Fractional Derivatives and Leibniz Rule

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and thus reach the point

$$p' = \frac{1}{2}\bar{d}(2f) = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{4f},$$

so

$$p - p' = \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2f-1} - \frac{1}{2f}\right).$$

Hence point p' falls short of point p by an amount approaching $\frac{1}{2}\log 2$, hence $\bar{d}(2f)$ falls short of $2d(f)$ by the distance log 2. We may then ask how much additional fuel Δf is required to cover this additional distance. For this, we must solve

$$\log 2 = \frac{1}{2f+1} + \frac{1}{2f+2} + \frac{1}{2f+\Delta f}\cdot$$

The sum on the right for large f is close to

$$\int_{2f}^{2f+\Delta f} \frac{1}{x}\,dx = \log((2f+\Delta f)/2f) = \log(1+\Delta f/2f),$$

hence Δf is approximately $2f$, and we see that for large distances round trips are nearly four times as expensive as one-way trips!

Knowing the answer, it is now clear why this should be so! For a round trip, it is necessary to set up very substantial fuel depots at the far end of the desert, whereas for two one-way trips, the closer one gets to the far end of the desert, the smaller the depots have to be.

MATHEMATICAL NOTES

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FRACTIONAL DERIVATIVES AND LEIBNIZ RULE

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1. Introduction. The fractional derivative is an extension of the familiar derivative, $d^n f(z)/dz^n$, to nonintegral values of n. Fractional differentiation is of use in the solution of ordinary [6], partial [12], and integral equations [2, 3], as well as in other contexts, a few of which are indicated in the bibliography. Although other methods of solution are available, the fractional derivative approach to these problems often suggests methods that are not obvious in a classical formulation. The fractional calculus forms a special chapter in the more general "operational calculus" which considers functions of the differential operator "D" more general than D^α, (see [5, p. 115] and [8, p. 28]).

We shall define the fractional derivative by generalizing Cauchy's integral formula. The remainder of this paper consists of a generalization of the Leibniz rule for the derivative of the product of two functions, and a use of the Leibniz rule to compute the value of the hypergeometric function of unit argument in terms of gamma functions. The methods used in this paper can be followed by students familiar with complex integration and the gamma and beta functions. The author has given this material in an undergraduate course in complex variables with success since it incorporates the topics of branch points, complex integration, Cauchy's integral formula, and Taylor's theorem.

2. The definition and some examples. The concept of a derivative whose order is not a natural number (fractional derivative) was introduced by Liouville [7] in 1832 when he observed that the relation $D^N e^{az}=a^N e^{az}$ could be generalized for arbitrary complex numbers α by $D^\alpha e^{az}=a^\alpha e^{az}$. Liouville then considered a general function $f(z)$ expanded in a Fourier series and differentiated fractionally term by term. We find this method of defining fractional derivatives discussed in [15, pp. 133–142].

In contrast to Liouville, we follow Nekrassov [9] and give a definition of fractional differentiation by generalizing the Cauchy integral formula

$$D_z^N f(z) = \frac{N!}{2\pi i}\oint_C f(t)(t-z)^{-N-1}dt,$$

where the contour C is a simple closed curve enclosing z in the positive sense and containing inside only regular points of $f(t)$. At first glance it seems easy enough to replace N by arbitrary α on the left side of this formula, and $N!$ and $(t-z)^{-N-1}$ by $\Gamma(\alpha+1)$ and $(t-z)^{-\alpha-1}$ respectively on the right. However, $(t-z)^{-\alpha-1}$ no longer has a pole at $t=z$, but a branch point. Thus we are no longer as free to deform C, for the position at which our contour cuts the branch line of this function determines the value of the integral. Instead we take the branch cut to be the semi-infinite straight line segment starting at $t=z$, passing through the origin, and continuing to infinity. Let the contour of integration C start at $t=0$, enclose $t=z$ once in the positive sense, and return to $t=0$ without enclosing any singularities of $f(t)$ or intersecting the branch cut at any point except $t=0$. It is standard to denote this contour integral by

$$D_z^\alpha f(z) = \frac{\Gamma(\alpha+1)}{2\pi i}\int_0^{(z^+)} f(t)(t-z)^{-\alpha-1}dt. \tag{1}$$

Since $(t-z)^{-\alpha-1}=\exp[(-\alpha-1)\ln(t-z)]$ is multiple valued, we remove this ambiguity by requiring $\ln(t-z)$ to be real when $t-z>0$. We summarize the results of this informal discussion in a definition.

DEFINITION: *Let* $f(z)=z^p g(z)$, *where* $g(z)$ *is analytic in a simply connected region* R *of the z-plane containing the origin, and let* $\mathrm{Re}(p)>-1$. *Then* (1) *defines the* ***fractional derivative*** $D_z^\alpha f(z)$ *of order* α *of* $f(z)$ *for* $\alpha\neq -1, -2, -3, \cdots$.

The reason we select functions $f(z)$ admitting branch point singularities at the origin, rather than simply functions analytic in some region containing the origin, is that these functions occur naturally in the application of fractional derivatives to the study of the special functions of mathematical physics [**5**, pp. 118–122; **6**; **10**]. We also require functions with singularities of the form z^p in the examination of the hypergeometric function of unit argument at the close of this paper. Naturally, we select $\mathrm{Re}(p)>-1$ so that the contour integral (1) is defined.

As an example of the above definition, we compute the fractional derivative of z^p. Since

$$D_z^N z^p = p!z^{p-N}/(N-p)!,$$

we guess that our definition (1) for fractional derivatives should yield

$$D_z^\alpha z^p = \Gamma(p+1)z^{p-\alpha}/\Gamma(p-\alpha+1). \tag{2}$$

To demonstrate that (2) follows from our defining relation for fractional derivative (1) in the case where $\mathrm{Re}(p)>-1$, we set $t=zs$ in (1) and obtain

$$D_z^\alpha z^p = \frac{\Gamma(\alpha+1)z^{p-\alpha}}{2\pi i}\int_0^{(1^+)} s^p(s-1)^{-\alpha-1}ds.$$

We deform the contour of this integral into a straight line segment from the origin to $1-\epsilon$, followed by the circle $|s-1|=\epsilon$ traversed in the positive sense, and then the real s axis back to the origin. We obtain

$$\begin{aligned} D_z^\alpha z^p &= \Gamma(\alpha+1)[1-\exp(-2\pi i(\alpha+1))](z^{p-\alpha}/2\pi i)\int_0^1 s^p(s-1)^{-\alpha-1}ds \\ &= \frac{z^{p-\alpha}}{\Gamma(-\alpha)}\int_0^1 s^p(1-s)^{-\alpha-1}ds \\ &= \Gamma(p+1)z^{p-\alpha}/\Gamma(p-\alpha+1), \end{aligned}$$

using well-known properties of the gamma and beta functions.

3. Leibniz rule. We consider next the Leibniz rule from elementary calculus for the derivative of the product of two functions $u(z)$ and $v(z)$:

$$D_z^N uv = \sum_{n=0}^{N}\binom{N}{n} D_z^{N-n} u D_z^n v.$$

A reasonable guess for the generalization of this result to fractional derivatives is

$$D_z^\alpha uv = \sum_{n=0}^{\infty} \binom{\alpha}{n} D_z^{\alpha-n} u D_z^n v. \tag{3}$$

This guess is indeed correct, and it was given as early as 1867 by A. K. Grunwald [4]. We now prove (3) (in a manner different from Grunwald) in the following theorem:

THEOREM. *Let $u(z)$, $v(z)$, and $u(z)v(z)$ satisfy the requirements given in the definition above for the existence of their fractional derivatives. Let the region R in this definition be the entire z-plane. Then the generalized Leibniz rule* (3) *is valid for all $\alpha \neq -1, -2, -3, \cdots$ and $z \neq 0$.*

Proof: We first expand $v(t)$ in a Taylor series about $t=z$:

$$v(t) = \sum_{n=0}^{\infty} D_z^n v(z)(t-z)^n/n!. \tag{4}$$

We multiply both sides of (4) by $\Gamma(\alpha+1)(t-z)^{-\alpha-1}u(t)/2\pi i$ and integrate along the contour C of (1) to get

$$\frac{\Gamma(\alpha+1)}{2\pi i}\int_0^{(z+)} \frac{u(t)v(t)dt}{(t-z)^{\alpha+1}} = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+1)}{2\pi i n!}\int_0^{(z+)} \frac{u(t)dt}{(t-z)^{\alpha-n+1}} D_z^n v(z).$$

By comparing the contour integrals in this last expression with the definition of fractional differentiation (1) we obtain (3) at once.

4. Hypergeometric function of unit argument. Fractional differentiation often provides a convenient tool for deriving relations among the special functions of mathematical physics ([5, p. 118], and [10]). As an example, we obtain the sum of the hypergeometric series [14, p. 281] of unit argument,

$$F(a, b; c; 1) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!},$$

where $(a)_n = a(a+1)(a+2) \cdots (a+n-1)$. If $u(z) = z^{c-a-1}$, $v(z) = z^{-b}$, and $\alpha = -a$ in the Leibniz rule (3), we then obtain

$$D_z^{-a} z^{c-a-b-1} = \sum_{n=0}^{\infty} \frac{\Gamma(1-a) D_z^{-a-n} z^{c-a-1} \cdot D_z^n z^{-b}}{\Gamma(1-n-a)n!}.$$

Using (2) to compute the three derivatives appearing in the above expression, we obtain

$$\frac{\Gamma(c-a-b)}{\Gamma(c-b)} = \sum_{n=0}^{\infty} \frac{\Gamma(1-a)\Gamma(c-a)\Gamma(1-b)}{\Gamma(1-a-n)\Gamma(1-b-n)\Gamma(c+n)n!}$$

after minor simplification. Since $\Gamma(1-x)\Gamma(x) = \pi \csc \pi x$, it follows that

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} = F(a, b; c; 1).$$

In applying the Leibniz rule, u, v, and uv are restricted by the condition that the fractional derivatives occurring are defined by (1). This means that $\mathrm{Re}(c-a)$, $\mathrm{Re}(1-b)$, and $\mathrm{Re}(c-a-b)$ must be positive. It is well known [**14**, p. 281] that only the last of these three is necessary.

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ANOTHER SOLUTION OF AN OLD PROBLEM OF PÓLYA

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In 1913, Pólya [3] proposed the following problem: Show that there is no uniform affixing of $\pm$ signs to the elements of the square matrices of order $n>2$ such that the permanent of the resulting matrix equals the determinant of the original one. (The permanent of $A=\|a_{ij}\|$ is per $A=\sum a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{n\sigma(n)}$, where σ runs over all permutations of $\{1, \cdots, n\}$.) Szegö [4] solved this problem by using parity arguments. Meanwhile a much more general result was established in [2, p. 381], namely, there is no linear map $A \to T(A)$ on the square