r^2 = t_{r+1}^2 + t_{r+2}^2 + t_{r+3}^2 + \cdots + t_n^2.$$

Since $t_k = p_{k-1} \tan\theta_k$ we have

$$p_n^2 - p_r^2 = p_r^2 \tan^2\theta_{r+1} + p_{r+1}^2 \tan^2\theta_{r+2} + p_{r+3}^2 \tan^2\theta_{r+4} + \cdots + p_{n-1}^2 \tan^2\theta_n.$$

Because $p_k < p_{k+1}$ and $\theta_k < \tan\theta_k$ we get

$$p_n^2 - p_r^2 > p_r^2\left(\theta_{r+1}^2 + \theta_{r+2}^2 + \theta_{r+3}^2 + \cdots + \theta_n^2\right),$$

and since $p_n < \pi / 2$ for all *n* we have

$$\frac{\pi^2}{2^2} - p_r^2 > p_r^2\left(\theta_{r+1}^2 + \theta_{r+2}^2 + \theta_{r+3}^2 + \cdots + \theta_n^2\right).$$

Recall that $\theta_k = \pi / 2^{k+1}$, so

$$\frac{\pi^2}{2^2} - p_r^2 > p_r^2\pi^2\left(\frac{1}{4^{r+2}} + \frac{1}{4^{r+3}} + \frac{1}{4^{r+4}} + \cdots + \frac{1}{4^{n+1}}\right).$$

Since *n* does not appear on the left side, we can take *n* arbitrarily large on the right and get

$$\frac{\pi^2}{2^2} - p_r^2 > \frac{p_r^2\pi^2}{16}\frac{1}{4^r}\left(1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots\right).$$

From the geometric series we know that $1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots = \frac{1}{1 - 1/4} = \frac{4}{3}$, so

$$\frac{\pi^2}{2^2} - p_r^2 > \frac{p_r^2 \pi^2}{12} \cdot \frac{1}{4^r}.$$

We now have

$$\frac{\pi}{2} - p_r > \frac{\frac{p_r^2 \pi^2}{12} \cdot \frac{1}{4^r}}{\frac{\pi}{2} + p_r},$$

and since $\frac{\pi}{2} + p_r < \pi$ we get a lower bound for the error

$$\text{(9)} \qquad \frac{\pi}{2} - p_r > \frac{p_r^2 \pi}{12 \cdot 4^r}.$$

6. Numerical calculations

The following table shows numerical values for the partial products p_n, the true error and our estimated lower and upper bounds for the error. Notice that (8) is true even for $n = 1, 2, 3$. Notice also that the lower bound is very close to the true error.

n	Partial Product p_n	Estimated Lower-bound for Error $\frac{p_n^2 \pi}{12 \cdot 4^n}$	True Error $\lvert \pi / 2 - p_n \rvert$	Estimated Upper-bound for Error $0.8046\,\theta_n^2$
1	1.414214	0.13089969	0.15658276	0.49631777
2	1.530734	0.03833963	0.04006260	0.12407944
3	1.560723	0.00996415	0.01007375	0.03101986
4	1.568274	0.00251520	0.00252208	0.00775497
5	1.570166	0.00063032	0.00063075	0.00193874
6	1.570639	0.00015767	0.00015770	0.00048469

7	1.570757	0.00003942	0.00003943	0.00012117
8	1.570786	0.00000986	0.00000986	0.00003029
9	1.570794	0.00000246	0.00000246	0.00000757
10	1.570796	0.00000062	0.00000062	0.00000189

New papers generalizing Vieta's product are [3, 4, 5, and 6].

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References

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