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A FURTHER EXTENSION OF THE LEIBNIZ RULE TO FRACTIONAL DERIVATIVES AND ITS RELATION TO PARSEVAL'S FORMULA*

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Abstract. The familiar Leibniz rule for the Nth derivative of the product of two functions is $D^N uv = \sum \binom{N}{n} D^{N-n}u D^n v$. A generalization of this formula for fractional derivatives is given as $D^\alpha uv = \sum a \binom{\alpha}{an+\gamma} D^{\alpha-an-\gamma}u D^{an+\gamma}v$, where α need not be a natural number and $0 < a \leqq 1$. (The special case, $a = 1$, appeared previously.) Further generalizations of the Leibniz rule are also given and are derived from a generalization of Taylor's series given previously by the author. It is shown that these new series are generalizations of Parseval's formula from the study of Fourier series. Finally, new series expansions relating the special functions of mathematical physics are derived as special cases of the generalizations of the Leibniz rule. These series include a generalized Dougall's formula, several series of the Cardinal type, and a series related to a problem of Ramanujan.

1. Introduction. The fractional derivative of order α of $f(z)$ with respect to $g(z)$ is written $D^\alpha_{g(z)}f(z)$ and is an extension of the familiar derivative $d^\alpha f(z)/dg(z)^\alpha$ to nonintegral values of α. Fractional derivatives have been employed successfully in finding solutions to ordinary [9], partial [6], [18], and integral [5] equations. In these applications, the fractional derivative is advantageous because certain critical operations which are not obvious in a classical formulation are suggested by the notation itself. Consider, for example, the result

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x (x-t)^{\alpha-1}\int_0^t f(u)(t-u)^{\beta-1}\,du\,dt = \frac{1}{\Gamma(\alpha+\beta)}\int_0^x f(t)(x-t)^{\alpha+\beta-1}\,dt,$$

$\mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0$. In the notation of fractional derivatives, this last result reads

$$D_x^{-\alpha}D_x^{-\beta}f(x) = D_x^{-\alpha-\beta}f(x),$$

a result which students of the calculus would guess.

Fractional derivatives are also of value in exploring the properties of the higher transcendental functions. Consider the known, but not commonly seen, formula for the Bessel function of order ν:

$$J_\nu(z) = (2z)^{-\nu}\pi^{-1/2}D_{z^2}^{-\nu-1/2}\frac{\cos z}{z}.$$

When ν is $-1/2, -3/2, -5/2, \cdots$, this formula shows that $J_\nu(z)$ is an elementary function. Since $J_\nu(z)$, and many of the important special functions, can be represented as fractional derivatives of elementary functions, it seems reasonable that important properties of the higher transcendental functions could be derived from a knowledge of rules for manipulating fractional derivatives. This observation has appeared previously [7], [9], [11], [12], [13], [14]. The author's papers [11], [12], [13], [14], [15] have been concerned with extending familiar rules for derivatives from the elementary calculus (chain rule, Leibniz rule, Taylor's series) to the higher calculus of fractional derivatives.

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As early as 1859, George Boole [2, Preface] wrote: "This question of the true value and proper place of symbolical methods is undoubtedly of great importance. Their convenient simplicity—their condensed power—must ever constitute their first claim upon attention." It is in this spirit that the Leibniz rule from the elementary calculus is extended in this paper and used in conjunction with fractional derivative representations of the special functions. In this way the "simplicity and condensed power" of the fractional derivative notation is exploited.

We list below successively more complex extensions of the familiar Leibniz rule,

$$D^N uv = \sum_{n=0}^{N} \binom{N}{n} D^{N-n}u D^n v.$$

Extension 1. If α is not a natural number, the Leibniz rule admits the simple generalization

$$D_z^\alpha u(z)v(z) = \sum_{n=0}^{\infty} \binom{\alpha}{n} D_z^{\alpha-n}u(z)D_z^n v(z) \tag{1.1}$$

which was known to Grunwald [8] as early as 1867. Other authors have also considered this formula [1], [11], [12], [15], [16], [19]. A simple derivation employing complex variable techniques and Taylor's series is given in the author's expository paper [15].

Extension 2. Equation (1.1) has a disturbing feature. If we interchange u and v, the formula remains unchanged on the left side, while on the right side this is not obvious since u is differentiated fractionally and v is differentiated in the usual elementary sense. A generalization of (1.1) in which the interchanging of u and v appears permissible on both sides is

$$D_z^\alpha u(z)v(z) = \sum_{n=-\infty}^{\infty} \binom{\alpha}{n+\gamma} D_z^{\alpha-n-\gamma}u(z)D_z^{n+\gamma}v(z), \tag{1.2}$$

where

$$\binom{\alpha}{\beta} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\beta+1)\Gamma(\beta+1)},$$

and γ is an arbitrary real or complex number. This series was first published by Watanabe [19] in 1931. The region of convergence of the series in the z-plane was first determined by the author [11], [12].

Extension 3. Our next extension shows that we can also differentiate fractionally with respect to an arbitrary function $g(z)$, and even more, the sum need not be over the integers n, but can be over a times n, where $0 < a \leqq 1$:

$$D_{g(z)}^\alpha u(z)v(z) = \sum_{n=-\infty}^{\infty} a \binom{\alpha}{an+\gamma} D_{g(z)}^{\alpha-an-\gamma}u(z)D_{g(z)}^{an+\gamma}v(z). \tag{1.3}$$

This result is new.

Extension 4. By introducing the function $\theta(\zeta; z) = (g(\zeta) - g(z))q(\zeta)$, we can generalize the previous result to

$$
D^{\alpha}_{g(z)}u(z)v(z) = \sum_{n=-\infty}^{\infty} a\binom{\alpha}{an+\gamma} D^{\alpha-an-\gamma}_{g(z)}[u(z)q(z)^{an+\gamma}] \cdot D^{an+\gamma}_{g(\zeta)}[v(\zeta)\theta_{g(\zeta)}(\zeta; z)q(\zeta)^{-an-\gamma-1}]\Big|_{\zeta=z}, \tag{1.4}
$$

where again $0 < a \leqq 1$ and γ is arbitrary. This new result can be simplified provided $v(g^{-1}(0)) = 0$ as

$$
D^{\alpha}_{g(z)}u(z)v(z) = \sum_{n=-\infty}^{\infty} a\binom{\alpha}{an+\gamma} D^{\alpha-an-\gamma}_{g(z)}[u(z)q(z)^{an+\gamma}] \cdot D^{an+\gamma-1}_{g(z)}\left[\frac{dv(z)}{dg(z)}q(z)^{-an-\gamma}\right]. \tag{1.4a}
$$

Extension 5. The product uv can be replaced by a general function of two variables. This leads to the generalization

$$
D^{\alpha}_{g(z)}f(z,z) = \sum_{n=-\infty}^{\infty} a\binom{\alpha}{an+\gamma} D^{\alpha-an-\gamma,an+\gamma}_{g(\xi),g(\zeta)} \cdot [f(\xi,\zeta)\theta_{g(\zeta)}(\zeta; z)q(\xi)^{an+\gamma}q(\zeta)^{-an-\gamma-1}]\Big|_{\substack{\xi=z\\ \zeta=z}}, \tag{1.5}
$$

where again $0 < a \leqq 1$, and $\theta(\zeta : z) = (g(\zeta) - g(z))q(\zeta)$. Here the $D^{\alpha,\beta}_{g(\xi),h(\zeta)}f(\xi,\zeta)$ means operate on $f(\xi,\zeta)$ with $D^{\beta}_{h(\zeta)}$ holding ξ fixed followed by $D^{\alpha}_{g(\xi)}$ holding ζ fixed. If we set $f(\xi,\zeta) = u(\xi)v(\zeta)$ in (1.5), we obtain (1.4). If, in addition, $f(\xi, g^{-1}(0)) = 0$, (1.5) simplifies to a form corresponding to (1.4a):

$$
D^{\alpha}_{g(z)}f(z,z) = \sum_{n=-\infty}^{\infty} a\binom{\alpha}{an+\gamma} D^{\alpha-an-\gamma,an+\gamma-1}_{g(\xi),g(\zeta)} \cdot [f_{g(\zeta)}(\xi,\zeta)[q(\xi)/q(\zeta)]^{an+\gamma}]\Big|_{\substack{\xi=z\\ \zeta=z}}. \tag{1.5a}
$$

The special case of (1.5) in which $q(\zeta) = 1$ and $a = 1$ (and thus $\theta_{g(\zeta)}(\zeta; z) = 1$) was given by the author in [11], [12]. All the preceding generalizations of the Leibniz rule are special cases of (1.5).

While derivations of special cases of our generalized Leibniz rule (1.5) have been given previously, the derivation presented in this paper is not an extension of previous methods. An entirely new technique is employed based on the author's previous generalization of Taylor's series to fractional derivatives [14].

The relationship between the generalized Leibniz rule and the familiar Parseval's formula [21, p. 37] from Fourier series is examined. We discover the interesting fact that Parseval's formula is a special case of the Leibniz rule in much the same way that a Fourier series is a special case of a Laurent series.

The paper concludes with an examination of several infinite series expansions derived from (1.5) by introducing specific functions for f, g, q, θ, and specific parameters for α, γ, and a. These series which relate the higher transcendental functions show one way in which fractional derivatives can be exploited in the study of the special functions.

In summary, then, this paper contributes the following items in mathematical analysis:

(i) The generalization of Leibniz rule (1.5) as well as its special cases (1.3), (1.4), (1.4a), and (1.5a) are new.

(ii) The derivation of the generalized Leibniz rule (1.5), based on a generalized Taylor's series, is new. (See § 4.)

(iii) The observation of the relation between the Leibniz rule and Parseval's formula is new. (See § 3.)

(iv) Several of the series expansions relating the special functions (see Table 5.2) appear to be new.

2. Fractional derivatives and special functions. In this section we review the definition of fractional differentiation and give examples of common special functions of mathematical physics represented by fractional derivatives of elementary functions.

The most common definition for the fractional derivative of $f(z)$ of order α found in the literature is the "Riemann–Liouville integral" [4], [5], [6], [7], [8], [9], [18]

$$D_z^{\alpha} f(z) = \Gamma(-\alpha)^{-1} \int_0^z f(t)(z-t)^{-\alpha-1}\,dt,$$

where $\operatorname{Re}(\alpha) < 0$. The concept of a fractional derivative with respect to an arbitrary function $g(z)$, $D^{\alpha}_{g(z)} f(z)$, was apparently introduced for the first time in the author's papers [11], [12], while the idea appeared earlier for certain specific functions $g(z)$ in [6]. The most convenient form of the definition for our purposes is given through a generalization of Cauchy's integral formula. A thorough motivation for the following precise definition is found in [11], [12].

Definition 2.1. Let $f(z)$ be analytic in the simply connected region R. Let $g(z)$ be regular and univalent on R, and let $g^{-1}(0)$ be an interior or boundary point of R. Assume also that $\int_C f(z)g'(z)\,dz = 0$ for any simple closed contour C in $R \cup \{g^{-1}(0)\}$ through $g^{-1}(0)$. Then if α is not a negative integer, and z is in R, we define the *fractional derivative of order* α *of* $f(z)$ *with respect to* $g(z)$ to be

$$D^{\alpha}_{g(z)} f(z) = \frac{\Gamma(\alpha+1)}{2\pi i} \int_{g^{-1}(0)}^{(z+)} \frac{f(\zeta)g'(\zeta)\,d\zeta}{(g(\zeta)-g(z))^{\alpha+1}}. \tag{2.1}$$

For nonintegral α, the integrand has a branch line which begins at $\zeta = z$ and passes through $\zeta = g^{-1}(0)$. The limits of integration imply that the contour of integration starts at $g^{-1}(0)$, encloses z once in the positive sense, and returns to $g^{-1}(0)$ without cutting the branch line or leaving $R \cup \{g^{-1}(0)\}$. (See Fig. 2.1).

If α is a negative integer $-N$, $\Gamma(\alpha+1) = \infty$ while the integral in (2.1) vanishes. If we interpret (2.1) as the limit as α approaches $-N$, it then defines the derivative of order $-N$, or perhaps we should say the "Nth iterated integral of $f(z)$ with respect to $g(z)$."

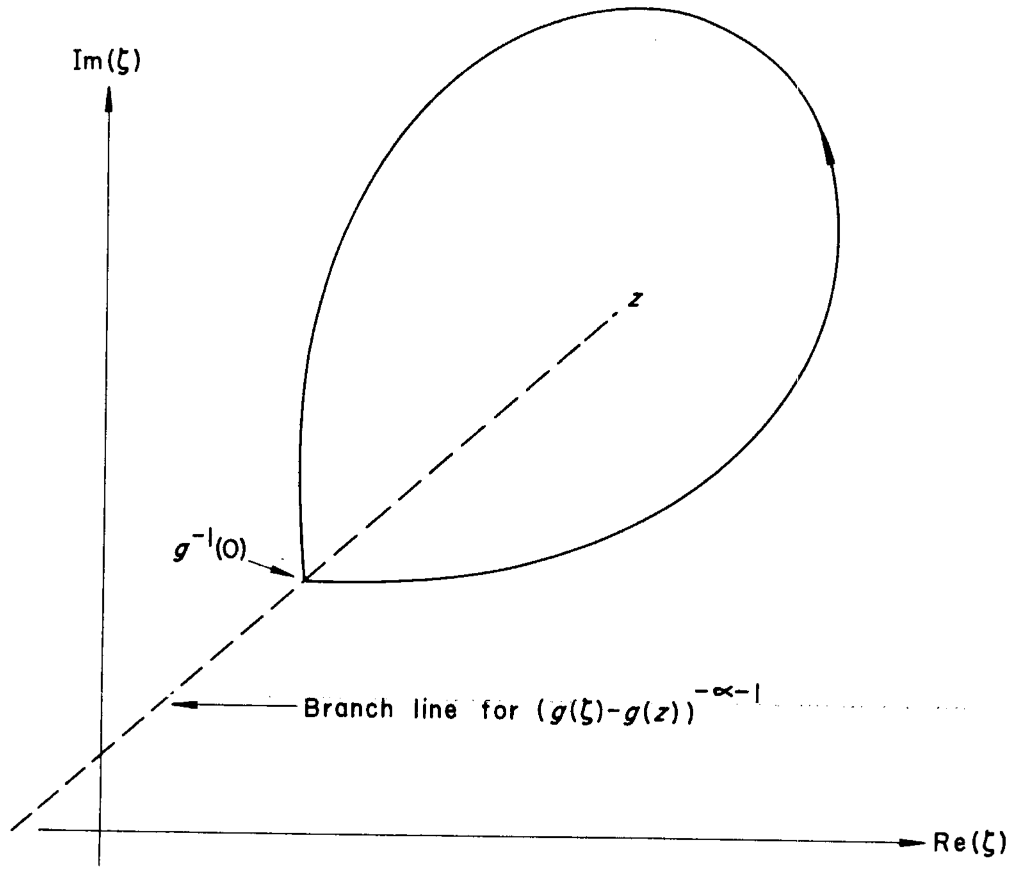

FIG. 2.1. *Branch line and contour of integration for Definition* 2.1 *of fractional differentiation*

It is important to notice that with the restrictions on $g(z)$ as given in Definition 2.1, the substitution $w = g(z)$ maintains the equality $D^\alpha_w f(w) = D^\alpha_{g(z)} f(g(z))$.

It is particularly interesting to set $g(z) = z - a$, for we find that

$$D^\alpha_{z-a} f(z) = \frac{\Gamma(\alpha+1)}{2\pi i}\int_a^{(z^+)} f(\zeta)(\zeta - z)^{-\alpha-1}\,d\zeta. \tag{2.2}$$

While ordinary derivatives with respect to z and $z - a$ are equal, (2.2) shows that this is not the case for fractional derivatives, since the value of the contour integral depends on the point $\zeta = a$ at which the contour crosses the branch line.

We also require fractional partial derivatives.

DEFINITION 2.2. Let $f(z, w)$ be an analytic function of two variables for z and w in the simply connected region R. Let $g(z)$ be regular and univalent on R, and let $g^{-1}(0)$ be an interior or boundary point of R. Assume also that $\int_C f(z, w)g'(w)\,dw = 0$ and $\int_C D^\beta_{g(w)} f(z, w)g'(z)\,dz = 0$ for any simple closed contour C in $R \cup \{g^{-1}(0)\}$ through $g^{-1}(0)$. Then if α and β are not negative integers, and z and w are in R we write

$$\begin{aligned} D^{\alpha,\beta}_{g(z),g(w)} f(z, w) &= D^\alpha_{g(z)}[D^\beta_{g(w)} f(z, w)] \\ &= \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{-4\pi^2}\int_{g^{-1}(0)}^{(z^+)} \frac{g'(\zeta)}{(g(\zeta) - g(z))^{\alpha+1}} \\ &\quad \cdot \int_{g^{-1}(0)}^{(w^+)} \frac{f(\zeta, \xi)g'(\xi)\,d\xi\,d\zeta}{(g(\xi) - g(w))^{\beta+1}}. \end{aligned} \tag{2.3}$$

Contour integrals of the type (2.1) occur often in the representations of special functions. These are particularly convenient for use with the generalized Leibniz rule (1.5). Fractional derivative representations of special functions are also found in [11], [12] and can be easily constructed from the tables in [4]. A few examples follow:

$$F(a,b;c;z) = \frac{\Gamma(c)z^{1-c}}{\Gamma(b)} D_z^{b-c} z^{b-1}(1-z)^{-a},$$

$${}_1F_1(a;c;z) = \frac{\Gamma(c)z^{1-c}}{\Gamma(a)} D_z^{a-c} e^z z^{a-1},$$

$$J_v(z) = \frac{z^{-v}}{2^v\sqrt{\pi}} D_{z^2}^{-v-1/2} \frac{\cos z}{z},$$

$$P_v^u(z) = \frac{(1-z^2)^{u/2}}{\Gamma(v+1)2^v} D_{1-z}^{v+u}(1-z^2)^v.$$

Having reviewed briefly the definition of fractional differentiation and its relation to the special functions, we proceed to show a formal correspondence between our generalized Leibniz rule and a familiar formula from the elementary study of Fourier series.

3. The Leibniz rule and Parseval's formula. In this section we formally examine the special case of the generalized Leibniz rule (1.4). By holding z fixed, and making a suitable change of variables, we shall see that Parseval's formula [21, p. 37], familiar from the study of Fourier series, emerges.

Let us begin by assuming that (1.4) is true. With $g(z) = z$ we have

$$\frac{1}{\Gamma(\alpha+1)} D_z^{\alpha} u(z)v(z) = \sum_{n=-\infty}^{\infty} \frac{a}{\Gamma(\alpha - an - \gamma + 1)} D_z^{\alpha - an - \gamma}[u(z)q(z)^{an+\gamma}]$$
$$\cdot \frac{1}{\Gamma(an+\gamma+1)} D_\zeta^{an+\gamma}[v(\zeta)q(\zeta)^{-an-\gamma-1}\theta_\zeta(\zeta;z)]\Big|_{\zeta=z},$$

where we recall that $\theta(\zeta;z) = (\zeta - z)q(\zeta)$ and $0 < a \leqq 1$. Making use of the contour integral representation for fractional derivatives (2.2), we get

$$\frac{1}{2\pi i}\int_0^{(z+)} \frac{u(t)v(t)\,dt}{(t-z)^{\alpha+1}} = \sum_{n=-\infty}^{\infty} \frac{-a}{4\pi^2}\int_0^{(z+)} \frac{u(t)q(t)^{an+\gamma}\,dt}{(t-z)^{\alpha-an-\gamma+1}} \tag{3.1}$$
$$\cdot \int_0^{(z+)} \frac{v(t)q(t)^{-an-\gamma-1}\theta_t(t;z)\,dt}{(t-z)^{an+\gamma+1}}.$$

We now fix z and select the contours of integration appearing above to coincide with the curve defined by $|\theta(t;z)| = |\theta(0;z)|$; that is, the contour which passes through the origin (in the t-plane) on which $\theta(t;z)$ has constant modulus. This contour we assume is a closed curve which can be parametrized by the variable ϕ such that

$$\theta(t;z) = |\theta(0;z)|e^{i\phi}, \tag{3.2}$$

with $\phi_0 < \phi < \phi_0 + 2\pi$. Using (3.2) to change the variable of integration from

t to ϕ in (3.1), and writing $u(t) = u[\phi], \cdots$, we get

$$
\begin{aligned}
&\frac{a}{2\pi}\int_{\phi_0}^{\phi_0 + 2\pi/a} f(\phi)h(\phi)\,d\phi \\
(3.3)\qquad &= \sum_{n=-\infty}^{\infty} \frac{a}{2\pi}\int_{\phi_0}^{\phi_0 + 2\pi/a} f(\phi)e^{ian\phi}\,d\phi \cdot \frac{a}{2\pi}\int_{\phi_0}^{\phi_0 + 2\pi/a} h(\phi)e^{-ian\phi}\,d\phi,
\end{aligned}
$$

where we have set

$$
f(\phi) = \begin{cases} \dfrac{u[\phi]q[\phi]^{\alpha+1}e^{i(\gamma-\alpha)\phi}}{\theta_t[\phi; z]} & \text{for } \phi_0 < \phi < \phi_0 + 2\pi, \\ 0 & \text{otherwise,} \end{cases}
$$

and

$$
h(\phi) = \begin{cases} v[\phi]e^{-i\gamma\phi} & \text{for } \phi_0 < \phi < \phi_0 + 2\pi, \\ 0 & \text{otherwise.} \end{cases}
$$

We recognize (3.3) as Parseval's formula [21, p. 37].

Now consider an analytic function $w(t)$ expanded in a Laurent series $w(t) = \sum a_n(t - z)^n$. If we restrict t to the circle $z + |z|e^{i\phi}$, the Laurent series becomes the Fourier series $w(z + |z|e^{i\phi}) = \sum a_n|z|^n e^{in\phi}$. Note the similarity between this and the previous calculation, in which we held z fixed and examined u and v on a particular closed curve in the t-plane. In fact, if $\theta(t; z) = t - z$, the contour of integration defined by (3.2) is identical to the circle on which the Laurent series was just examined. We conclude that by holding z fixed, the generalized Leibniz rule reduces to Parseval's formula in the same way that Laurent's series reduces to a Fourier series. Thus our extended Leibniz rule is a generalization of Parseval's formula.

4. The extended Leibniz rule.

In formally examining the special case of the extended Leibniz rule (1.4) in the previous section, we have seen that it is related to the Parseval's formula familiar from the study of Fourier series. We now proceed to derive the extended Leibniz rule rigorously. We shall see that the derivation of the Leibniz rule follows from the generalized Taylor's series in much the same way that the Parseval's relation follows from the Fourier series.

We begin by stating and proving the special case of the extended Leibniz rule in which $g(z) = z$.

THEOREM 4.1. (i) *Let R be a simply connected region in the complex plane having the origin as an interior or boundary point.*

(ii) *Let $f(\xi, \zeta)$ satisfy the conditions of Definition* 2.2 *for the existence of $D^{\alpha,b}_{\xi,\zeta} f(\xi, \zeta)$ and $D^{\alpha}_{z} f(z, z)$ for ξ, ζ, and z in R.*

(iii) *Let $\theta(\zeta; z) = (\zeta - z)q(\zeta)$ be a given function such that $q(\zeta)$ is analytic for $\zeta \in R$, and $q(\zeta)$ is never zero on R.*

(iv) *Assume that the curves $C(z) = \{\zeta \,|\, |\theta(\zeta; z)| = |\theta(0; z)|\}$ are simple and closed for each z such that $C(z) \subset R \cup \{0\}$. Assume also that each curve defined by $\{\zeta \,|\, |\theta(\zeta; z)| = \text{const.}\}$ interior to $C(z)$ is simple and closed.*

(v) *Call $S = \{z | C(z) \subset R \cup \{0\}\}$.*

Then for $z \in S$, $0 < a \leqq 1$, *and all* α *and* γ *such that* $\binom{\alpha}{an+\gamma}$ *is defined,*

$$D_z^\alpha f(z,z) = \sum_{n=-\infty}^{\infty} a \binom{\alpha}{an+\gamma} D_{\xi,\zeta}^{\alpha-an-\gamma,\,an+\gamma} \cdot [f(\xi,\zeta)\theta_\zeta(\zeta;z)q(\xi)^{an+\gamma}q(\zeta)^{-an-\gamma-1}]|_{\xi=\zeta=z}, \tag{4.1}$$

where $\theta_\zeta(\zeta;z) = d\theta(\zeta;z)/d\zeta$.

Proof. The $C(z)$ are the curves in the complex ζ-plane which pass through the origin, over which the amplitude of $\theta(\zeta;z)$ is constant. For example, if $\theta(\zeta;z) = \zeta - z$, then $C(z)$ is the circle centered at $\zeta = z$ passing through the origin. By restricting z to S (described in (v)), we insure that the curves $C(z)$ are contained in the region $R \cup \{0\}$ on which $f(\xi,\zeta)$ is sufficiently regular for manipulations which follow. In particular, $f(\xi,\zeta)$ can be expanded in a generalized Taylor's series for $\zeta \in C(z)$ in powers of $\theta(\zeta;z)$ since (ii), (iii) and (iv) are all that is required for its validity [14]. We obtain

$$f(\xi,\zeta) = \sum_{n=-\infty}^{\infty} \frac{aD_\zeta^{an+\gamma}[f(\xi,\zeta)\theta_\zeta(\zeta;z)q(\zeta)^{-an-\gamma-1}]|_{\zeta=z}(q(\zeta)(\zeta-z))^{an+\gamma}}{\Gamma(an+\gamma+1)}.$$

Multiply both sides of this last expression by $\Gamma(\alpha+1)(\xi-z)^{-\alpha-1}/(2\pi i)$ and set $\zeta = \xi$:

$$\frac{\Gamma(\alpha+1)}{2\pi i}\frac{f(\xi,\xi)}{(\xi-z)^{\alpha+1}} = \sum_{n=-\infty}^{\infty} \frac{a\Gamma(\alpha+1)q(\xi)^{an+\gamma}}{\Gamma(an+\gamma+1)2\pi i(\xi-z)^{\alpha-an-\gamma+1}} \cdot D_\zeta^{an+\gamma}[f(\xi,\zeta)\theta_\zeta(\zeta;z)q(\zeta)^{-an-\gamma-1}]|_{\zeta=z}. \tag{4.2}$$

Since (4.2) converges for ξ on the curve $C(z)$ in the complex ξ-plane, we can integrate both sides along the contour $C(z)$ with respect to ξ starting and ending at $\xi = 0$. It is clear that we can integrate term by term along the contour $C(z)$, since (4.2) is really a Fourier series in the variable ϕ when we replace $\theta(\xi;z)$ by $|\theta(0;z)|e^{i\phi}$:

$$\frac{\Gamma(\alpha+1)}{2\pi i}\int_{C(z)} \frac{f(\xi,\xi)\,d\xi}{(\xi-z)^{\alpha+1}} = \sum_{n=-\infty}^{\infty} \frac{a\Gamma(\alpha+1)}{\Gamma(an+\gamma+1)2\pi i} \cdot \int_{C(z)} \frac{q(\xi)^{an+\gamma}D_\zeta^{an+\gamma}[f(\xi,\zeta)\theta_\zeta(\zeta;z)q(\zeta)^{-an-\gamma-1}]|_{\zeta=z}\,d\xi}{(\xi-z)^{\alpha-an-\gamma+1}}.$$

Comparing the integrals above with the definitions of fractional differentiation (2.2) and (2.3) we see at once that the generalized Leibniz rule (4.1) is obtained.

Equation (4.1) can be simplified to

$$D_z^\alpha f(z,z) = \sum_{n=-\infty}^{\infty} a \binom{\alpha}{an+\gamma} D_{\xi,\zeta}^{\alpha-an-\gamma,\,an+\gamma-1}[f_\zeta(\xi,\zeta)[q(\xi)/q(\zeta)]^{an+\gamma}]|_{\xi=\zeta=z} \tag{4.3}$$

if we add the restriction that $f(\xi,0) = 0$.

COROLLARY 4.1. *With the hypothesis of Theorem* 4.1 *and* $f(\xi,0) = 0$, *the relation* (4.3) *is valid.*

Proof. Comparing (4.1) and (4.3) it is clear we must show that

$$E = D_{\xi,\zeta}^{\alpha - an - \gamma, an + \gamma}[f(\xi, \zeta)\theta_\zeta(\zeta; z)q(\xi)^{an + \gamma}q(\zeta)^{-an-\gamma-1}]|_{\zeta=\xi=z}$$
$$= D_{\xi,\zeta}^{\alpha - an - \gamma, an + \gamma - 1}[f_\zeta(\xi, \zeta)[q(\xi)/q(\zeta)]^{an+\gamma}]|_{\xi=\zeta=z}.$$

The left-hand side of this last relation can be written as

$$E = D_\xi^{\alpha - an - \gamma}\left[q(\xi)^{an+\gamma}\frac{\Gamma(an+\gamma+1)}{2\pi i}\int_0^{(z^+)}\frac{f(\xi,\zeta)\theta_\zeta(\zeta; z)\,d\zeta}{\theta(\zeta; z)^{an+\gamma+1}}\right]\Bigg|_{\xi=z}$$

using (2.2) and (2.3). Integrating by parts we get

$$E = D_\xi^{\alpha - an - \gamma}\left[q(\xi)^{an+\gamma}\frac{\Gamma(an+\gamma)}{2\pi i}\int_0^{(z^+)} f_\zeta(\xi,\zeta)\theta(\zeta; z)^{-an-\gamma}\,d\zeta\right]\Bigg|_{\xi=z},$$

where the jump term vanished because $f(\xi, 0) = 0$. Rewriting this last integral using the definitions of fractional differentiation (2.2) and (2.3), and $\theta(\zeta; z) = (\zeta - z)q(\zeta)$, we see at once that the corollary is proved.

We complete our derivation by extending the Leibniz rule to the case in which we differentiate with respect to an arbitary function $g(z)$.

COROLLARY 4.2. *Assume the hypothesis of Theorem* 4.1 *and the additional conditions*

(i) $g(w)$ *is regular and univalent for* $w \in g^{-1}(R)$,
(ii) $F(s, t) = f(g(s), g(t))$,
(iii) $\Xi(s; w) = \theta(g(s); g(w)) = (g(s) - g(w))Q(s)$,
(iv) $q(g(s)) = Q(s)$.

Then

$$D_{g(w)}^\alpha F(w, w) = \sum_{n=-\infty}^{\infty} a\binom{\alpha}{an+\gamma} D_{g(s),g(t)}^{\alpha - an - \gamma, an + \gamma} \cdot \left[F(s, t)\frac{d\Xi(t; w)}{dg(t)}Q(s)^{an+\gamma}Q(t)^{-an-\gamma-1}\right]\Bigg|_{s=t=w} \tag{4.4}$$

for $w \in g^{-1}(S)$, $0 < a \leqq 1$, *and all* α *and* γ *for which* $\binom{\alpha}{an+\gamma}$ *is defined.*

If in addition we have

(v) $F(s, g^{-1}(0)) = 0$,

then (4.4) *can be simplified to*

$$D_{g(w)}^\alpha F(w, w) = \sum_{n=-\infty}^{\infty} a\binom{\alpha}{an+\gamma} D_{g(s),g(t)}^{\alpha - an - \gamma, an + \gamma - 1} \cdot \left[\frac{\partial F(s, t)}{\partial g(t)}[Q(s)/Q(t)]^{an+\gamma}\right]\Bigg|_{s=t=w}. \tag{4.5}$$

Proof. The proof of this corollary follows at once upon replacing z by $g(w)$ in Theorem 4.1 and Corollary 4.1, since $D_z^\alpha f(z) \equiv D_{g(w)}^\alpha f(g(w))$.

Remark. In Theorem 4.1 and the above corollaries, the Leibniz rule is valid for all α and γ for which

$$(4.6) \qquad \binom{\alpha}{an+\gamma} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha - an - \gamma + 1)\Gamma(an+\gamma+1)}$$

is defined. Since $\Gamma(z)$ is analytic except for poles at $z = 0, -1, -2, \cdots$; and since $1/\Gamma(z)$ is entire, the only values of α for which (4.6) is suspect are $\alpha = -1, -2, -3, \cdots$. It is well known that when α is a negative integer, $a = 1$, and $\gamma = 0$, the binomial coefficient (4.6) is defined; however, if α is a negative integer and $\gamma \neq 0$, then (4.6) is not defined for every integer n. A glance at (4.2) shows that the troublesome $\Gamma(\alpha + 1)$ appears in the numerator on both sides of the equation. If we divide both sides of our Leibniz rule by $\Gamma(\alpha + 1)$, this difficulty disappears. Thus we conclude that the restrictions on α and γ for the validity of the generalized Leibniz rule are needed only because the notation $\binom{\alpha}{an+\gamma}$ is convenient. When we use the Leibniz rule in the next section, to derive series expansions relating the special functions, we will divide by $\Gamma(\alpha + 1)$ and conclude that restrictions on α and γ are unnecessary.

We have completed our rigorous examination of the generalized Leibniz rule, and now turn to examples of infinite series relating special functions which are derived from it.

5. Examples. Before ending our discussion, it seems appropriate to examine direct consequences of our new formulas. We select specific functions for $f(\xi, \zeta)$, $g(z)$, $q(\zeta)$, and specific parameters for α, γ and a in our generalized Leibniz rule (1.5). A list of the selections is given in Table 5.1. The fractional derivatives encountered can be computed with the help of the extensive table in [4, vol. 2, pp. 181–200] and also with the short table of fractional derivative representations for special functions in [12, p. 668]. The results of this simple procedure appear in Table 5.2. A similar table, restricted to the special case of (1.5) in which $q(\zeta) = 1$ and $a = 1$, appeared in [11], [12]. The notation for the special functions used is that of Erdélyi et al. [3], [4].

We call particular attention to the following series from Table 5.2.

Extension of Dougall's formula. Series 9 is a generalization of "Dougall's formula" [3, vol. 1, p. 7]. Dougall's formula is the special case of series 9 in which $a = 1$.

Series of the Cardinal type. Series 2 through 8 are of the Cardinal type [20, pp. 62–71]. A Cardinal series gives the values of a function $f(\alpha)$ when the values of $f(\alpha)$ are known only at $\alpha = an + \gamma$, where $0 < a \leqq 1$ and γ are fixed and $n = 0, \pm 1, \pm 2, \cdots$. If we set $u(z) = 1$ in (1.3), we obtain

$$(5.1) \qquad \frac{g(z)^\alpha D^\alpha_{g(z)} v(z)}{\Gamma(\alpha+1)} = \sum_{n=-\infty}^{\infty} \frac{a \sin \pi(\alpha - an - \gamma)}{\pi(\alpha - an - \gamma)} \frac{g(z)^{an+\gamma} D^{an+\gamma}_{g(z)} v(z)}{\Gamma(an+\gamma+1)}.$$

Thus if $f(\alpha)$ is of the form

$$f(\alpha) = \frac{g(z)^\alpha D^\alpha_{g(z)} v(z)}{\Gamma(\alpha+1)},$$

TABLE 5.1

Choices for functions and parameters in the generalized Leibniz rule (1.5) from which the series in Table 5.2 are derived

Series No.	$f(\xi, \zeta)$	$q(\zeta)$	$g(z)$	α	γ
1	$e^{A\xi}\zeta$	$e^{B\zeta}$	z	$N = 1, 2, 3, \cdots$	0
2	$\zeta^{B-1}(1-\zeta)^{-A}$	1	z	$B - C$	γ
3	$e^{\zeta}\zeta^{A-1}$	1	z	$A - B$	$A - C$
4	$(\cos \zeta)/\zeta$	1	z^2	$-\nu - 1/2$	$-1/2 - B$
5	$(\cosh \zeta)/\zeta$	1	z^2	$-\nu - 1/2$	$-1/2 - B$
6	$(\sin \zeta)/\zeta$	1	z^2	$-\nu - 1/2$	$-1/2 - B$
7	$(\sinh \zeta)/\zeta$	1	z^2	$-\nu - 1/2$	$-1/2 - B$
8	$(1-\zeta^2)^{\nu}$	1	$1 - z$	$\nu + \mu$	γ
9	$\xi^{B+C-2}\zeta^{A+D-2}$	1	z	$A + C - 2$	$A - 1$
10	$\xi^{b-1}(1-\xi)^{-e}\zeta^{B-1}(1-\zeta)^{-E}$	1	z	$b + B - d - D$	$B - D$
11	$\xi^{D-A-1}\zeta^{C-B-1}(\xi\zeta+1)^{-E}$	1	z	$C - A - 1$	$C - 1$
12	$e^{A\xi - B\zeta}\xi^{P-1}$	$e^{C\zeta}$	z	α	0
13	ξ^{B-1}	$\zeta + A$	z	α	0
14	$\xi^{A-1}\zeta^{B}$	$\zeta^k + P^k$	z	α	γ
15	$\xi^{A-1}\zeta^{B}$	$\exp(\zeta^k)$	z	α	γ
16	$\xi^{A}\zeta^{B-1}{}_rF_s(a_1, \cdots, a_r;$ $b_1, \cdots, b_s; \zeta)$	1	z	α	γ

then (5.1) yields the Cardinal series

$$f(\alpha) = \sum_{n=-\infty}^{\infty} \frac{a \sin \pi(\alpha - an - \gamma)}{\pi(\alpha - an - \gamma)} f(an + \gamma).$$

A problem of Ramanujan. The series 12 is a generalization of the series

$$\varphi(z) = e^{-z} \sum_{n=0}^{\infty} \frac{(n+1)^n (ze^{-z})^n}{n!}$$

considered by Ramanujan [17, p. 332, Question 738]. Ramanujan set as a portion of a problem the demonstration that $\varphi(z) \equiv 1$ for $0 \leqq z \leqq 1$. This problem can be

TABLE 5.2

Series expansions derived from the generalized Leibniz rule

Note: Unless otherwise stated, $0 < a \leqq 1$ *in all series*

Series No.	Series Expansion
1	$NA^{N-1} = \sum_{n=1}^{N} \binom{N}{n}(A + Bn)^{N-n}(-Bn)^{n-1}, N = 1, 2, 3 \cdots,$ $a = 1$
2	$\frac{\pi\,{}_2F_1(A, B; C; z)}{a\Gamma(C)\Gamma(B - C + 1)} = \sum_{n=-\infty}^{\infty} \frac{\sin((an + \gamma + C - B)\pi)\,{}_2F_1(A, B; B - \gamma - an; z)}{(an + \gamma + C - B)\Gamma(an + \gamma + 1)\Gamma(B - \gamma - an)},$ $\mathrm{Re}(z) < 1/2, \quad 0 < \mathrm{Re}(B)$
3	$\frac{\pi\,{}_1F_1(A; B; z)}{a\Gamma(B)\Gamma(A - B + 1)} = \sum_{n=-\infty}^{\infty} \frac{\sin((an + B - C)\pi)\,{}_1F_1(A; C - an; z)}{(an + B - C)\Gamma(an + A - C + 1)\Gamma(C - an)},$ $\mathrm{Re}(A) > 0$
4 through 7	$\mathscr{F}_\nu(z) = \frac{a\Gamma(1/2 - \nu)(z/2)^{\nu - B}}{\pi} \sum_{n=-\infty}^{\infty} \frac{\sin((an + \nu - B)\pi)\mathscr{F}_{B-an}(z)}{(an + \nu - B)\Gamma(an - B + 1/2)}\left(\frac{z}{2}\right)^{an},$ where $\mathscr{F}_\nu = J_\nu, I_\nu, \mathbf{H}_\nu$ and $\mathbf{L}_\nu$, respectively, for series 4, 5, 6, and 7
8	$\frac{\pi P_\nu^\mu(z)}{a\Gamma(\nu + \mu + 1)} = \sum_{n=-\infty}^{\infty} \frac{\sin((an + \gamma - \nu - \mu)\pi)}{(an + \gamma - \nu - \mu)} \cdot \frac{P_\nu^{an+\gamma-\nu}(z)}{\Gamma(an + \gamma + 1)}\left(\frac{1 - z}{1 + z}\right)^{(an+\gamma-\nu-\mu)/2},$ $-1 < \mathrm{Re}(\nu), \quad 0 < \mathrm{Re}(z)$
9	$\frac{\Gamma(A + B + C + D - 3)}{a\Gamma(A + C - 1)\Gamma(A + D - 1)\Gamma(B + C - 1)\Gamma(B + D - 1)}$ $= \sum_{n=-\infty}^{\infty} \frac{1}{\Gamma(an + A)\Gamma(an + B)\Gamma(C - an)\Gamma(D - an)},$ $1 < \mathrm{Re}(B + C), \quad 1 < \mathrm{Re}(A + D), \quad 3 < \mathrm{Re}(A + B + C + D).$
10	$\frac{\Gamma(b + B - 1)\,{}_2F_1(e + E, b + B - 1; d + D - 1; z)}{a\Gamma(d + D - 1)\Gamma(b + B - d - D + 1)\Gamma(b)\Gamma(B)}$ $= \sum_{n=-\infty}^{\infty} \frac{{}_2F_1(e, b; d + an; z)\,{}_2F_1(E, B; D - an; z)}{\Gamma(an + B - D + 1)\Gamma(an + d)\Gamma(b - d - an + 1)\Gamma(D - an)},$ $0 < \mathrm{Re}(b), \quad 0 < \mathrm{Re}(B), \quad 1/2 > \mathrm{Re}(z), \quad 1 < \mathrm{Re}(b + B)$
11	$\frac{\Gamma(C + D - A - B - 1)\,{}_3F_2\left[\begin{matrix} E, (C + D - A - B - 1)/2, (C + D - A - B)/2; -z^2 \\ (D - B)/2, (D - B + 1)/2 \end{matrix}\right]}{a\Gamma(C - A)\Gamma(C - B)\Gamma(D - A)\Gamma(D - B)}$ $= \sum_{n=-\infty}^{\infty} \frac{{}_3F_2\left[\begin{matrix} E, C - B, D - A; -z^2 \\ D + an, 1 - B - an \end{matrix}\right]}{\Gamma(an + C)\Gamma(an + D)\Gamma(1 - A - an)\Gamma(1 - B - an)},$ $0 < \mathrm{Re}(D - A), \quad \mathrm{Re}(C - B), \quad 1 < \mathrm{Re}(C + D - A - B)$
12	${}_1F_1(P; P - \alpha; (A - B)z) = e^{-(C+B)z} \sum_{n=0}^{\infty} \frac{(-\alpha)_n\,{}_1F_1(P; P - \alpha + n; (A + Cn)z)}{(P - \alpha)_n n!(Cn + B + C)^{-n}(ze^{-Cz})^{-n}},$ $a = 1 \quad$ in (1.5), $\quad \mathrm{Re}(P) > 0$

TABLE 5.2 (Cont.)

Series No.	Series Expansion
13	$\frac{1}{2} = \sum_{n=0}^{\infty} \frac{(2n)!(-\alpha)_n(-Az)^n(z+A)^{-2n}{}_2F_1(-n, B; B-\alpha+n; -z/A)}{(B-\alpha)_n(n!)^2}$, $a = 1$ in (1.5)
14	$\frac{\Gamma(A+B)}{a\Gamma(A+B-\alpha)\Gamma(\alpha+1)\Gamma(A)\Gamma(B+1)}$ $= \sum_{n=-\infty}^{\infty} \frac{{}_{k+1}F_k\left[\begin{matrix} -an-\gamma, A/k, (A+1)/k, \cdots, (A+k-1)/k; -z^k/P^k \\ (an+A+\gamma-\alpha)/k, (an+A+\gamma-\alpha+1)/k, \cdots, (an+A+\gamma-\alpha+k-1)/k \end{matrix}\right]}{\Gamma(an+\gamma+1)\Gamma(an+A+\gamma-\alpha)\Gamma(\alpha-\gamma-an+1)\Gamma(B-\gamma-an+1)}$ $\cdot {}_{k+1}F_k\left[\begin{matrix} an+\gamma, B/k, (B+1)/k, \cdots, (B+k-1)/k; -z^k/P^k \\ (B-an-\gamma+1)/k, \cdots, (B-an-\gamma+k)/k \end{matrix}\right]$, $-1 < \mathrm{Re}(A), \quad 0 < \mathrm{Re}(B), \quad k = 1, 2, 3, \cdots$
15	$\frac{\Gamma(A+B)}{a\Gamma(A+B-\alpha)\Gamma(\alpha+1)\Gamma(A)\Gamma(B+1)}$ $= \sum_{n=-\infty}^{\infty} \frac{{}_kF_k\left[\begin{matrix} B/k, (B+1)/k, \cdots, (B+k-1)/k; -(an+\gamma)z^k \\ (B-an-\gamma+1)/k, (B-an-\gamma+2)/k, \cdots, (B-an-\gamma+k)/k \end{matrix}\right]}{\Gamma(an+A+\gamma-\alpha)\Gamma(an+\gamma+1)\Gamma(\alpha-\gamma-an+1)\Gamma(B-\gamma-an+1)}$ $\cdot {}_kF_k\left[\begin{matrix} A/k, (A+1)/k, \cdots, (A+k-1)/k; (an+\gamma)z^k \\ (an+A+\gamma-\alpha)/k, (an+A+\gamma-\alpha+1)/k, \cdots, (an+A+\gamma-\alpha+k-1)/k \end{matrix}\right]$, $-1 < \mathrm{Re}(A), \quad 0 < \mathrm{Re}(B), \quad k = 1, 2, 3, \cdots$
16	$\frac{\Gamma(A+B)_{r+1}F_{s+1}\left[\begin{matrix} A+B, a_1, \cdots, a_r; z \\ A+B-\alpha, b_1, \cdots, b_s \end{matrix}\right]}{a\Gamma(A+B-\alpha)\Gamma(\alpha+1)\Gamma(A+1)\Gamma(B)}$ $= \sum_{n=-\infty}^{\infty} \frac{{}_{r+1}F_{s+1}\left[\begin{matrix} B, a_1, \cdots, a_r; z \\ B-\gamma-an, b_1, \cdots, b_s \end{matrix}\right]}{\Gamma(an+\gamma+1)\Gamma(an+A-\alpha+\gamma+1)\Gamma(\alpha-\gamma-an+1)\Gamma(B-\gamma-an)}$, $-1 < \mathrm{Re}(A), \quad 0 < \mathrm{Re}(B)$

solved easily from series 12 if we set $\alpha = -1$, $B = 0$, $C = 1$, and $A = P + 1$ and obtain

$$
\begin{aligned}
&{}_1F_1(P; P+1; (P+1)z) \\
&\quad = e^{-z} \sum_{n=0}^{\infty} \frac{(n+1)^n(ze^{-z})^n}{n!} \frac{n!}{(P+1)_n} {}_1F_1(P; P+n+1; (P+n+1)z)
\end{aligned}
\tag{5.2}
$$

where $\mathrm{Re}(P) > 0$. If we could set $P = 0$ on both sides of (5.2), we would answer Ramanujan's question at once since ${}_1F_1(0; c; x) = 1$. However, the restriction $\mathrm{Re}(P) > 0$ does not permit us to set $P = 0$. Instead, we show that for fixed z,

$0 \leqq z < 1$, the series (5.2) converges uniformly in P, for $0 \leqq P \leqq 1$. This uniform convergence permits us to let P approach zero term by term in (5.2) and thereby solve Ramanujan's problem. As n approaches infinity, for fixed z, $0 \leqq z \leqq 1$, and all P such that $0 \leqq P \leqq 1$,

$$
\begin{aligned}
&{}_1F_1(P; P+n+1; (P+n+1)z) \\
&\quad = (1-z)^{-P}\left[1 - \frac{P(P+1)}{2(P+n+1)}\left(\frac{z}{1-z}\right)^2 + O(|P+n+1|^{-2})\right],
\end{aligned}
$$

[3, vol. 1, p. 280]. Thus

$$
\left|\frac{n!}{(P+1)_n}\,{}_1F_1(P; P+n+1; (P+n+1)z)\right|
$$

is bounded and the series (5.2) converges uniformly in P, for $0 \leqq P \leqq 1$, by the familiar test of Weierstrass. Thus we have shown that $\varphi(z) \equiv 1$ for $0 \leqq z < 1$ and have answered Ramanujan's question.

Note on restrictions in Table 5.2. The restrictions obtained from the hypothesis of Theorem 4.1 for the validity of the series in Table 5.2 are sometimes too strong. Consider, for example, series 9. It is known that only the restriction $\mathrm{Re}(A + B + C + D) > 3$ is necessary. The restrictions $\mathrm{Re}(B + C) > 1$ and $\mathrm{Re}(A + D) > 1$ are not needed and emerge from item (ii) of the hypothesis of Theorem 4.1 in which we require that $D^{a,b}_{\xi,\zeta} f(\xi, \zeta)$ be defined. Since Table 5.2 is provided to illustrate our general expansions, all restrictions emerging from the theorems of this paper are listed.

6.1. Concluding thoughts. In 1695 Leibniz [10], in a letter to J. Bernoulli, expressed his interest in the fact that the binomial series

$$
(A + B)^N = \sum_{n=0}^{N} \binom{N}{n} A^{N-n} B^n
$$

and the rule for the derivative of a product

$$
D^N uv = \sum_{n=0}^{N} \binom{N}{n} D^{N-n} u D^n v
$$

look so similar:

"There are yet many things latent in these progressions of summation and differentiation, which will gradually appear. There is thus notable agreement between the numerical powers of binomial and differential expansions; and I believe that I do not know what is hidden there."

Bernoulli answered: "Nothing is more elegant than the agreement which you have observed between the numerical power of the binomial and differential expansions; there is no doubt that something is hidden there."

Indeed the giants of analysis were correct. At the time of these letters, Newton had invented an extension of the binomial theorem to fractional powers, but the extension of the product rule had to await the invention of the fractional calculus.

Furthermore, the binomial series admits the generalization [14]

$$(6.1)\qquad (A + B)^\alpha = \sum_{n=-\infty}^{\infty} a \binom{\alpha}{an + \gamma} A^{\alpha - an - \gamma} B^{an + \gamma},$$

where $0 < a \leqq 1$ and $|A/B| = 1$. Equation (6.1) resembles our generalized Leibniz rule (6.2):

$$(6.2)\qquad D^\alpha uv = \sum_{n=-\infty}^{\infty} a \binom{\alpha}{an + \gamma} D^{\alpha - an - \gamma} u D^{an + \gamma} v.$$

Moreover, (6.1) is a special case of the generalized Taylor series from which (6.2) is derived in this paper. Thus a reason for the similarity in the two series is made evident.

It is already clear, however, that even further results "lay hidden." The Leibniz rule for functions of the operator D more general than D^α was given as early as 1930 by Emil Post [16]. Undoubtedly Post's form of the product rule can be generalized to reveal further connections prophesied by Leibniz.

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