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SIAM Review, Volume 18, Issue 2 (Apr., 1976), 240-268.

SIAM REVIEW
Vol. 18, No 2, April 1976

FRACTIONAL DERIVATIVES AND SPECIAL FUNCTIONS*

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Abstract. The fractional derivative operator is an extension of the familiar derivative operator D^n to arbitrary (integer, rational, irrational, or complex) values of n. The most important representations which have been proposed for this concept are reviewed in this paper. In particular, those representations which appear to be of greatest interest for use in exploring the special functions, are presented in detail. A list of selected formulas and theorems on fractional differentiation is presented. Applications to the summation of series and the evaluation of definite integrals incorporating special functions are mentioned.

1. Introduction. The most commonly encountered explicit representations for the special functions of mathematical physics are power series and definite or contour integrals. As an example, the Bessel function is given by the power series

$$J_\nu(z)=\sum_{k=0}^{\infty}\frac{(-1)^k z^{2k+\nu}}{2^{2k+\nu}k!\Gamma(\nu+k+1)},$$

and by several definite integrals, one of which is

$$J_\nu(z)=\frac{z^{-\nu}2^{-\nu+1}}{\sqrt{\pi}\Gamma(\nu+\frac{1}{2})}\int_0^z\frac{\cos t\,dt}{(z^2-t^2)^{-\nu+1/2}}.$$

Less often, the so called "Rodrigues type formulas" are used to define the function by means of repeated differentiation as in

$$J_{-n-1/2}(z)=\frac{z^{n+1/2}2^{n+1/2}}{\sqrt{\pi}}\frac{d^n}{(dz^2)^n}\left(\frac{\cos z}{z}\right).$$

At first glance, this representation for $J_\nu(z)$ with $\nu=-n-\frac{1}{2}$ seems to be valid only when $n=0,1,2,\cdots$. However, by appropriately defining the "fractional derivative of arbitrary order α with respect to z^2," we can generalize $d^n/(dz^2)^n$ to arbitrary (integer, rational, irrational or complex) order α which we denote by $D^\alpha_{z^2}$. This leads to

$$J_{-\alpha-1/2}(z)=\frac{z^{\alpha+1/2}2^{\alpha+1/2}}{\sqrt{\pi}}D^\alpha_{z^2}\frac{\cos z}{z},$$

which is valid for all α.

At times the representation by fractional derivatives is more convenient than that by power series and by definite integrals because the notation itself suggests manipulations which would otherwise not seem obvious. As an example, the

* Received by the editors December 7, 1973, and in revised form March 11, 1975. This work was supported by the National Research Council of Canada under Grant A4027.

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definite integral

$$\int_0^z J_{-\beta-1/2}(t)(z^2-t^2)^{-\alpha-1}t^{-\beta+1/2}\,dt = 2^{-\alpha+1/2}z^{-\alpha-\beta-1/2}J_{-\alpha-\beta-1/2}(z)$$

in fractional derivative notation is nothing more than the obvious relation (see § 19)

$$D_{z^2}^{\alpha}D_{z^2}^{\beta}\frac{\cos z}{z} = D_{z^2}^{\alpha+\beta}\frac{\cos z}{z}.$$

The mathematician has much experience with D^n and fortunately this experience often carries over to D^α. Expressions like $D^{\alpha+\beta}=D^\alpha D^\beta$, the Taylor's series, the Leibniz rule, etc., familiar from the elementary calculus do indeed have counterparts in the fractional calculus which can often be guessed quite easily [10], [17] through [25].

In Part I of this paper we review various representations which have been proposed for fractional differentiation. Those representations which experience has shown are of much interest in describing the special functions are considered in detail. These include the power series (§ 3), the Riemann–Liouville integrals (§ 5), and the Cauchy and Pochhammer type contour integrals (§§ 11, 12 and 13). Those representations which have been of less interest in the study of the special functions are treated lightly. These include the Weyl fractional derivatives (§ 6) and the definitions generated by Fourier and Laplace transforms (§ 7).

The simplest representations are presented first. These include the fractional derivatives of the two functions e^{az} and z^p. The surprising fact that these two representations are inconsistent leads to the conclusion that fractional differentiation, like integration, must be viewed as taking place between lower and upper limits! Where it seems instructive, brief mention of historically related notes is included. Gradually the representations become more involved, until we arrive at the Pochhammer integral in § 13, which incorporates a four loop contour integral in the complex plane. The merits of the various representations are compared with reference to their suitability for studying the special functions. Since the Cauchy and Pochhammer complex contour type integrals have proved to be of great help in formulating rigorous proofs [10] of properties of fractional derivatives, care has been taken to present their features in detail. (Sections 12 and 13 might be a bit tedious, and can be omitted on a first reading.)

An unusual feature of the paper is the inclusion of § 8 on the generalization of the familiar finite difference quotients for fractional derivatives. This method of representing the derivative was given in 1867 by Grunwald, but his approach was not mathematically rigorous. The first rigorous demonstration was given by the Russian A. V. Letnikov in 1868, and his methods are repeated here.

In § 16 we examine the fractional derivatives of elementary functions. In § 17 a list of the fractional derivative representations of selected special functions of both one and two variables appears.

While most of the representations for fractional differentiation which have been of interest to researchers are at least mentioned here, the authors do not claim that every representation is presented. In particular, the representation introduced by Gaer and Rubel [5] (for functions analytic on the extended real line and vanishing at infinity) is not considered because it does not appear to be of wide use in describing the special functions.

In Part II of this paper we present a list of formulas and theorems on fractional differentiation. This list includes: analyticity properties (§ 18), the law of exponents (§ 19), generalizations of the Leibniz rule (§ 21) and the generalized chain rule (§ 22). In § 23 we show how certain infinite series can be summed and definite integrals evaluated with the help of fractional derivative formulas.

Recently, the first book devoted to the study of the fractional calculus was authored [16] by Oldham and Spanier. This paper differs from that book in that it centers interest on the application of fractional derivatives to the study of the special functions. Thus Pochhammer type contour integrals, studied here in detail are not found in the book by Oldham and Spanier.

PART I: VARIOUS REPRESENTATIONS FOR THE FRACTIONAL DERIVATIVE

2. $D^\alpha e^{az}$. The simplest approach to a definition of fractional differentiation begins with the formula

$$\frac{d^n(e^{az})}{dz^n} = a^n e^{az}.$$

If we replace $n = 0, 1, 2, \cdots$, by arbitrary (integral, rational, irrational or complex) α we get

$$\frac{d^\alpha e^{az}}{dz^\alpha} = D^\alpha e^{az} = a^\alpha e^{az}. \tag{2.1}$$

This idea appeared surprisingly early in the history of the calculus. It is mentioned in a letter from G. W. Leibniz to G. F. A. l'Hospital dated September 30, 1695.

It is of great importance to consider the meaning of (2.1) when α is a negative integer. It seems reasonable that D^{-1} should mean $\int dz$ and D^{-2} should mean $\iint dz\, dz$, etc. This is indeed the case, for (2.1) gives

$$\begin{aligned} D^{-1} e^{az} &= a^{-1} e^{az} = \int e^{az}\, dz, \\ D^{-2} e^{az} &= a^{-2} e^{az} = \iint e^{az}\, dz\, dz, \\ &\vdots \qquad\quad \vdots \qquad\qquad \vdots \end{aligned}$$

Between 1832 and 1836 J. Liouville wrote eight extensive memoirs on fractional derivatives based on (2.1). (See the bibliography in [1] for Liouville's papers.) To represent the fractional derivative of an arbitrary function $F(z)$, Liouville assumed that the function could be expanded in a series of exponentials

$$F(z) = \sum_{n=0}^{\infty} c_n e^{a_n z}$$

and then he differentiated termwise to get

$$D^{\alpha}F(z)=\sum_{n=0}^{\infty} c_n a_n^{\alpha} e^{a_n z}.$$

Although (2.1) appears to be a fundamental result, we shall see that it is inconsistent with the representation for fractional differentiation which we shall ultimately find of greatest interest for the study of the special functions.

3. $d_z^{\alpha} z^p$. In 1731, L. E. Euler considered the concept of fractional differentiation when he extended the familiar formula

$$\frac{d^n z^p}{dz^n}=p(p-1)(p-2)\cdots(p-n+1)z^{p-n}=\frac{p!}{(p-n)!}z^{p-n}$$

to $n=\alpha$, where α is as usual arbitrary by writing

$$D_z^{\alpha} z^p=\frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)}z^{p-\alpha}. \tag{3.1}$$

In fact, it was this formula which led Euler to invent the gamma function for fractional values of the factorial: $\Gamma(p+1)=p!$.

Relation (3.1) is of great interest in this paper. Suppose we wish to find the fractional derivative of the function $z^p f(z)$, where $f(z)$ is analytic at $z=0$. Expanding $f(z)$ in a Maclaurin series, it seems reasonable to write

$$\begin{aligned} D_z^{\alpha}(z^p f(z)) &= D_z^{\alpha}\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}z^{p+n} \\ &= \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}D_z^{\alpha} z^{p+n} \\ &= \sum_{n=0}^{\infty}\frac{f^{(n)}(0)\Gamma(p+n+1)}{n!\Gamma(p+n-\alpha+1)}z^{p+n-\alpha}. \end{aligned} \tag{3.2}$$

If the Maclaurin series for $f(z)$ converges for $|z|<R$, then the ratio test shows that the series (3.2) converges for $0<|z|<R$. Thus (3.2) is a suitable representation for $D_z^{\alpha}(z^p f(z))$. Relation (3.2) fails to have meaning in two cases: (i) z outside the circle of convergence, and (ii) p a negative integer. Relation (3.2) is useful for computing fractional derivatives.

4. $D^{\alpha} e^{az}$ not consistent with $D_z^{\alpha} z^p$. Surprisingly, (2.1) and (3.2) are inconsistent. To see this, we note that it seems natural to write

$$D^{\alpha} e^{z}=D_z^{\alpha}\sum_{n=0}^{\infty} z^n/n!.$$

From (2.1) we see that the left-hand side of this expression is e^z, but (3.2) gives for the right-hand side

$$\sum_{n=0}^{\infty} D_z^{\alpha} z^n/n!=\sum_{n=0}^{\infty} z^{n-\alpha}/\Gamma(n-\alpha+1).$$

Now this last expression is e^z only when $\alpha=0, 1, 2, \cdots$.

This inconsistency caused much concern to researchers in the middle nineteenth century. We find A. De Morgan commenting in his "Differential and Integral Calculus" in 1842 on the two different systems of fractional differentiation which evolve from (2.1) and (3.2): "at present I incline (and incline only, in deference to the well-known ability of the supporters of the opposed systems), to the conclusion that neither system has any claim to be considered as giving *the* form of $D^n x^n$, though either may be *a* form." De Morgan's anticipation that (2.1) and (3.2) are two different special cases of a large family of fractional derivatives was indeed correct.

Why should there be two different values of the fractional derivative of e^z? After all, there is only one value for $D^n e^z$ when n is a positive integer. To answer this question, it is first instructive to examine (2.1) and (3.1) for the special values $\alpha = -1, -2, -3, \cdots$. Recall that in this case D^α really means "integration," not differentiation. Integrals are between limits. Let us look at (2.1):

$$D^{-1} e^z = e^z = \int_{z_0}^{z} e^t \, dt.$$

We see that the lower limit z_0 must be $-\infty$ for this to be valid. Next examine (3.1):

$$D_z^{-1} z^p = \frac{z^{p+1}}{(p+1)} = \int_{z_0}^{z} t^p \, dt.$$

We see that we must take $z_0 = 0$ for (3.1) to be valid. Since the "lower limits of differentiation" are $-\infty$ for (2.1), and 0 for (3.1), we are not surprised that they give different results. We shall see later that derivatives of arbitrary order are in general "between limits." Each fractional derivative features a fixed lower limit z_0 and the variable upper limit z. Only when the order of the derivative is a nonnegative integer does the dependence on the lower limit vanish.

To distinguish between different limits of fractional derivatives, many authors have chosen the notation

$${}_{z_0}D_z^\alpha F(z),$$

where the left subscript indicates the "lower limit of differentiation." We shall not write our fractional derivatives this way. Instead, we shall write

$$D_{z-z_0}^\alpha F(z).$$

In this notation (2.1) becomes

$$D_{z+\infty}^\alpha e^{az} = a^\alpha e^{az},$$

and (3.1) becomes

$$D_z^\alpha z^p = \Gamma(p+1) z^{p-\alpha} / \Gamma(p-\alpha+1).$$

We shall see the advantages of this notation later when we define $D_{g(z)}^\alpha F(z)$, for arbitrary functions $g(z)$, as a generalization of the familiar derivative $d^n F(z)/(dg(z))^n$.

5. Riemann–Liouville integral. Another approach to the fractional calculus begins with "Cauchy's iterated integral" [6, p. 295]

$$D_{x-x_0}^{-n}F(x)=\int_{x_0}^{x}\int_{x_0}^{t_n}\cdots\int_{x_0}^{t_3}\int_{x_0}^{t_2}F(t_1)\,dt_1\,dt_2\cdots dt_n=\int_{x_0}^{x}F(t)\frac{(x-t)^{n-1}\,dt}{(n-1)!}.$$

Replacing $-n$ by α we get

$$D_{x-x_0}^{\alpha}F(x)=\frac{1}{\Gamma(-\alpha)}\int_{x_0}^{x}F(t)(x-t)^{-\alpha-1}\,dt,\qquad \operatorname{Re}(\alpha)<0. \tag{5.1}$$

When $x_0=0$, (5.1) is called the Riemann–Liouville integral. Some authors denote (5.1) by the symbols $I^{-\alpha}F(x)$ and ${}_{x_0}I_x^{-\alpha}F(x)$ which are called fractional integrals of order $-\alpha$. Note that (5.1) is undefined unless $\operatorname{Re}(\alpha)<0$. When $x_0=0$, and $F(x)$ is of the form $x^p f(x)$, where $f(x)$ is analytic at $x=0$, then (5.1) and (3.2) are equivalent, provided $\operatorname{Re}(\alpha)<0$. However, (5.1) does not require analyticity of $F(x)$, but only that $F(x)$ be "integrable."

To remove the restriction $\operatorname{Re}(\alpha)<0$, many authors write

$$D_{x-x_0}^{\alpha}F(x)=\frac{d^m}{dx^m}D_{x-x_0}^{\alpha-m}F(x), \tag{5.2}$$

where $m-1\leqq\operatorname{Re}(\alpha)<m$, and $m=1,2,3,\cdots$. Equation (5.2) is a common representation for fractional differentiation used in many research papers. When (5.2) is used, the author is usually only interested in real x, (although this is not necessary), and $F(x)$ is not constrained to be an analytic function of the complex variable x.

6. Weyl fractional derivatives. When the fixed limit of differentiation x_0 takes on the singular values ∞ or $-\infty$ in (5.1) or (5.2) we obtain expressions which are often called "Weyl fractional derivatives." They are

$$D_{x-\infty}^{\alpha}F(x)=\frac{(-1)^{-\alpha}}{\Gamma(-\alpha)}\int_{x}^{\infty}F(t)(t-x)^{-\alpha-1}\,dt$$

(where some suitable branch of $(-1)^{-\alpha}$ must be specified) and

$$D_{x+\infty}^{\alpha}F(x)=\frac{1}{\Gamma(-\alpha)}\int_{-\infty}^{x}F(t)(x-t)^{-\alpha-1}\,dt.$$

The symbol $K^{-\alpha}$ is often used for $D_{x-\infty}^{\alpha}$ as well as ${}_xD_{\infty}^{\alpha}$.

Because the integrals defining these Weyl derivatives are improper, greater restrictions must usually be placed on the function $F(x)$ than are needed when x_0 is finite. Also, general theorems about Weyl derivatives are often more difficult to formulate and prove than are corresponding theorems for Riemann–Liouville derivatives. Because of this difficulty, the authors suspect that the Weyl fractional derivative is an inferior tool for exploring the special functions, and thus it is given little consideration in this paper.

It is, however, worthy of remark that for certain elementary functions $F(x)$, $D_{x-\infty}^{\alpha}F(x)$ is again an elementary function whereas $D_z^{\alpha}F(x)$ is not. In particular we note that

$$D^{\alpha}_{x-\infty}\, e^{ax} = a^{\alpha}\, e^{ax}, \qquad \operatorname{Re}(a)<0$$

whereas

$$D^{\alpha}_{x}\, e^{ax} = \frac{x^{-\alpha}}{\Gamma(1-\alpha)}\, {}_1F_1(1; 1-\alpha; ax)$$

and

$$D^{\alpha}_{x-\infty}(x+a)^{p} = \frac{\Gamma(\alpha-p)}{\Gamma(-p)}(x+a)^{p-\alpha}$$

whereas

$$D^{\alpha}_{x}(x+a)^{p} = \frac{a^{p}x^{-\alpha}}{\Gamma(1-\alpha)}\, {}_2F_1\left(\begin{matrix}1, & -p\\ & 1-\alpha\end{matrix}\,\middle|\, -\frac{x}{a}\right).$$

Thus the Weyl fractional derivative is not always more involved than the Riemann–Liouville derivative.

7. Fourier and Laplace transforms. The familiar representations of a function by means of Fourier and Laplace transforms are respectively

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ivx}\int_{-\infty}^{\infty} e^{-iuv}F(u)\, du\, dv$$

and

$$F(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{sx}\int_{0}^{\infty} e^{-su}F(u)\, du\, ds.$$

Both these representations suggest at once representations for the fractional derivative $D^{\alpha}F(x)$, for if we operate with D^{α} under the integral signs we encounter $D^{\alpha}\, e^{ivx}$ and $D^{\alpha}\, e^{sx}$ which experience from § 2 shows should be replaced by $(iv)^{\alpha}\, e^{ivx}$ and $s^{\alpha}\, e^{sx}$ respectively. We get

$$D^{\alpha}F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} (iv)^{\alpha}\, e^{ivx}\int_{-\infty}^{\infty} e^{-iuv}F(u)\, du\, dv \tag{7.1}$$

and

$$D^{\alpha}F(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} s^{\alpha}\, e^{sx}\int_{0}^{\infty} e^{-su}F(u)\, du\, ds. \tag{7.2}$$

The relation (7.1) was given by Joseph Fourier in 1822 [4, p. 437].

In using (7.1) and (7.2), special attention must be paid to the range of values of α admitted and to the behavior of $F(x)$ so that the improper integrals are defined. In particular, for (7.2) to be valid, $F(x)$ must satisfy the relations

$$D_x^{\alpha-m}F(x)=0,$$
$$\frac{d}{dx}D_x^{\alpha-m}F(x)=0,$$
$$\vdots$$
$$\frac{d^{(m-1)}}{dx^{(m-1)}}D_x^{\alpha-m}F(x)=0,$$

where m is the smallest integer greater than or equal to α. Since (7.1) and (7.2) have not been of great interest in the representation of the special functions, further details regarding their validity will not be considered here.

An important question remains however: "What types of fractional derivatives do (7.1) and (7.2) generate?" In other words, "What is the value of the fixed limit of differentiation x_0, implied by these representations?" The Fourier relation (7.1) is either of the Weyl fractional derivatives $D_{x-\infty}^{\alpha}F(x)$ or $D_{x+\infty}^{\alpha}F(x)$, depending upon which is defined. The Laplace relation (7.2) is the Riemann–Liouville derivative $D_x^{\alpha}F(x)$ with lower limit $x_0=0$.

For further details on (7.1) consult [2], and for a discussion of (7.2) see [32].

8. Grunwald's limit of finite difference quotients. Perhaps the most difficult, yet in some ways the most natural, approach to a representation for fractional differentiation was initiated by Anton Karl Grunwald in 1867 [7]. Grunwald began by writing the familiar finite difference quotients

$$\frac{F(z)-F(z-h)}{h},$$
$$\frac{F(z)-2F(z-h)+F(z-2h)}{h^2},$$
$$\frac{F(z)-3F(z-h)+3F(z-2h)-F(z-3h)}{h^3},$$

the limits of which, as h approaches zero, yield the first, second and third derivatives with respect to z of $F(z)$. Generalizing from these formulas Grunwald wrote

$$D_{z-z_0}^{\alpha}F(z)=\lim_{n\to\infty}h^{-\alpha}\sum_{k=0}^{n}(-1)^k\binom{\alpha}{k}f(z-kh). \tag{8.1}$$

As n approaches infinity, we want h to approach zero in (8.1). To see how these two limits should take place simultaneously, it is instructive to examine (8.1) for the special case in which $\alpha=-1$, i.e., the case where

$$D_{z-z_0}^{-1}F(z)=\int_{z_0}^{z}F(t)\,dt.$$

We get

$$\int_{z_0}^{z} F(t)\,dt = \lim_{\substack{n\to\infty\\ h\to 0}} \sum_{k=0}^{n} F(z-kh)h$$

$$= \int_0^{\lim nh} F(z-t)\,dt = \int_{z-\lim nh}^{z} F(t)\,dt.$$

But $z - \lim nh$ must equal z_0, and we see that $z - z_0 = \lim nh$, which means that we should take

$$h = \frac{z - z_0}{n}. \tag{8.2}$$

It is interesting to note that (8.1) combined with (8.2) is quite free from the restrictions which have been peculiar to the definitions used so far. It does not require that $F(z)$ be analytic, as does (3.2), nor does it require that Re $(\alpha) < 0$ as does the Riemann–Liouville integral (5.1).

Next we show that (8.1) reduces to the Riemann–Liouville integral when Re $(\alpha) < 0$. First note that

$$(-1)^k \binom{\alpha}{K} = k^{-\alpha-1} \frac{(-\alpha)(-\alpha+1)(-\alpha+2)\cdots(-\alpha-1+k)}{k!k^{-\alpha-1}}$$

$$= k^{-\alpha-1} x_k,$$

where we know that

$$\lim_{k\to\infty} x_k = \Gamma(-\alpha)^{-1}$$

[12, vol. 1, p. 9]. Thus

$$D^{\alpha}_{z-z_0} F(z) = \lim_{\substack{n\to\infty\\ h=(z-z_0)/n}} \sum_{k=0}^{n} x_k (hk)^{-\alpha-1} F(z-kh)h. \tag{8.3}$$

Next assume that $F(z)$ is continuous (and therefore bounded) on the closed line segment joining z_0 and z. Since x_k approaches $\Gamma(-\alpha)^{-1}$, it seems likely that we can remove x_k from the sum in (8.3) and get

$$d^{\alpha}_{z-z_0} F(z) = \lim \Gamma(-\alpha)^{-1} \sum_{k=0}^{n} (hk)^{-\alpha-1} F(z-kh)h. \tag{8.4}$$

We shall justify this last manipulation below, but for now, we assume (8.4) is true and get at once, by virtue of (8.2) and the "partial sum of the Riemann integral" (8.4),

$$\begin{aligned} D^{\alpha}_{z-z_0} F(z) &= \Gamma(-\alpha)^{-1} \int_0^{z-z_0} t^{-\alpha-1} F(z-t)\,dt \\ &= \Gamma(-\alpha)^{-1} \int_{z_0}^{z} F(t)(z-t)^{-\alpha-1}\,dt. \end{aligned} \tag{8.5}$$

Thus the sequence of generalized finite difference quotients (8.1) is seen to reduce to the Riemann–Liouville integral when $\mathrm{Re}(\alpha)<0$ provided we can deduce (8.4) from (8.3) rigorously. Grunwald was able to see the above intuitive derivation, but he erred when he attempted the rigorous explanation of how (8.4) follows from (8.3). Only one year after the appearance of Grunwald's paper, the Russian A. V. Letnikov [12] gave the necessary exact derivation which we follow below.

First Letnikov derived the following lemma:

LEMMA. *Suppose $x_k \to X$ and*

(i) $a_{nk} \to 0$ *when* $n \to \infty$,

(ii) $\sum_{k=0}^{n} |a_{nk}| < K$ *(K independent of n),*

(iii) $\sum_{k=0}^{n} a_{nk} = A_n \to A$;

then

$$\lim_{n\to\infty} \sum_{k=0}^{n} x_k a_{nk} = xA.$$

This lemma appears as Theorem 5 on page 75 of [9]. (Note that Knopp [9] credits this lemma to O. Toeplitz in 1911, but it was given by Letnikov 43 years earlier.) In our case, $X=\Gamma(-\alpha)^{-1}$, and $a_{nk}=(hk)^{-\alpha-1}F(z-kh)h$. Condition (i) requires that $\mathrm{Re}(\alpha)<0$, and the boundedness of $F(z)$ assures condition (ii). Condition (iii) follows from the continuity of $F(z)$ and the fact that $A=\int_{z_0}^{z} F(t)(z-t)^{-\alpha-1}\,dt$. Thus it follows that our conclusion (8.5) is justified.

Letnikov was perhaps the most prolific writer on the subject of fractional differentiation. His papers were the first to exhibit a strong attention to details and mathematical rigor.

9. The functions of interest in this paper. In this paper, our interest centers on those functions whose fractional derivatives yield the classical "special functions of mathematical physics." Experience shows that we shall require fractional derivatives of two types of functions, $(z-z_0)^p f(z)$ and $(z-z_0)^p \ln(z-z_0) f(z)$, where $f(z)$ is analytic at z_0, and z_0 is the "lower limit of fractional differentiation." Table 16.1 shows the fractional derivatives of several such functions. The notation used for the special functions is that of Erdélyi et al. [3]. Table 16.1 shows how we can represent the higher transcendental functions by taking fractional derivatives of more elementary functions. For an extensive table of fractional derivatives see [3, vol. 2, pp. 185–200].

10. $D_z^\alpha z^p \ln z f(z)$. We saw in (3.2) that if $f(z)$ is analytic at $z=0$, then $D_z^\alpha(z^p f(z))$ can be computed by expanding $f(z)$ in a Maclaurin series, valid for $|z|<R$, and differentiating fractionally termwise. The resulting series converges on the same circle with $z=0$ removed, $0<|z|<R$.

A similar result is true for $D_z^\alpha(z^p \ln zf(z))$. Using (16.1) we write

$$D_z^\alpha(z^p \ln zf(z)) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} D_z^\alpha z^{p+n} \ln z$$

(10.1)
$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)\Gamma(p+n+1)z^{p+n-\alpha}}{n!\Gamma(p+n-\alpha+1)} \cdot [\ln z + \psi(p+n+1) - \psi(p+n-\alpha+1)],$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$. It is easily shown, using the ratio test, that (10.1) converges for $0 < |z| < R$.

It is also easy to show that (10.1) and the Riemann–Liouville integral ((5.1) with $x_0 = 0$) are equivalent when $\operatorname{Re}(\alpha) < 0$. Writing $z^p \ln zf(z)$ as $z^p \ln z$ times the Maclaurin series for $f(z)$, we simply apply (5.1) termwise to this series.

There is a simple formal verification of the formula for $D_z^\alpha z^p \ln z$ shown in (16.1). Taking the derivative with respect to p of both sides of (3.1), and assuming that d/dp and D_z^α commute we get

$$\frac{d}{dp} D_z^\alpha z^p = D_z^\alpha \frac{dz^p}{dp} = D_z^\alpha z^p \ln z = \frac{d}{dp} \frac{\Gamma(p+1)z^{p-\alpha}}{\Gamma(p-\alpha+1)}$$

$$= \frac{\Gamma'(p+1)z^{p-\alpha}}{\Gamma(p-\alpha+1)} + \frac{\Gamma(p+1)z^{p-\alpha}\ln z}{\Gamma(p-\alpha+1)} - \frac{\Gamma(p+1)\Gamma'(p-\alpha+1)z^{p-\alpha}}{\Gamma(p-\alpha+1)^2}.$$

Using the definition $\psi(x) = \Gamma'(x)/\Gamma(x)$ we see that the above expression reduces to the same result as is in (16.1). A rigorous derivation of this result could be obtained from the Riemann–Liouville integral.

11. Cauchy's integral formula. It is well known that contour integration in the complex plane is a powerful tool in classical analysis. Now we consider representations of fractional differentiation employing contour integration. We begin by generalizing the familiar "Cauchy's integral formula"

(11.1)
$$D^n F(z) = \frac{n!}{2\pi i} \oint F(t)(t-z)^{-n-1}\, dt$$

to arbitrary values of n. Before continuing, let us adopt certain conventions to be used throughout the remainder of this paper.

CONVENTIONS 11.1. (i) $\mathscr{R}$ is an open, simply connected set in the complex plane containing the origin.

(ii) $f(z)$ is an analytic function for $z \in \mathscr{R}$.

(iii) The notations

$$\int_{C(z_0,z^+)} g(t)\, dt = \int_{C(z_0,z^+;g_1,g_2)} g(t)\, dt$$

denote integrals over closed contours which start at $t = z_0$, where the integrand takes on the value $g(z_0) = g_1$, encircle $t = z$ once in the positive sense, and return to $t = z_0$ where now the integrand assumes the value $g(z_0) = g_2$. We assume that the contour remains in the region $\mathscr{R}$ and that the integrand $g(t)$ varies "continuously" as we traverse the contour.

(iv) The integrand will contain multiple-valued factors such as t^p, $\ln t$, (or $(t-z)^q$), etc. The branch cut for these functions always passes through the beginning and ending point of the contour of integration, but never cuts the contour otherwise. Unless otherwise stated, these functions denote the *principal branch*, which is that continuous range of the function for which $\arg t$ (or $\arg(t-z)$) is zero when t (or $t-z$) is real and positive. In the event that the branch line is $\arg t = 0$ (or $\arg(t-z)=0$), then we define the principal branch by $-2\pi < \arg t$ (or $\arg(t-z)) \leqq 0$.

Returning to (11.1), we consider the consequences of replacing n by arbitrary α. Naturally, $n!$ will be replaced by $\Gamma(\alpha+1)$, but replacing $(t-z)^{-n-1}$ by $(t-z)^{\alpha-1}$ is more involved. $(t-z)^{-n-1}$ has an "isolated singularity" at $t=z$, while $(t-z)^{-\alpha-1}$ has a "branch point" at $t=z$. Let the branch line for $(t-z)^{-\alpha-1}$ start at $t=z$ and pass through the fixed point $t=z_0$. Let the contour of integration be $C(z_0, z^+)$. Notice that the value of the integral now depends on the position of the starting and ending point z_0 of the contour. We shall see that z_0 is the "lower limit of differentiation," and thus we write

$$D^{\alpha}_{z-z_0}F(z) = \frac{\Gamma(\alpha+1)}{2\pi i}\int_{C(z_0,z^+)} F(t)(t-z)^{-\alpha-1}\,dt. \tag{11.2}$$

Next we show that (11.2) is equivalent to (5.1) when $\operatorname{Re}(\alpha) < 0$. First, deform $C(z_0, z^+)$ into the union of three contours $C(z_0, z^+) = C_1 \cup C_2 \cup C_3$, where C_1 is a straight line segment from z_0 almost to z, C_2 is a small circle centered at $t=z$, C_3 is C_1 traversed in the opposite direction. See Fig. 11.1.

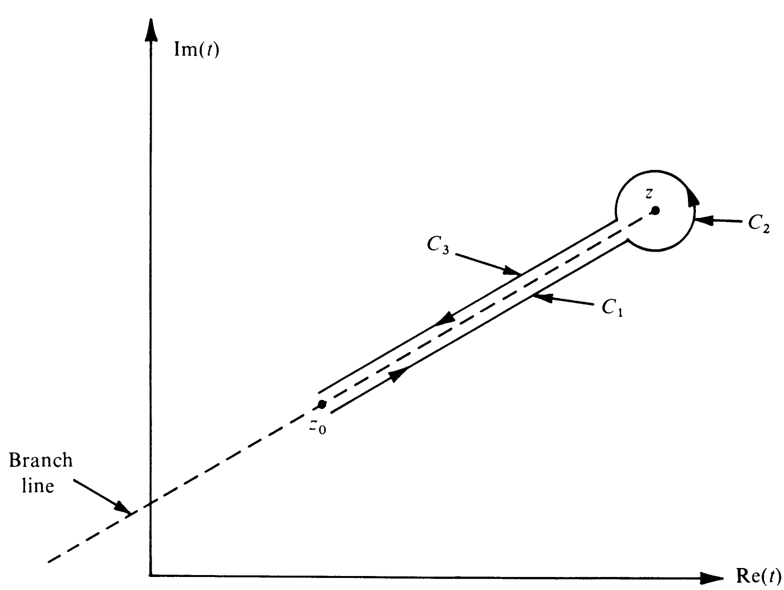

FIG. 11.1. *Contour of integration used with Cauchy integral* (11.2)

Thus

$$\int_{C(z_0,z^+)} F(t)(t-z)^{-\alpha-1}\,dt = \int_{C_1} + \int_{C_2} + \int_{C_3}.$$

On C_1, $(t-z)^{-\alpha-1} = \exp[(-\alpha-1)(\ln|t-z| + i(\arg(z-t)-\pi))]$. On C_3, $(t-z)^{-\alpha-1} = \exp[(-\alpha-1)(\ln|t-z| + i(\arg(z-t)+\pi))]$. If Re $(\alpha) < 0$, the integral over C_2 approaches zero as the radius of the contour tends to zero. Thus we get

$$\int_{C(z_0,z^+)} F(t)(t-z)^{-\alpha-1}\,dt = (-e^{i\pi\alpha} + e^{-i\pi\alpha}) \int_{z_0}^{z} F(t)(z-t)^{-\alpha-1}\,dt.$$

Combining this last result with (11.2) we get

$$D^{\alpha}_{z-z_0}F(t) = -\frac{(e^{\pi\alpha i} - e^{-\pi\alpha i})\Gamma(\alpha+1)}{2\pi i} \int_{z_0}^{z} F(t)(z-t)^{-\alpha-1}\,dt.$$

Since $[(e^{\pi\alpha i} - e^{-\pi\alpha i})/2i]\Gamma(\alpha+1) = \sin(\pi\alpha)\Gamma(\alpha+1) = -\pi/\Gamma(-\alpha)$, we see at once that (11.2) is equivalent to (5.1) when Re $(\alpha) < 0$.

Notice that (11.2) is defined for all values of α. The apparent singularities at $\alpha = -1, -2, -3, \cdots$ due to the factor $\Gamma(\alpha+1)$ are removable since in this case the integrand is analytic, and thus the integral is zero.

This form (11.2) of a contour integral representation for the fractional derivative was used previously in [18] through [25] to study derivatives of the form $D^{\alpha}_{z}z^{p}f(z)$. In this case we have

$$D^{\alpha}_{z}z^{p}f(z) = \frac{\Gamma(\alpha+1)}{2\pi i} \int_{C(0,z^+)} t^{p}f(t)(t-z)^{-\alpha-1}\,dt. \tag{11.3}$$

Since the contour of integration defining $D^{\alpha}_{z}z^{p}f(z)$ passes through the origin, we must take Re $(p) > -1$ so that the integral will converge. This is a serious defect. We will attempt other contour integral representations below in the hope of removing this restriction on p.

A representation for fractional differentiation like (11.2) was employed as early as 1888 by Nekrassov [14].

12. Another generalized Cauchy integral.[1] The representation for $D^{\alpha}_{z}z^{p}f(z)$ given by (11.3) restricts the parameter p to the right half-plane Re $(p) > -1$, while it is quite generous with the parameter α, since it has only removable singularities at $\alpha = -1, -2, -3, \cdots$. Can we write another contour integral for $D^{\alpha}_{z}z^{p}f(z)$ which gives its analytic continuation with respect to the parameter p into the remainder of the complex p-plane? The answer is yes, provided we restrict p such that $p \neq -1, -2, -3, \cdots$. The result is

$$D^{\alpha}_{z}z^{p}f(z) = \frac{e^{\pi p i}\csc \pi p}{2i\Gamma(-\alpha)} \int_{C(z,0^+)} t^{p}f(t)(z-t)^{-\alpha-1}\,dt. \tag{12.1}$$

Notice that in (12.1) the contour of integration does not pass through $t=0$, and thus the integral gives no restriction on p. However, the factor csc πp implies that p may not be an integer. It can be shown that the apparent singularities at $p = 0, 1, 2, \cdots$, are removable, while those at $p = -1, -2, -3, \cdots$, are in general

[1] This section can be omitted on a first reading.

simple poles (but for certain special values of α are removable singular points). While (12.1), when compared to (11.3), has improved the range of validity of $D_z^\alpha z^p f(z)$ with regard to the parameter p, it has made the range of validity narrower with regard to α. Now we must take $\operatorname{Re}(\alpha)<0$ since $C(z, 0^+)$ passes through $t=z$.

The demonstration of the validity of (12.1) is similar to that of (11.3). Deform $C(z, 0^+)$ into $C_1 \cup C_2 \cup C_3$ as shown in Fig. 12.1. Now

$$\int_{C(z,0^+)} = \int_{C_1} + \int_{C_2} + \int_{C_3}.$$

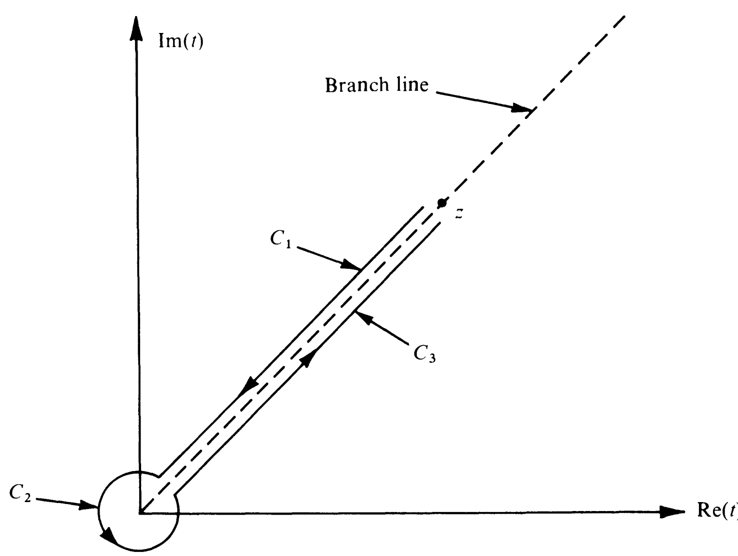

FIG. 12.1. *Contour of integration used with Cauchy integral* (12.1)

We must take care that the correct branch of the function t^p is used. By Conventions 11.1, t^p denotes the "principal value" of this multiple-valued function. (This means that with $\arg t$ as shown in Fig. 12.1, we have $0 \leqq \arg t < \pi/2$.) On C_1 we write $t^p = e^{p(\ln|t|+i(\arg t - 2\pi i))}$, and on C_3 we write $t^p = e^{p(\ln|t|+i(\arg t))}$. Let the radius of C_2 approach zero and note that if we restrict p such that $\operatorname{Re}(p) > -1$, then the integral over C_2 will vanish. Thus we get

$$(12.2) \qquad \int_{C(z,0^+)} t^p f(t)(z-t)^{-\alpha-1}\,dt = (-e^{-2\pi p i}+1)\int_0^z t^p f(t)(z-t)^{-\alpha-1}\,dt.$$

Multiplying (12.2) by $e^{\pi p i} \csc \pi p/[2i\Gamma(-\alpha)]$ we see that the right-hand side of (12.1) is the Riemann–Liouville integral (5.1) for $D_z^\alpha z^p f(z)$.

13. Pochhammer contour integrals.[2] We have given two single loop contour integral representations for $D_z^\alpha z^p f(z)$. The first, (11.3), could not be used if

[2] This section can be omitted on a first reading.

$\operatorname{Re}(p) \leqq -1$, and the second, (12.1), could not be used if $\operatorname{Re}(\alpha) \geqq 0$. Now we present a contour integral for $D_z^\alpha z^p f(z)$ which has no "half-plane" restrictions. The price we must pay for this greater generality with respect to the parameters α and p is that we will employ a more complicated contour of integration. It is the opinion of the authors that the Pochhammer representation presented here is the most efficient tool for proving general formulas and properties for fractional derivatives. The use of other representations often requires consideration of first this case and then that case, while the Pochhammer representation completes the proof in one stroke.

Recall from § 9 that our interest centers on finding fractional derivatives of $z^p(\ln z)^\delta f(z)$, where δ is either 0 or 1. We begin by examining

$$\int_P F(t)\,dt = \int_P t^p(\ln t)^\delta f(t)(t-z)^{-\alpha-1}\,dt, \tag{13.1}$$

where the contour P is called the "Pochhammer contour," and is given by $P = C_1 \cup C_2 \cup C_3 \cup C_4$. Thus

$$\int_P = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}.$$

Figure 13.1 shows these four components of P, and also shows how the two branch lines of the integrand of (13.1) both pass through the point $t = a$ without crossing P at any other point.

Let $F(a) = a^p(\ln a)^\delta f(a)(a-z)^{-\alpha-1}$ (where $\delta = 0$ or 1) denote the value of the integrand of (13.1) when we begin to traverse P, i.e., $F(a)$ is the principal value as defined in Conventions 11.1. Then, using the notation for contours adopted in Conventions 11.1 we have

$$C_1 = C(a, z^+; F(a), F(a)\,e^{-2\pi i\alpha}),$$

$$C_2 = C\left(a, 0^+; F(a)\,e^{-2\pi i\alpha}, F(a)\left(1+\frac{2\pi i\delta}{\ln a}\right)e^{2\pi i(p-\alpha)}\right),$$

$$C_3 = C\left(a, z^-; F(a)\left(1+\frac{2\pi i\delta}{\ln a}\right)e^{2\pi i(p-\alpha)}, F(a)\left(1+\frac{2\pi i\delta}{\ln a}\right)e^{2\pi ip}\right),$$

$$C_4 = C\left(a, 0^-; F(a)\left(1+\frac{2\pi i\delta}{\ln a}\right)e^{2\pi ip}; F(a)\right).$$

Notice that after completely traversing all four components of P, the integrand of (13.1) returns to the value of $F(a)$ with which it started.

Next we rewrite the integral over C_3 in terms of the integral over C_1.

$$\int_{C_3} F(t)\,dt = -\int_{C(a,z^+;F(a)(1+2\pi i\delta/\ln a)\,e^{2\pi ip};F(a)(1+2\pi i\delta/\ln a)\,e^{2\pi i(p-\alpha)})} f(t)\,dt$$

$$= -e^{2\pi ip}\int_{C(a,z^+;F(a)(1+2\pi i\delta/\ln a),F(a)(1+2\pi i\delta/\ln a)\,e^{-2\pi i\alpha})} F(t)\,dt.$$

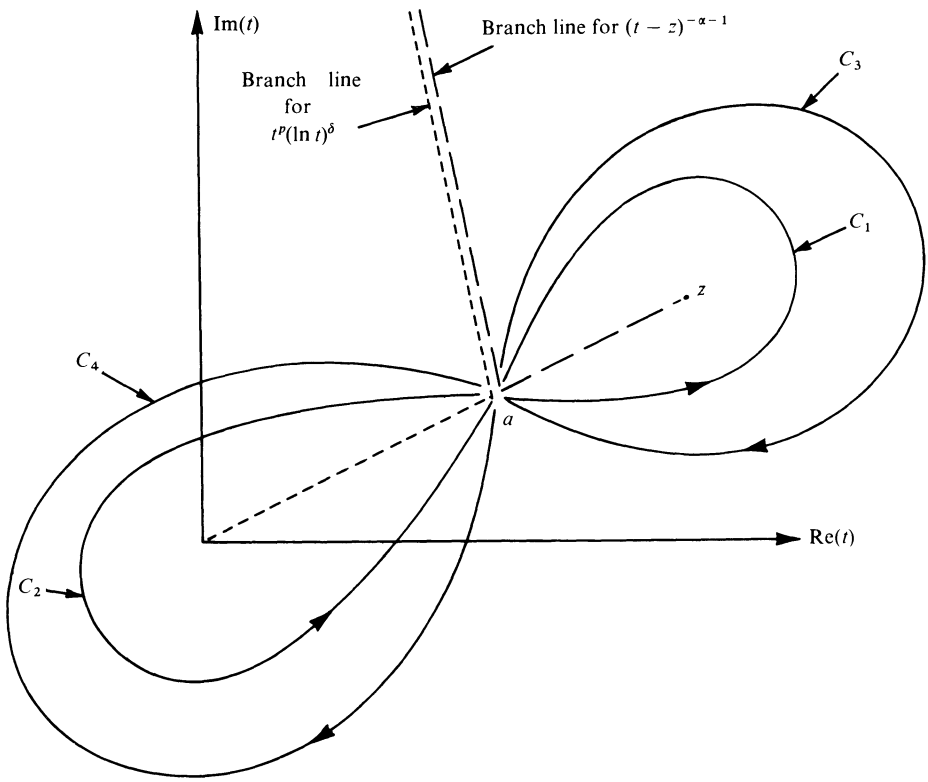

FIG. 13.1. *The four components of the Pochhammer contour*

Now if $F(t)$ denotes the "principal value," then $F(t)(1+2\pi\delta i/\ln t) = F(t)+2\pi\delta i t^p f(t)(t-z)^{-\alpha-1}$ and the above integral becomes

$$(13.2)\qquad = -e^{2\pi ip}\int_{C_1} F(t)\,dt - 1\pi i\delta\, e^{2\pi ip}\int_{C(a,z^+)} t^p f(t)(t-z)^{-\alpha-1}\,dt.$$

We next rewrite the integral over C_4 in terms of the integral over C_2.

$$\int_{C_4} F(t)\,dt = -\int_{C(a,0^+;F(a),F(a)(1+2\pi i\delta/\ln a)\,e^{2\pi ip})} F(t)\,dt$$

$$(13.3)\qquad = -e^{2\pi i\alpha}\int_{C(a,0^+;F(a)\,e^{-2\pi i\alpha},F(a)(1+2\pi i\delta/\ln a)\,e^{2\pi i(p-\alpha)})} F(t)\,dt$$

$$= -e^{2\pi i\alpha}\int_{C_2} F(t)\,dt.$$

Combining (13.1), (13.2) and (13.3) we get

$$(13.4)\quad \begin{aligned}\int_P F(t)\,dt &= (1-e^{2\pi ip})\int_{C_1} F(t)\,dt \\ &+(1-e^{2\pi i\alpha})\int_{C_2} F(t)\,dt - 2\pi i\delta\, e^{2\pi ip}\int_{C(a,z^+)} t^p f(t)(t-z)^{-\alpha-1}\,dt.\end{aligned}$$

Next we try to identify (13.4) with the fractional derivative. If we take $\operatorname{Re}(p)>-1$, we can let a approach zero and then the integral over C_2 above will vanish. Thus we get

$$(13.5)\quad \int_P F(t)\,dt = (1-e^{2\pi ip})\int_{C(0,z^+)} F(t)\,dt - 2\pi i\delta\, e^{2\pi ip}\int_{C(0,z^+)} t^p f(t)(t-z)^{-\alpha-1}\,dt.$$

If we set $\delta=0$ in (13.5), we get

$$\int_{C(0,z^+)} t^p f(t)(t-z)^{-\alpha-1}\,dt = \frac{1}{(1-e^{2\pi ip})}\int_P t^p f(t)(t-z)^{-\alpha-1}\,dt.$$

Multiplying both sides by $\Gamma(\alpha+1)/2\pi i$ and comparing the result with (11.3) we get

REPRESENTATION 13.1. Using the notation adopted in Conventions 11.1, we have for α not a negative integer, p not an integer, and z on $\mathcal{R}-\{0\}$,

$$(13.6)\quad D_z^\alpha z^p f(z) = \frac{\Gamma(\alpha+1)\,e^{-\pi ip}}{4\pi\sin\pi p}\int_P t^p f(t)(t-z)^{-\alpha-1}\,dt.$$

If we set $\delta=1$ in (13.5) we get

$$\int_{C(0,z^+)} F(t)\,dt = \frac{1}{(1-e^{2\pi ip})}\int_P F(t)\,dt + \frac{2\pi i\,e^{2\pi ip}}{1-e^{2\pi ip}}\int_{C(0,z^+)} t^p f(t)(t-z)^{-\alpha-1}\,dt.$$

Multiplying both sides of the above expression by $\Gamma(\alpha+1)/2\pi i$ and comparing the result with (11.2) we get

$$\begin{aligned}D_z^\alpha z^p \ln z f(z) &= \frac{e^{-\pi ip}\Gamma(\alpha+1)}{4\pi\sin\pi p}\int_P t^p \ln t f(t)(t-z)^{-\alpha-1}\,dt \\ &-\frac{\pi\,e^{\pi ip}}{\sin\pi p} D_z^\alpha z^p f(z).\end{aligned}$$

Comparing this last result with (13.6) we get the following representation.

REPRESENTATION 13.2. With the notation adopted in Conventions 11.1, we have for α not a negative integer, p not an integer, and z on $\mathcal{R}-\{o\}$,

$$(13.7)\quad \begin{aligned}D_z^\alpha z^p \ln z f(z) &= \frac{e^{-\pi ip}\Gamma(\alpha+1)}{4\pi\sin\pi p}\int_P t^p \ln t f(t)(t-z)^{-\alpha-1}\,dt \\ &-\frac{\Gamma(\alpha+1)}{4\sin^2\pi p}\int_P t^p f(t)(t-z)^{-\alpha-1}\,dt.\end{aligned}$$

The Pochhammer contour integrals (13.6) and (13.7) appear to provide us with representations for the fractional derivative which give the widest possible range of values to z, α and p.

OBSERVATION 13.1. In Representations 13.1 and 13.2, we required $f(z)$ to be analytic at $z=0$. It is interesting to note here that we could also allow $f(z)$ to have an essential singularity at $z=0$, and still relations (13.6) and (13.7) would have meaning. This extension follows since the terms involving negative powers of z in the expansion of $f(z)$ do not affect the multivalued nature of the integrands of (13.6) and (13.7).

Multiloop contour integrals, like those studied above, have a long history. The first reference to them seems to be Riemann's [29] in 1857. They were studied by Jordan [8] in 1887 in his work on linear differential equations, and by P. A. Nekrassov [15] in 1891. L. Pochhammer published a few papers on this type of contour [26], [27], [28] and that may be the reason why they are named after him.

14. Derivatives of composite functions. Next we consider replacing the order "n" in the familiar "derivative with respect to $g(z)$," $d^nF(z)/(dg(z))^n$, by arbitrary "α" and denote it by $D^\alpha_{g(z)}F(z)$. To define $D^\alpha_{g(z)}F(z)$, we make the natural substitutions $g(z)=w$, $F(z)=F(g^{-1}(w))$, and write

$$D^\alpha_{g(z)}F(z)=D^\alpha_wF(g^{-1}(w)), \tag{14.1}$$

where the right-hand side of (14.1) was defined previously. In order to insure that $g^{-1}(w)$ is well-defined, we assume that $g(z)$ is analytic and univalent over a region of the z-plane which includes the point $z=g^{-1}(0)$ as an interior or boundary point. We can now rewrite integrals like (11.2) to get representations of the form

$$D^\alpha_{g(z)}F(z)=\frac{\Gamma(\alpha+1)}{2\pi i}\int_{C(g^{-1}(0),z^+)}F(t)(g(t)-g(z))^{-\alpha-1}g'(t)\,dt.$$

Note in particular, that for the special case $g(z)=z-z_0$, we get

$$D^\alpha_{z-z_0}F(z)=\frac{\Gamma(\alpha+1)}{2\pi i}\int_{C(z_0,z^+)}F(t)(t-z)^{-\alpha-1}\,dt.$$

Because the above formula is a natural outgrowth of our definition of differentiation with respect to $g(z)$, we adopted the notation $D^\alpha_{z-z_0}$ in preference to ${}_{z_0}D^\alpha_z$ in § 4. For ordinary derivatives, $D^n_z=D^n_{z-z_0}$, but this is not the case for fractional derivatives. Derivatives with respect to $g(z)$ were considered in [17], [18], [21], [23], [24].

15. Final remarks on the representations of D^α. Throughout this paper, we generated several different representations for $D^\alpha_z z^p f(z)$ and $D^\alpha_z z^p \ln zf(z)$. In particular, we were interested in obtaining representations which would explicitly define the fractional derivative for the widest range of the three variables z, α and p. The following Table 15.1 summarizes our results. Recall that $\mathcal{R}$ denotes the region of analyticity of $f(z)$ (which contains the origin), and that R denotes the radius of convergence of the Maclaurin series for $f(z)$.

For further information on the history of fractional derivatives and other generalized derivatives see [1], [30].

TABLE 15.1
Range of validity of various representations for the fractional derivative

Representation for $D_z^\alpha z^p f(z)$ and $D_z^\alpha z^p \ln z f(z)$	Restrictions on the variables z	α	p
1. Power series (3.2) and (10.1)	$0<\|z\|<R$	no restriction	$p \neq$ negative integer
2. Riemann–Liouville integral (5.1)	$z \in \mathcal{R}-\{o\}$	$\mathrm{Re}(\alpha)<0$	$\mathrm{Re}(p)>-1$
3. Cauchy integrals (11.2) and (11.3)	$z \in \mathcal{R}-\{0\}$	$\alpha \neq$ negative integer	$\mathrm{Re}(p)>-1$
4. Cauchy integral (12.1)	$z \in \mathcal{R}-\{o\}$	$\mathrm{Re}(\alpha)<0$	$p \neq$ integer
5. Pochhammer integrals (13.6) and (13.7)	$z \in \mathcal{R}-\{0\}$	$\alpha \neq$ negative integer	$p \neq$ integer

16. Fractional derivatives of elementary functions. To find the fractional derivative of an elementary function, one of the simplest techniques is to expand the function in a power series and differentiate fractionally termwise. For example, to find $D_z^\alpha(z+a)^P$ we first write

$$(z+a)^P = a^P(1+z/a)^P$$
$$= a^P \sum_{n=0}^{\infty} \frac{(-P)_n}{n!}\left(\frac{-z}{a}\right)^n,$$

where $(-P)_n = (-P)(-P+1)(-P+2)\cdots(-P+n-1)$. We now differentiate termwise and use

$$D_z^\alpha z^n = \frac{n!z^{n-\alpha}}{\Gamma(n-\alpha+1)}$$
$$= \frac{n!z^{n-\alpha}}{\Gamma(1-\alpha)(1-\alpha)_n}$$

to get

$$D_z^\alpha(z+a)^P = a^P \sum_{n=0}^{\infty} \frac{(-P_n(-1)^n}{n!a^n} D_z^\alpha z^n$$

$$= \frac{a^P z^{-\alpha}}{\Gamma(1-\alpha)} \sum_{n=0}^{\infty} \frac{(-P)_n(1)_n}{(1-\alpha)_n n!} \left(\frac{-z}{a}\right)^n.$$

We now recognize this last series as a hypergeometric series of the form

$${}_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n x^n}{(c)_n n!}$$

so that we get

$$D_z^\alpha(z+a)^P = \frac{a^P z^{-\alpha}}{\Gamma(1-\alpha)} {}_2F_1(1, -p; 1-\alpha; -z/a).$$

Other examples include

$$D_z^\alpha e^{az} = \frac{z^{-\alpha}}{\Gamma(1-\alpha)} {}_1F_1(1; 1-\alpha; az)$$

and

(16.1) $$D_z^\alpha z^P \ln z = \frac{\Gamma(p+1)z^{p-\alpha}}{\Gamma(p-\alpha+1)}[\ln z + \psi(p+1) - \psi(p-\alpha+1)],$$

$$p \neq -1, -2, -3, \cdots.$$

A table of fractional derivatives of elementary and higher transcendental functions is found in [3, vol. 2, pp. 185–200]. It is interesting to note that while the ordinary derivative of an elementary function is again an elementary function, the fractional derivative of an elementary function usually yields a higher transcendental function.

17. The special functions. Table 17.1 lists the fractional derivative representations of several of the special functions of one variable. These are generalized "Rodrigues type" formulas.

It is also possible to represent special functions of several variables by means of fractional differentiation. In this case, it is sometimes necessary to employ fractional partial differentiation which is quite easily explained by

$$D_{z,w}^{\alpha,\beta} = D_z^\alpha D_w^\beta.$$

The following list illustrates these representations for functions of several variables. The notations used for the special functions are those of Erdélyi et al. [3].

The Appell functions.

$$F_1(\alpha,\beta,\beta';\gamma;xt,yt)=\frac{\Gamma(\gamma)}{\Gamma(\alpha)}t^{1-\gamma}D_t^{\alpha-\gamma}t^{\alpha-1}(1-xt)^{-\beta}(1-yt)^{-\beta'},$$

$$F_2(\alpha,\beta,\beta';\gamma,\gamma';x,y)=\frac{\Gamma(\gamma)\Gamma(\gamma')}{\Gamma(\beta)\Gamma(\beta')}x^{1-\gamma}y^{1-\gamma'}D_{x,y}^{\beta-\gamma,\beta'-\gamma'}x^{\beta-1}y^{\beta'-1}(1-x-y)^{-\alpha},$$

$$F_3(\alpha,\alpha',\beta,\beta';\gamma;x,y)=\frac{\Gamma(\gamma-\alpha')\Gamma(\gamma-\beta')}{\Gamma(\alpha)\Gamma(\beta)}x^{1+\alpha'-\gamma}D_x^{\alpha+\alpha'-\gamma}x^{\alpha+\beta'-\gamma}D_x^{\beta+\beta'-\gamma}$$

$$\cdot\left\{x^{\beta-1}(1-x)^{\alpha'+\beta'-\gamma}{}_2F_1\left(\begin{matrix}\alpha',\beta'\\ \gamma\end{matrix}\middle|x+y-xy\right)\right\}$$

$$=\frac{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}{\Gamma(\alpha')\Gamma(\beta')}y^{1+\alpha-\gamma}D_y^{\alpha+\alpha'-\gamma}y^{\alpha'+\beta-\gamma}D_y^{\beta+\beta'-\gamma}$$

$$\cdot\left\{y^{\beta'-1}(1-y)^{\alpha+\beta-\gamma}{}_2F_1\left(\begin{matrix}\alpha,\beta\\ \gamma\end{matrix}\middle|x+y-xy\right)\right\},$$

$$F_4(\alpha,\beta;\gamma,\gamma';x,y)=\frac{\Gamma(\gamma)\Gamma(\gamma')}{\Gamma(\beta)\Gamma(\beta)}\ x^{1-\gamma}y^{1-\gamma'}D_{x,y}^{\beta-\gamma,\beta-\gamma'}(xy)^{\beta-1}(1-x-y)^{-\alpha}$$

$$\cdot\,{}_2F_1\left(\begin{matrix}\frac{\alpha}{2},\frac{\alpha+1}{2}\\ \beta\end{matrix}\middle|\frac{4xy}{(1-x-y)^2}\right),$$

$$=\frac{\Gamma(\gamma)\Gamma(\gamma')}{\Gamma(\alpha)\Gamma(\alpha)}y^{1-\gamma'}x^{1-\gamma}D_{x,y}^{\alpha-\gamma,\alpha-\gamma'}(xy)^{\alpha-1}(1-x-y)^{-\beta}$$

$$\cdot\,{}_2F_1\left(\begin{matrix}\frac{\beta}{2},\frac{\beta+1}{2}\\ \alpha\end{matrix}\middle|\frac{4xy}{(1-x-y)^2}\right).$$

The confluent functions of Humbert.

$$\phi_1(\alpha,\beta;\gamma;xt,yt)=\frac{\Gamma(\gamma)}{\Gamma(\alpha)}t^{1-\gamma}D_t^{\alpha-\gamma}t^{\alpha-1}(1-xt)^{-\beta}\,e^{yt},$$

$$\phi_2(\alpha,\beta;\gamma;xt,yt)=\frac{\Gamma(\gamma-\alpha)}{\Gamma(\beta)}y^{1+\alpha-\gamma}D_y^{\beta+\alpha-\gamma}y^{\beta-1}\,e^{yt}{}_1F_1\left(\begin{matrix}\alpha\\ \gamma\end{matrix}\middle|t(x-y)\right)$$

$$=\frac{\Gamma(\gamma)}{\Gamma(\alpha+\beta)}t^{1-\gamma}D_t^{\alpha+\beta-\gamma}t^{\alpha+\beta-1}\,e^{yt}{}_1F_1\left(\begin{matrix}\alpha\\ \alpha+\beta\end{matrix}\middle|t(x-y)\right)$$

$$\psi_1(\alpha,\beta;\gamma,\gamma';x,y)=\frac{\Gamma(\gamma)\Gamma(\gamma')}{\Gamma(\alpha)\Gamma(\beta)}x^{1-\gamma}y^{1-\gamma'}D_{x,y}^{\beta-\gamma,\alpha-\gamma'}x^{\beta-1}\,y^{\alpha-1}(1-x)^{-\alpha}\,e^{y/(1-x)},$$

$$\psi_2(\alpha,\beta;\gamma;x,y)=\frac{\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\alpha)}x^{1-\beta}y^{1-\gamma}D_{x,y}^{\alpha-\beta,\alpha-\gamma}(xy)^{\alpha-1}\,e^{x+y}{}_0F_1\left(\begin{matrix}-\\ \alpha\end{matrix}\middle|xy\right).$$

TABLE 17.1
Special functions expressed as fractional derivatives

Name	Derivative Representation
Hypergeometric function	${}_2F_1(a, b; c; z) = \frac{\Gamma(c)z^{1-c}}{\Gamma(b)} D_z^{b-c}[z^{b-1}(1-z)^{-a}]$
Confluent hypergeometric function	${}_1F_1(a; c; z) = \frac{\Gamma(c)z^{1-c}}{\Gamma(a)} D_z^{a-c}[e^z z^{a-1}]$
Generalized hypergeometric function	${}_{p+1}F_{q+1}\left[\begin{matrix} a_1, \cdots, a_p, c \\ b_1, \cdots, b_q, d \end{matrix}\middle| z\right] = \frac{\Gamma(d)z^{1-d}}{\Gamma(c)} \cdot D_z^{c-d}\left\{z^{c-1}{}_pF_q\left[\begin{matrix} a_1, \cdots, a_p \\ b_1, \cdots, b_q \end{matrix}\middle| z\right]\right\}$
Bessel function	$J_v(z) = \pi^{-1/2}2^{1-v}z^{-v}D_{z^2}^{-v+(1/2)} \sin z$
Legendre function	$P_v(z) = \frac{1}{\Gamma(v+1)2^v} D_{1-z}^v(1-z^2)^v$
Psi function	$\psi(x) = -\gamma + \ln z - \Gamma(x)z^{1-x}D_z^{1-x} \ln z$
Incomplete gamma function	$\gamma(a, z) = \Gamma(a)\, e^{-z}D_z^{-a}\, e^z$

PART II: SUMMARY OF FORMULAS AND THEOREMS ON FRACTIONAL DERIVATIVES

The various representations for the fractional derivative given in the first part of this paper have been investigated with an eye toward ultimately obtaining useful new information concerning the special functions. In particular, certain infinite series and definite integrals incorporating special functions have been determined in closed form with the aid of fractional derivatives. Now we list several formulas and theorems which have been used by the authors to obtain these series and integrals. No proofs or derivations are given, as these are referred to in the literature.

18. Analyticity properties of fractional derivatives. The functions $D_z^\alpha z^p f(z)$ and $D_z^\alpha z^p \ln zf(z)$ can both be viewed as functions of the three complex variables z, α and p. Holding any two of these variables fixed, the fractional derivative is then an analytic function of the remaining variable. What are its analyticity properties? Is it entire, or meromorphic? Where are the singularities? These questions are answered by the following two theorems which are discussed in full in [10].

THEOREM 18.1. *Let $f(z)$ be analytic on the simply connected open set $\mathcal{R}$ which contains the point $z = 0$. Also let $f(0) \neq 0$.*

(i) *If $z \in \mathcal{R} - \{0\}$ and $p \neq -1, -2, -3, \cdots$, then $D_z^\alpha z^p f(z)$ is an entire function of α (with z and p held fixed).*

(ii) *if* $z \in \mathcal{R}-\{0\}$, *and* $\alpha = 0, 1, 2, \cdots$, *then* $D_z^\alpha z^p f(z)$ *is an entire function of* p (*with* z *and* α *fixed*). *When* $\alpha \neq 0, 1, 2, \cdots$, *and* $z \in \mathcal{R}-\{0\}$ *then* $D_z^\alpha z^p f(z)$ *is a meromorphic function of* p *whose only singularities are simple poles at the points* $p = -1, -2, -3, \cdots$, *or a subset thereof.*

(iii) *If* $p \neq -1, -2, -3, \cdots$, *then* $D_z^\alpha z^p f(z) = z^{p-\alpha} g(\alpha, p; z)$, *where* $g(\alpha, p; z)$ *is an analytic function of* z *on* $\mathcal{R}$.

THEOREM 18.2. *Let* $f(z)$ *be an analytic function for* z *on the simply connected open set* $\mathcal{R}$, *which contains the point* $z = 0$. *Also* $f(0) \neq 0$.

(i) *If* $z \in \mathcal{R}-\{0\}$ *and* $p \neq -1, -2, -3, \cdots$, *then* $D_z^\alpha z^p \ln(z) f(z)$ *is an entire function of* α (*with* z *and* p *fixed*).

(ii) *If* $z \in \mathcal{R}-\{0\}$ *and* $\alpha = 0, 1, 2, \cdots$, *then* $D_z^\alpha z^p \ln(z) f(z)$ *is an entire function of* p (*with* z *and* α *fixed*). *If* $z \in \mathcal{R}-\{0\}$ *and* $\alpha \neq 0, 1, 2, \cdots$, *then* $D_z^\alpha z^p \ln(z) f(z)$ *is a meromorphic function of* p *whose only singularities are simple or double poles at* $p = -1, -2, -3, \cdots$, *or a subset thereof.*

(iii) *if* $p \neq -1, -2, -3, \cdots$, *then* $D_z^\alpha z^p \ln(z) f(z) = z^{p-\alpha}[\ln(z) A(\alpha, p; z) + B(\alpha, p; z)]$, *where* $A(\alpha, p; z)$ *and* $B(\alpha, p; z)$ *are analytic functions of* z *on* $\mathcal{R}$.

19. The law of exponents. The relation $D^\beta D^\alpha = D^{\beta+\alpha}$ is always true when α and β are natural numbers, but it is not true for arbitrary values of α and β. The simplest example is

$$D_z^{-1} D_z^1 f(z) = \int_0^z f'(z)\, dz = f(z) - f(0) \neq D_z^0 f(z) = f(z).$$

The general situation is examined in the following two theorems. Derivations of these theorems are found in [10].

THEOREM 19.1. *Let* $f(z)$ *be analytic on the simply connected open set* $\mathcal{R}$ *containing the point* $z = 0$. *Also assume* $f(0) \neq 0$, *and that* $p \neq -1, -2, -3, \cdots$. *If* $p - \alpha \neq -1, -2, -3, \cdots$, *then for* $z \in \mathcal{R}-\{0\}$ *we have*

$$D_z^\beta D_z^\alpha z^p f(z) = D_z^{\beta+\alpha} z^p f(z).$$

If $p - \alpha = -N$, $N = 1, 2, 3, \cdots$, *then for* $z \in \mathcal{R}-\{0\}$,

$$D_z^\beta D_z^\alpha z^p f(z) = D_z^{\beta+\alpha} z^p f(z) - \sum_{n=0}^{N-1} \frac{f^{(n)}(0)\Gamma(p+n+1) z^{p-\alpha-\beta+n}}{n!\,\Gamma(p-\alpha-\beta+n+1)}.$$

THEOREM 19.2. *Let* $f(z)$ *be analytic on the simply connected open set* $\mathcal{R}$ *containing the point* $z = 0$. *Also assume* $f(0) \neq 0$, *and that* $p \neq -1, -2, -3, \cdots$. *If* $z \in \mathcal{R}-\{0\}$, *and* $p - \alpha \neq -1, -2, -3, \cdots$, *then*

$$D_z^\beta D_z^\alpha z^p \ln(z) f(z) = D_z^{\beta+\alpha} z^p \ln(z) f(z). \tag{19.1}$$

If, however, $p - \alpha$ *is a negative integer, then* $D_z^\beta D_z^\alpha z^p \ln(z) f(z)$ *is undefined unless* $\beta = 0, 1, 2, \cdots$, *in which case* (19.1) *remains true.*

20. Generalizations of Taylor's series. One of the remarkable features of the fractional calculus is that often formulas familiar from the elementary calculus incorporating the symbol D^n can be generalized by simply replacing the n by some simple expression which is no longer a positive integer. An example of such a formula is the familiar Taylor's series

$$f(z)=\sum_{n=0}^{\infty}\frac{D^n f(z)|_{z=z_0}}{n!}(z-z_0)^n. \tag{20.1}$$

Replacing n by $n+\gamma$, where γ is an arbitrary complex number, and extending the summation from minus infinity to infinity we get the correct formula:

$$f(z)=\sum_{n=-\infty}^{\infty}\frac{D_z^{n+\gamma} f(z)|_{z=z_0}}{\Gamma(n+\gamma+1)}(z-z_0)^{n+\gamma}. \tag{20.2}$$

This formula in general converges only for z on the circle $|z-z_0|=|z_0|$.

We can generalize (20.2) by replacing n by an and by multiplying the series by a. The result is

$$f(z)=\sum_{n=-\infty}^{\infty}\frac{D_z^{an+\gamma} f(z)|_{z=z_0}}{\Gamma(an+\gamma+1)}(z-z_0)^{an+\gamma}a, \tag{20.3}$$

provided a is a real number restricted to $0<a\leqq 1$. If $1<a$, the left-hand side of (20.3) becomes more complicated (see [18]). Again, (20.3) converges only for z on the circle $|z-z_0|=|z_0|$.

We can also generalize the above formulas so that they include the familiar Lagrange's expansion

$$\frac{f(y)}{1-t\phi'(y)}=\sum_{n=0}^{\infty}D_x^n\{f(x)[\phi(x)]^n\}\frac{t^n}{n!}, \tag{20.4}$$

where $y=x+t\phi(y)$. If $0<a\leqq 1$ we have

$$\frac{f(y)}{1-t\phi'(y)}=\sum_{n=-\infty}^{\infty}D_x^{an+\gamma}\{f(x)[\phi(x)]^{an+\gamma}\}\frac{t^{an+\gamma}}{\Gamma(an+\gamma+1)}a \tag{20.5}$$

provided that $|(y-x)/\phi(y)|=|x/\phi(0)|$. This formula and its generalization to the case where $1<a$ is discussed in [18] where another notation is used.

Formulas (20.3) and (20.5) convert from series to integrals when we let the parameter a tend to zero. In this case, $\sum_{n=-\infty}^{\infty}$ is replaced by $\int_{-\infty}^{\infty}$, an is replaced by ω, and a is replaced by $d\omega$. As an example, (20.3) becomes

$$f(z)=\int_{-\infty}^{\infty}\frac{D_z^{\omega+\gamma} f(z)|_{z=z_0}}{\Gamma(\omega+\gamma+1)}(z-z_0)^{\omega+\gamma}\,d\omega. \tag{20.6}$$

This formula and similar generalizations are discussed in [21].

21. Generalizations of the Leibniz rule. In the previous section we gave several series and integral generalizations of the familiar Taylor's series. Now we give similar series and integral generalizations of the Leibniz rule

$$D^N u(z)v(z)=\sum_{n=0}^{N}\binom{N}{n}D^{N-1}u(z)D^n v(z). \tag{21.1}$$

Replacing N by arbitrary (real or complex) α we get

$$D_z^\alpha u(z)v(z)=\sum_{n=0}^{\infty}\binom{\alpha}{n}D_z^{\alpha-n}u(z)D_z^n v(z). \tag{21.2}$$

This formula has a strange feature. We can obviously interchange u and v on the left-hand side of the formula, but the interchange is not obvious on the right-hand side since u is differentiated fractionally while v is differentiated in the usual elementary sense! We suspect that (21.2) is the special case of a more general formula in which the interchange is clearly possible.

The necessary generalization is

$$D_z^\alpha u(z)v(z) = \sum_{n=-\infty}^{\infty} \binom{\alpha}{n+\gamma} D_z^{\alpha-n-\gamma} u(z) D_z^{n+\gamma} v(z). \tag{21.3}$$

In this series γ is an arbitrary complex number. The series (21.3) has an interesting region of convergence. Let $z_1, z_2, z_3, \cdots$, be singular points of the functions $u(z)$ and $v(z)$ in the complex z-plane. (We assume here that the collection of singular points can be uncountably infinite in number in spite of the subscripts indicated.) From each singularity z_i, draw the line segment to the origin and call it L_i. Next consider the half-plane HP_i whose boundary is the perpendicular bisector of L_i and which contains the point $z = 0$. The intersection of all such open half-planes forms the region of convergence of (21.3) (see Fig. 21.1). A full discussion of (21.3) is found in [17] and [24].

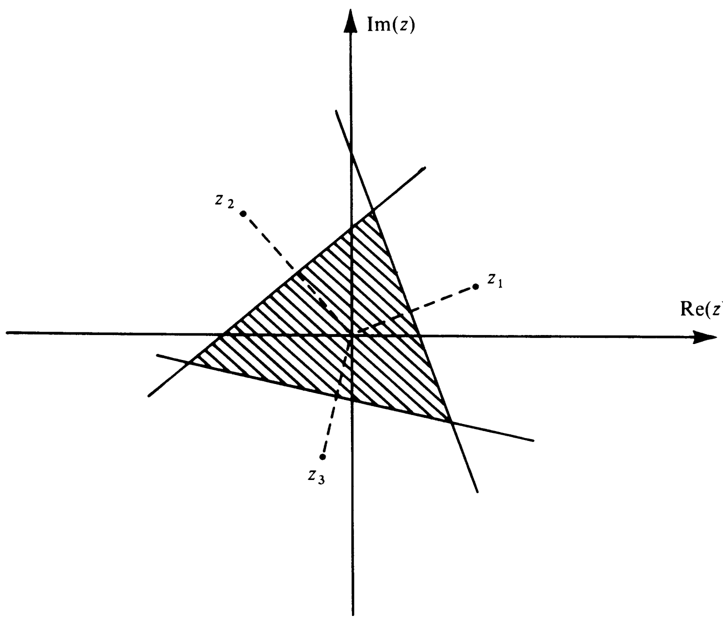

FIG. 21.1. *The region of convergence of the generalizations of the Leibniz rule*

By introducing the parameter a, where $0 < a \leqq 1$, we can further generalize (21.3) as

$$D_z^\alpha u(z)v(z) = \sum_{n=-\infty}^{\infty} \binom{\alpha}{an+\gamma} D_z^{\alpha-an-\gamma} u(z) D_z^{an+\gamma} v(z) a. \tag{21.4}$$

This series is discussed in [20] and [23].

Letting a approach zero in (21.4) we get the integral form of the Leibniz rule

$$D_z^\alpha u(z)v(z)=\int_{-\infty}^{\infty}\binom{\alpha}{\omega+\gamma}D_z^{\alpha-\omega-\gamma}u(z)D_z^{\omega+\gamma}v(z)\,d\omega. \tag{21.5}$$

This integral analogue of the Leibniz rule is discussed in [22] along with other generalizations.

22. The generalized chain rule. Another formula which yields a simple generalization to fractional derivatives is the formula for the Nth derivative of a composite function

$$D_z^N f(h(z))=\sum_{n=0}^{N}U_n(z)D_{h(z)}^n f(h(z))/n!, \tag{22.1}$$

where

$$U_n(z)=\sum_{r=0}^{n}\binom{n}{r}(-h(z))^r D_z^N[h(z)^{n-r}].$$

In fact, we can simply replace the natural number N in this formula by arbitrary α and we get the correct result for fractional derivatives

$$D_z^\alpha f(h(z))=\sum_{n=0}^{\infty}U_n(z)D_{h(z)}^n f(h(z))/n!, \tag{22.2}$$

where

$$U_n(z)=\sum_{r=0}^{n}\binom{n}{r}(-h(z))^r D_z^\alpha[h(z)^{n-r}].$$

In (22.2) we require that $h^{-1}(0)=0$. This formula for the fractional derivative of a composite function, as well as other formulas, is discussed in detail in [17].

23. The summation of infinite series and the evaluation of definite integrals. We conclude this survey of formulas and theorems from the fractional calculus by giving a few examples of how certain of these formulas can be used to obtain closed form expressions for series and integrals involving the special functions.

Example 23.1. *A Fourier integral.* The integral analogue of Taylor's series (20.5) can be used to generate Fourier transforms. As an example, set $f(z)=z^p$ in (20.5) and get

$$z^p=\int_{-\infty}^{\infty}\frac{D_z^{\omega+\gamma}z^p|_{z=z_0}}{\Gamma(\omega+\gamma+1)}(z-z_0)^{\omega+\gamma}\,d\omega. \tag{23.1}$$

From the first entry of Table 16.1 we have

$$D_z^{\omega+\gamma}z^p|_{z=z_0}=\frac{\Gamma(p+1)z_0^{p-\omega-\gamma}}{\Gamma(p-\omega-\gamma+1)}.$$

Inserting this last expression into (23.1) and simplifying we have

$$(z/z_0)^p = \int_{-\infty}^{\infty} \binom{p}{\omega+\gamma}((z/z_0)-1)^{\omega+\gamma}\, d\omega, \tag{23.2}$$

where we recall that

$$\binom{a}{b} = \frac{\Gamma(a+1)}{\Gamma(a-b+1)\Gamma(b+1)}.$$

Since this integral is valid only for z on the circle $|z-z_0|=|z_0|$, we can set $z=z_0+z_0\,e^{-i\phi}$, for $|\phi|<\pi$. Now (23.2) becomes

$$\int_{-\infty}^{\infty} \binom{p}{\omega+\gamma} e^{-i\phi(\omega+\gamma)}\, d\omega = (1+e^{-i\phi})^p \quad \text{for } |\phi|<\pi. \tag{23.3}$$

It can be shown that this integral vanishes for $|\phi|>\pi$. The integral (23.3) tells us that the Fourier transform of the function $\binom{p}{\omega}$, (thought of as a function of ω), is $(1+e^{-i\phi})^p$, for $|\phi|<\pi$, where the transform variable is ϕ.

In the same manner, by selecting other functions for $f(z)$ in the integral analogue of Taylor's series (20.5), we generate additional Fourier integrals. A list of such integrals is given in [21].

Example 23.2. *Generalizations of Dougall's formula.* Using the notation

$$L = \frac{\Gamma(A+B+C+D-3)}{\Gamma(A+C-1)\Gamma(A+D-1)\Gamma(B+C-1)\Gamma(B+D-1)},$$

and

$$R(x) = \frac{1}{\Gamma(A+x)\Gamma(B+x)\Gamma(C-x)\Gamma(D-x)},$$

we can write the so-called Dougall's formula as

$$L = \sum_{n=-\infty}^{\infty} R(n), \tag{23.4}$$

where

$$3 < \text{Re }(A+B+C+D).$$

Dougall's formula is itself a generalization of the formula for the sum of the hypergeometric series of unit argument. To see this, set $B=1$ in (23.4), and after straightforward manipulation get

$$\frac{\Gamma(A+C+D-2)\Gamma(A)}{\Gamma(A+C-1)\Gamma(A+D-1)} = \sum_{n=0}^{\infty} \frac{(1-C)_n(1-D)_n}{(A)_n n!}.$$

Dougall's formula (23.4) can be obtained easily from the generalized Leibniz rule (21.3) by taking $u(z)=z^{B+C-2}$, $v(z)=z^{A+D-2}$, $\alpha=A+C-2$ and $\gamma=A-1$ (see [17]). To get direct generalizations of the Dougall's formula, we can employ

(21.4) and (21.5) with the same substitutions and get

$$L=\sum_{n=-\infty}^{\infty} R(an)a=\int_{-\infty}^{\infty} R(\omega)\,d\omega, \tag{23.5}$$

(see [21] and [23]).

Examples 23.3. *A higher generalization of Dougall's formula.* Numerous higher generalizations of the Dougall's formula given in the previous example can be obtained by selecting other functions for $u(z)$ and $v(z)$ in (21.4) and (21.5). As an example, select $u(z)=z^{B+C-2}(1-z)^{-e}$, $v(z)=z^{A+D-2}(1-z)^{-E}$, $\alpha=A+C-2$ and $\gamma=A-1$ and get

$$L\,{}_2F_1\left[\begin{matrix} e+E,\ A+B+C+D-3 \\ A+D-1 \end{matrix}\middle| z\right]$$

$$\begin{aligned} &=\sum_{n=-\infty}^{\infty} R(an)\,{}_2F_1\left[\begin{matrix} e,\ A+C-1 \\ A+an \end{matrix}\middle| z\right]{}_2F_1\left[\begin{matrix} E,\ B+D-1 \\ D-an \end{matrix}\middle| z\right]a \qquad (0<a\leqq 1) \\ &=\int_{-\infty}^{\infty} R(\omega)\,{}_2F_1\left[\begin{matrix} e,\ A+C-1 \\ A+\omega \end{matrix}\middle| z\right]{}_2F_1\left[\begin{matrix} E,\ B+D-1 \\ D-\omega \end{matrix}\middle| z\right]d\omega. \end{aligned} \tag{23.6}$$

Another series and integrals are given in [21] and [23].

24. Acknowledgment. The authors wish to acknowledge the contribution of Professor J. B. Diaz of Rensselaer Polytechnic Institute who brought the work of A. V. Letnikov described in § 8 to their attention.

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