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THE FRACTIONAL DERIVATIVE OF A COMPOSITE FUNCTION*

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1. Introduction. In the elementary calculus one considers the derivative of order N of the composite function $f(z) = F(h(z))$ and obtains the formula [5, p. 19]

$$D_z^N f(z) = \sum_{n=0}^{N} \frac{U_n(z) D_{h(z)}^n f(z)}{n!}, \tag{1.1}$$

where

$$U_n(z) = \sum_{r=0}^{n} \binom{n}{r} (-h(z))^r D_z^N h(z)^{n-r}.$$

In this paper we consider the extension of (1.1) to fractional derivatives. We derive the fundamental result

$$D_{g(z)}^{\alpha} f(z) = D_{h(z)}^{\alpha} \left\{ \frac{f(z) g'(z)}{h'(z)} \left(\frac{h(z) - h(w)}{g(z) - g(w)} \right)^{\alpha+1} \right\} \Bigg|_{w=z}, \tag{1.2}$$

where the notation $D_{g(z)}^{\alpha} f(z)$ means the fractional derivative of order α of $f(z)$ with respect to $g(z)$. The Leibniz rule for fractional derivatives is then applied to (1.2) to obtain the new series expansion

$$\begin{aligned} D_{g(z)}^{\alpha} f(z) = &\sum_{n=-\infty}^{\infty} \binom{\alpha}{\gamma + n} D_{h(z)}^{\gamma+n} \frac{f(z)}{F(z, w)} \\ &\cdot D_{h(z)}^{\alpha-\gamma-n} \left\{ \frac{F(z, w) g'(z)}{h'(z)} \left(\frac{h(z) - h(w)}{g(z) - g(w)} \right)^{\alpha+1} \right\} \Bigg|_{w=z}, \end{aligned} \tag{1.3}$$

where

$$\binom{\alpha}{\gamma + n} = \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha - \gamma - n + 1)\Gamma(\gamma + n + 1)}.$$

The formula (1.1) from the elementary calculus is shown to be a special case of the "generalized chain rule" (1.3). A few specific examples of these general results are studied.

The concept of the fractional derivative with respect to an arbitrary function has been used in recent papers [3], [4]. However, to the best of the author's knowledge, the full definition and notation $D_{g(z)}^{\alpha} f(z)$, introduced in his paper [8], are new. Indeed, it is this new notation which suggests the possibility of investigating the fractional derivative of a composite function so as to generalize the calculus formula (1.1).

The Leibniz rule for fractional derivatives was first studied by Watanabe [10] in 1931. Recently the author [8] found a new proof for this Leibniz rule which revealed its precise region of convergence in the z-plane. Since the Leibniz rule plays a central role in the investigation of the series expansions in this paper, the reader should consult [8] before proceeding.

Finally, a few specific examples of (1.3) are examined. Novel derivations of the known result

$$F(\alpha, 1-\alpha; p-\alpha+1; 1/2) = \frac{2^\alpha \Gamma(p-\alpha+1)\Gamma(p/2+1)}{\Gamma(p/2-\alpha+1)\Gamma(p+1)}$$

and Kummer's formula

$$F(\alpha+1, p; p-\alpha; -1) = \frac{\Gamma(p/2)\Gamma(p-\alpha)}{2\Gamma(p)\Gamma(p/2-\alpha)}$$

are obtained as well as new results.

The generalized chain rule, the Leibniz rule, the relation $D^\alpha D^\beta = D^{\alpha+\beta}$, and other operations illustrate that the fractional calculus exists as a natural extension of the elementary calculus. Recent papers illustrating the application of the fractional calculus to problems in ordinary, partial and integral equations [3], [4], [6], [7], [9] demonstrate that it is a highly suggestive tool. Higgins [7] has observed that "although results using fractional integral operators can always be obtained by other methods, the succinct simplicity of the formulation may often suggest approaches which might not be evident in a classical approach." It is hoped that this paper will further reveal the uses of fractional derivatives.

2. Fractional derivatives and Leibniz rule. We now review briefly the definition of fractional derivative and the statement of the Leibniz rule. A full discussion of these ideas is found in [8].

DEFINITION 1. The *fractional derivative* or order α of $f(z)$ with respect to $h(z)$ is

$$D^\alpha_{h(z)} f(z) = \frac{\Gamma(\alpha+1)}{2\pi i} \int_{h^{-1}(0)}^{(z+)} \frac{f(t)h'(t)\,dt}{(h(t)-h(z))^{\alpha+1}}. \tag{2.1}$$

The branch line for $(h(t)-h(z))^{\alpha+1}$ starts at $t = z$, passes through $t = h^{-1}(0)$, and continues to infinity. The contour of integration starts at $t = h^{-1}(0)$, encircles $t = z$ in the positive sense once, and returns to $t = h^{-1}(0)$ without crossing the branch line of $(h(t)-h(z))^{\alpha+1}$. $f(z)$ and $h(z)$ are assumed to possess sufficient regularity to give the integral (2.1) meaning.

The critical use of the Leibniz rule for fractional derivatives,

$$D^\alpha_{h(z)} u(z)v(z) = \sum_{n=-\infty}^{\infty} \binom{\alpha}{\gamma+n} D^{\alpha-\gamma-n}_{h(z)} u(z) D^{\gamma+n}_{h(z)} v(z), \tag{2.2}$$

requires a description of the region in the z-plane over which the series (2.2) converges. To simplify the following discussion we describe this region of convergence as the "Leibniz region" and give its definition.

DEFINITION 2. Let $u(h^{-1}(z))$ and $v(h^{-1}(z))$ be defined and analytic on the simply connected region $\mathscr{R}$. Let $z = 0$ be an interior or boundary point of $\mathscr{R}$.

derivatives of elementary functions. A table of fractional derivatives or integrals such as that found in [2, vol. 2, pp. 185–214] is useful for this purpose.

The notation for the special functions used is that of Erdélyi, Magnus, Oberhettinger and Tricomi [1], [2].

Example 1. Setting $f(z) = z^{p-2}$, $g(z) = z^2$ and $h(z) = z$ in the fundamental relation (3.1), we obtain

$$D_z^\alpha z^{p-2} = D_z^\alpha 2z^{p-1}(z+w)^{-\alpha-1}|_{w=z}.$$

The left-hand side is evaluated with the aid of the relation

$$D_z^\alpha z^q = \frac{\Gamma(q+1)z^{q-\alpha}}{\Gamma(q-\alpha+1)},$$

after replacing z by z^2. Using [2, vol. 2, no. 9, p. 186] we obtain Kummer's formula

$$F(\alpha+1, p; p-\alpha; -1) = \frac{\Gamma(p/2)\Gamma(p-\alpha)}{2\Gamma(p)\Gamma(p/2-\alpha)}.$$

Example 2. Letting $f(z) = z^p$, $g(z) = z^2$, $h(z) = z$, $F(z, w) = z^p(z+w)^{\alpha+1}$ and $\gamma = 0$ in the generalized chain rule (3.3), we obtain

$$\frac{\Gamma(p/2+1)}{\Gamma(p/2-\alpha+1)} = \sum_{n=0}^{\infty} \frac{\Gamma(1-\alpha+n)\Gamma(p+1)(-1)^n(2)^{-\alpha-n}}{\Gamma(1-\alpha-n)\Gamma(p-\alpha+n+1)n!},$$

which reduces to

$$\frac{\Gamma(p-\alpha+1)\Gamma(p/2+1)2^\alpha}{\Gamma(p/2-\alpha+1)\Gamma(p+1)} = F(\alpha, 1-\alpha; p-\alpha+1; 1/2). \tag{4.1}$$

It may be noted that this novel method for determining the known relation (4.1) provides a direct evaluation of the hypergeometric series of argument $z = 1/2$.

Example 3. Setting $g(z) = z$, $h(z) = z^k$ and $F(z, w) = z^q$ in (3.4), we obtain

$$D_z^\alpha f(z) = \frac{\Gamma(q+1)z^{q-\alpha}}{\Gamma(q-\alpha+1)} \sum_{n=0}^{\infty} \frac{(-z^k)^n}{n!} D_{z^k}^n f(z)z^{-q} \cdot {}_{k+1}F_k\left(-n, \frac{q+1}{k}, \frac{q+2}{k}, \cdots, \frac{q+k}{k}; \frac{q-\alpha+1}{k}, \frac{q-\alpha+2}{k}, \cdots, \frac{q-\alpha+k}{k}; 1\right)$$

with the aid of [2, vol. 2, no. 11, p. 186]. This is the generalized chain rule for the fractional derivative of the composite function $f(z) = F(z^k)$ in terms of derivatives with respect to z^k.

Example 4. Setting $g(z) = z^p$, $h(z) = z$ and $F(z, w) = z^{q-p+1}$ in (3.4), we obtain

The Leibniz rule (2.2) yields the desired generalized chain rule at once:

$$(3.3)\qquad D^{\alpha}_{g(z)}f(z) = \sum_{n=-\infty}^{\infty} \binom{\alpha}{\gamma+n} D^{\gamma+n}_{h(z)} \frac{f(z)}{F(z,w)} \cdot D^{\alpha-\gamma-n}_{h(z)} \left\{ \frac{F(z,w)g'(z)}{h'(z)} \left(\frac{h(z)-h(w)}{g(z)-g(w)} \right)^{\alpha+1} \right\} \Bigg|_{w=z}.$$

The precise conclusion is stated as a theorem.

THEOREM 3. *Let $f(z)$, $g(z)$ and $h(z)$ satisfy the conditions of Theorem 2. Let $F(h^{-1}(z), h^{-1}(w))$ be regular on $\mathscr{R} \times \mathscr{R}$. Let $u(z)$ and $v(z)$ be defined by (3.2) and satisfy the conditions of Theorem 1. Then the generalized chain rule (3.3) is valid for z in the Leibniz region $\mathscr{L}(u, v; h)$ and arbitrary γ for which $\binom{\alpha}{\gamma+n}$ is defined.*

We conclude the analytical investigation of the generalized chain rule by converting (3.3) into a form somewhat like the elementary calculus formula (1.1). The new form is

$$(3.4)\qquad D^{\alpha}_{g(z)}f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} \{D^{\alpha}_{g(z)}F(z,w)(h(z)-h(w))^n\} D^{n}_{h(z)} \frac{f(z)}{F(z,w)} \Bigg|_{w=z},$$

where

$$(3.5)\qquad D^{\alpha}_{g(z)}F(z,w)(h(z)-h(w))^n|_{w=z} = \sum_{r=0}^{n} \binom{n}{r} (-h(z))^r D^{\alpha}_{g(z)}F(z,w)h(z)^{n-r} \Bigg|_{w=z}.$$

The elementary calculus formula (1.1) is seen as the special case in which α is an integer, $g(z) \equiv z$ and $F(z, w) \equiv 1$.

THEOREM 4. *With the hypothesis of Theorem 3, the relations* (3.4) *and* (3.5) *are valid.*

Proof. Set $\gamma = 0$ in the generalized chain rule (3.3). The summation now extends from $n = 0$ to ∞ rather than from $-\infty$ to ∞. We see at once that

$$\binom{\alpha}{n} D^{\alpha-n}_{h(z)} \left\{ \frac{F(z,w)g'(z)}{h'(z)} \left(\frac{h(z)-h(w)}{g(z)-g(w)} \right)^{\alpha+1} \right\} \Bigg|_{w=z} = \frac{1}{n!} D^{\alpha}_{g(z)}F(z,w)(h(z)-h(w))^n \Bigg|_{w=z}$$

upon writing both sides as a contour integral by (2.1). The relation (3.5) is obtained at once upon expanding $(h(z) - h(w))^n$ by the binomial theorem.

It is useful to note that Theorem 4 is valid even when $h^{-1}(0)$ and $g^{-1}(0)$ are not equal. This is seen at once upon replacing $h(z)$ by $h(z) - h(g^{-1}(0))$ in (3.4) and observing that $h'(z)$ and $D^{n}_{h(z)}$ do not change.

We have demonstrated that the formulas from the elementary calculus for the derivatives of a composite function generalize to fractional derivatives in a natural way. We proceed to investigate some consequences of the generalized chain rule through the study of a few specific examples.

4. Examples. We conclude this paper with an examination of a few special cases of the generalized chain rule. These require the evaluation of the fractional

$\mathscr{S}(\mathscr{R})$ denotes the set of all z such that the closed disk $|t - z| \leqq |z|$ contains only points t in $\mathscr{R} \cup \{0\}$. We call the set of all z in $h^{-1}(\mathscr{S}(\mathscr{R}))$ the *Leibniz region of u and v with respect to h* and denote it by $\mathscr{L}(u, v; h)$.

We state the precise version of the Leibniz rule from [8] as a theorem for future reference.

THEOREM 1 (Leibniz rule). *Let $u(h^{-1}(z))$ and $v(h^{-1}(z))$ be defined and analytic in the simply connected region $\mathscr{R}$. Let $\oint u(h^{-1}(z))\, dz$, $\oint v(h^{-1}(z))\, dz$ and $\oint u(h^{-1}(z)) \cdot v(h^{-1}(z))\, dz$ vanish over any simple closed contour in $\mathscr{R} \cup \{0\}$ passing through the origin. Then the Leibniz rule* (2.2) *is true for z in $\mathscr{L}(u, v; h)$ and arbitrary γ for which $\binom{\alpha}{\gamma + n}$ is defined.*

Having reviewed the definition of fractional derivative and the Leibniz rule we proceed to study the fractional derivative of a composite function. We shall see that the results are easy applications of the Leibniz rule and Definition 1.

3. The generalized chain rule. We begin by deriving the fundamental relation

$$D^{\alpha}_{g(z)} f(z) = D^{\alpha}_{h(z)} \left\{ \frac{f(z) g'(z)}{h'(z)} \left(\frac{h(z) - h(w)}{g(z) - g(w)} \right)^{\alpha + 1} \right\} \Bigg|_{w = z}. \tag{3.1}$$

This relation combined with the Leibniz rule yields the generalized chain rule for fractional derivatives.

THEOREM 2. *Let $f(g^{-1}(z))$ and $f(h^{-1}(z))$ be defined and analytic on the simply connected region $\mathscr{R}$, and let the origin be an interior or boundary point of $\mathscr{R}$. Suppose also that $g^{-1}(z)$ and $h^{-1}(z)$ are regular univalent functions on $\mathscr{R}$ and that $h^{-1}(0) = g^{-1}(0)$. Let $\oint f(g^{-1}(z))\, dz$ vanish over every simple closed contour in $\mathscr{R} \cup \{0\}$ through the origin. Then the fundamental relation* (3.1) *is valid.*

Proof. The result follows immediately upon converting both sides of the fundamental relation (3.1) to contour integrals by means of the definition of fractional derivative (2.1).

The Leibniz rule can be applied to the right-hand side of (3.1) once we select $u(z)$ and $v(z)$ such that

Using the Cauchy integral formula for fractional derivatives (2.1), we easily see that

$$\frac{(-1)^n}{n!} D_{z^p}^{\alpha} z^q (w-z)^n \bigg|_{w=z} = \frac{(-1)^{n-\alpha-1} p}{\Gamma(-\alpha)} D_z^{-n-1} (z^p - w^p)^{-\alpha-1} z^q \bigg|_{w=z}.$$

Finally, using [2, vol. 2, no. 11, p. 186] we obtain

$$D_{z^p}^{\alpha} f(z) = \frac{p\Gamma(q+1) z^{q-\alpha p-p+1}}{\Gamma(-\alpha)} \cdot \sum_{n=0}^{\infty} \frac{(-z)^n D_z^n f(z) z^{p-q-1}}{\Gamma(q+n+2)} {}_{p+1}F_p$$

$$\cdot \left(\alpha+1, \frac{q+1}{p}, \frac{q+2}{p}, \cdots, \frac{q+p}{p}; \frac{q+n+2}{p}, \frac{q+n+3}{p}, \cdots, \frac{q+n+p+1}{p}; 1\right).$$

Computation of the coefficient of $D_z^n f(z) z^{p-q-1}$ by means of (3.5) rather than by the procedure outlined above reveals that

$${}_{p+1}F_p\left(\alpha+1, \frac{q+1}{p}, \frac{q+2}{p}, \cdots, \frac{q+p}{p}; \frac{q+n+2}{p}, \frac{q+n+3}{p}, \cdots, \frac{q+n+p+1}{p}\ 1\right)$$

equals the finite sum of gamma functions

$$\frac{\Gamma(-\alpha)\Gamma(q+n+2)}{p\Gamma(q+1)} \sum_{r=0}^{n} \frac{(-1)^{n+r}\Gamma((q+n-r+1)/p)}{(n-r)!\,\Gamma((q+n-r+1-p\alpha)/p)}.$$

Replacement of z by $z^{1/p}$ in (4.2) yields the generalized chain rule for the fractional derivative of $f(z^{1/p})$ in terms of derivatives with respect to $z^{1/p}$.

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