Binomials to Binomials

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The familiar binomial theorem expands $(a+b)^n$ into a series involving $n+1$ terms.

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k .$$

The result may then be reduced to the form of a binomial once again, in examples such as

$$\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)^9 = 38+17\sqrt{5} \quad \text{and} \quad (2+i)^6 = -117+44i .$$

When the binomial theorem is used to evaluate this last example, first seven terms are calculated, then the four real numbers are added to get -117 and finally the three imaginary numbers are added to get $44i$. In general we write $(a+b)^n = a_n + b_n$, where

$a_n = \sum_{k\ even} \binom{n}{k} a^{n-k} b^k$ and $b_n = \sum_{k\ odd} \binom{n}{k} a^{n-k} b^k$. We are looking for a way to recursively generate the terms of these "a and b sequences". Note that $(a+b)^0 = a_0 + b_0 = 1+0$ and $(a+b)^1 = a_1 + b_1 = a+b$ so $\{a_n\} = \{1, a, a_2, a_3, \cdots\}$ and $\{b_n\} = \{0, b, b_2, b_3, \cdots\}$.

We will describe our method for generating successive terms of the a and b sequences, and then show why it works. Suppose you are given the numbers a and b . Calculate the values $C = 2a$, and $D = b^2 - a^2$. Now successive values of the a and b sequences can be calculated from the recursion relations

(1) $$a_n = C a_{n-1} + D a_{n-2} \quad \text{and} \quad b_n = C b_{n-1} + D b_{n-2} .$$

To justify (1) we first observe that

$$a_n = \tfrac{1}{2}\left((a+b)^n + (a-b)^n\right) \text{ and } b_n = \tfrac{1}{2}\left((a+b)^n - (a-b)^n\right).$$

This is so because $(-b)^{even} = b^{even}$, while $(-b)^{odd} = -b^{odd}$. Now

$$\left(a+b\right)^n = \left(a+b\right)^2\left(a+b\right)^{n-2} = (a^2+2ab+b^2)(a+b)^{n-2} =$$

$$\left(2a^2+2ab+b^2-a^2\right)\left(a+b\right)^{n-2} = \left(2a\left(a+b\right)+\left(b^2-a^2\right)\right)\left(a+b\right)^{n-2} =$$

$$2a\left(a+b\right)^{n-1} + (b^2-a^2)\left(a+b\right)^{n-2}.$$

In the notation of the a and b sequences, it follows that

$$a_n + b_n = 2a\left(a_{n-1}+b_{n-1}\right) + \left(b^2-a^2\right)\left(a_{n-2}+b_{n-2}\right) =$$

$$C\left(a_{n-1}+b_{n-1}\right) + D\left(a_{n-2}+b_{n-2}\right) = (Ca_{n-1}+Da_{n-2}) + (Cb_{n-1}+Db_{n-2}).$$

In exactly the same way,

$$a_n - b_n = (Ca_{n-1}+Da_{n-2}) - (Cb_{n-1}+Db_{n-2}).$$

Solving for a_n a_n and b_n gives (1).

For example, let us evaluate $\left(2+i\right)^6$. Here $a=2$ and $b=i$. So $C=4$ and $D=-5$. Now the recursion relations (1) are

$$-5a_{n-2} + 4a_{n-1} = a_n \text{ and } -5b_{n-2} + 4b_{n-1} = b_n.$$

Since the first two terms of the a sequence are 1 and 2, the next is found to be 3 from the recursion relation. The first two terms of the b sequence are 0 and i, so the next term is $4i$. Continuing in this way we get further terms of the sequences:

$n =$	0	1	2	3	4	5	6
$a_n =$	1	2	3	2	-7	-38	-117
$b_n =$	0	i	$4i$	$11i$	$24i$	$41i$	$44i$

Thus we have all the powers of $2 + i$ up to the sixth calculated in succession and

$$(2+i)^6 = -117 + 44i .$$

Notice that the real and imaginary parts are calculated independently. If we only need the real part of $(2+i)^6$ we can ignore the calculation of the b sequence completely.

For another example we show that $\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)^9 = 38 + 17\sqrt{5}$. Here $a = 1/2$ and $b = \sqrt{5}/2$. ($a + b$ is the golden section.) So $2a = C = 1$ and $b^2 - a^2 = D = 1$. The recursion relations are particularly nice, being the same as for the Fibonacci sequence:

$$a_{n-2} + a_{n-1} = a_n \quad \text{and} \quad b_{n-2} + b_{n-1} = b_n .$$

The first two terms of the a sequence are 1 and 1/2, so the next is 3/2. The first two terms of the b sequence are 0 and $\sqrt{5}/2$ so the next is $\sqrt{5}/2$. The following terms of the sequences are

$n =$	0	1	2	3	4	5	6	7	8	9
$a_n =$	1	1/2	3/2	2	7/2	11/2	9	29/2	47/2	38
$b_n =$	0	$\frac{\sqrt{5}}{2}$	$\frac{\sqrt{5}}{2}$	$\sqrt{5}$	$\frac{3\sqrt{5}}{2}$	$\frac{5\sqrt{5}}{2}$	$4\sqrt{5}$	$\frac{13\sqrt{5}}{2}$	$\frac{21\sqrt{5}}{2}$	$17\sqrt{5}$

We have easily calculated all the powers of the golden section up to the ninth.

The reader may have observed that both of our examples are of the following type: Let F be a field and let $d \in F$ be chosen so that $\sqrt{d} \notin F$. If α and β are elements of F and n is a positive integer, we want to express $(\alpha + \beta\sqrt{d})^n$ in the form $\alpha_n + \beta_n\sqrt{d}$, where α_n and β_n are in F. Our method applies in all such cases, producing the sequences $a_n = \alpha_n$ and $b_n = \beta_n\sqrt{d}$.