01/05/04

THE GENERAL VIETA-WALLIS PRODUCT FOR PI

Thomas J. Osler
Mathematics Department
Rowan University
Glassboro NJ 08028

osler@rowan.edu

Abstract

It has been shown recently that the two oldest infinite product representations of π, Vieta's product of radicals, and Wallis's product of rationals, are both special cases of another infinite product called the Vieta-Wallis product (VWP). In this new product there is a parameter p. When $p = 0$, we get Wallis's product, and as p grows to infinity, the Wallis product is morphed into the Vieta product. In this paper we find a new infinite product which we call the general Vieta-Wallis product (GVWP). The GVWP includes the VWP as a special case. At the heart of this new result is an unusual product representation of the sine function. We explore many special cases of this GVWP, some of which were found recently, while many seem to be new.

1. Introduction

The two oldest representations for the number π are infinite product expansions. The first

(1.1) $$\frac{2}{\pi} = \sqrt{\frac{1}{2}}\ \sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\ \sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdots,$$

is due to Vieta [7] in 1592. The second is Wallis's product [8] dating from 1655

(1.2) $$\frac{2}{\pi} = \frac{1\cdot 3}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\frac{5\cdot 7}{6\cdot 6}\cdot\frac{7\cdot 9}{8\cdot 8}\cdots.$$

Both are usually included in any list of interesting expressions for π [2]. (For more history see [1] and [3].)

In a recent note in the American Mathematical Monthly [4] a (possibly new) product was given which contained both of the above classical results as special cases.

The VWP:

(1.3) $$\frac{2}{\pi}=\prod_{n=1}^{p}\underbrace{\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\cdots+\frac{1}{2}\sqrt{\frac{1}{2}}}}}}_{(n\ radicals)}\quad\prod_{n=1}^{\infty}\frac{2^{p+1}n-1}{2^{p+1}n}\cdot\frac{2^{p+1}n+1}{2^{p+1}n}\ .$$

We will call (1.3) the VWP (Vieta-Wallis product). While (1.1) and (1.2) seem unrelated, the expression (1.3) shows that they are both special cases of the VWP which is a more general "double product". The first product in the VWP consists of the first p factors of Vieta's original infinite product (1.1). The second product in the VWP is a Wallis-like product. We say this because the case where p = 0 gives us the original Wallis's product (1.2), and for other values of p it is the original Wallis's product with factors deleted. Notice also that the Wallis-like product in the VWP provides us with the error factor needed to make the Vieta product (1.1) exact when only a finite number of factors are used . We will return to the VWP in the next section and examine these features in detail.

In the derivation of the VWP in [4], there is a step in which a general angle θ is replaced by $\pi/2$. By choosing other appropriate values for θ, we can obtain many variations on (1.3). A few examples of this type were found in [5], and an example of one of these variations is

$$\frac{3}{\pi}=\prod_{n=1}^{p}\underbrace{\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\cdots+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)}}}}}_{\leftarrow---n\ radicals---\rightarrow}\ \prod_{n=1}^{\infty}\left(\frac{3\cdot 2^{p+1}n-1}{3\cdot 2^{p+1}n}\cdot\frac{3\cdot 2^{p+1}n+1}{3\cdot 2^{p+1}n}\right).$$

In this paper we make another change in the original derivation [4] of the VWP which will yield further extensions. This change occurs when the function $\sin z$ is replaced by its familiar infinite product

(1.4) $$\sin z = z\prod_{k=1}^{\infty}\left(1-\frac{z^2}{k^2\pi^2}\right).$$

In this paper we use an unusual product, ([6] and [9]), for $\sin z$ given by

(1.5) $$\sin z=\exp\left(\frac{z}{\pi}\log\frac{M}{N}\right)z\prod_{k=0}^{\infty}\left(1-\frac{z/\pi}{kM+1}\right)\left(1-\frac{z/\pi}{kM+2}\right)\cdots\left(1-\frac{z/\pi}{kM+M}\right)\times \left(1+\frac{z/\pi}{kN+1}\right)\left(1+\frac{z/\pi}{kN+2}\right)\cdots\left(1+\frac{z/\pi}{kN+N}\right).$$

Here *M* and *N* are any positive integers. (The more familiar product (1.4) is obtained from (1.5) by letting $M = N = 1$.) In this way new extensions of (1.3) are obtained. Our most general result will be the following relation:

The GVWP:

(1.6) $$\frac{s\sin(r\pi/s)}{r\pi}\exp\left(\frac{r}{2^p s}\log\frac{N}{M}\right)=\prod_{n=1}^{p}\underbrace{\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\cdots+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\cos(r\pi/s)}}}}}_{<------(n\ radicals)------>}\times$$
$$\prod_{k=0}^{\infty}\left(\frac{2^p s(kM+1)-r}{2^p s(kM+1)}\right)\left(\frac{2^p s(kM+2)-r}{2^p s(kM+2)}\right)\cdots\left(\frac{2^p s(kM+M)-r}{2^p s(kM+M)}\right)\times \left(\frac{2^p s(kN+1)+r}{2^p s(kN+1)}\right)\left(\frac{2^p s(kN+2)+r}{2^p s(kN+2)}\right)\cdots\left(\frac{2^p s(kN+N)+r}{2^p s(kN+N)}\right).$$

We call (1.6) the general Vieta-Wallis product (GVWP). Here *M, N, r,* and *s* are positive integers, and *p* is a non-negative integer. By giving special values to the five integer parameters just described, many infinite products are obtained from the GVWP. Some of these products appeared recently as special cases of (1.5) (see section 4) and as slight generalizations of the VWP (1.3), (see section 5), while new results are shown in section 6. We will show how to derive the GVWP (1.6) in section 3.

2. A close look at the VWP

We now examine in detail how the VWP has the features mentioned in the introduction. The VWP given in (1.3) yields Vieta's product (1.1) as the limiting case as *p* goes to infinity, and the Wallis's product (1.2) as the case *p*=0. For each intermediate value of *p* = 1, 2, 3, ... we obtain "united Vieta-Wallis-like products":

p=0: $$\frac{2}{\pi}=\frac{1\cdot 3}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\frac{5\cdot 7}{6\cdot 6}\cdot\frac{7\cdot 9}{8\cdot 8}\cdot\frac{9\cdot 11}{10\cdot 10}\cdot\frac{11\cdot 13}{12\cdot 12}\cdots$$ (original Wallis's product)

p=1: $$\frac{2}{\pi}=\sqrt{\frac{1}{2}}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\frac{7\cdot 9}{8\cdot 8}\cdot\frac{11\cdot 13}{12\cdot 12}\cdot\frac{15\cdot 17}{16\cdot 16}\cdot\frac{19\cdot 21}{20\cdot 20}\cdots$$

p=2: $$\frac{2}{\pi}=\sqrt{\frac{1}{2}}\cdot\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\cdot\frac{7\cdot 9}{8\cdot 8}\cdot\frac{15\cdot 17}{16\cdot 16}\cdot\frac{23\cdot 25}{24\cdot 24}\cdot\frac{31\cdot 33}{32\cdot 32}\cdots$$

p=3: $$\frac{2}{\pi}=\sqrt{\frac{1}{2}}\cdot\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdot\frac{15\cdot 17}{16\cdot 16}\cdot\frac{31\cdot 33}{32\cdot 32}\cdot\frac{47\cdot 49}{48\cdot 48}\cdot\frac{63\cdot 65}{64\cdot 64}\cdots$$

...

$p\to\infty$: $$\frac{2}{\pi}=\sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdots$$ (Vieta's original product).

An examination of the above special cases of the VWP shows that each time we increase p by one, we increase the Vieta's product by one new radical factor, and remove alternate factors from the Wallis-like product. The author unexpectedly discovered the VWP while trying to derive Vieta's product (1.1).

3. The derivation of the general Vieta-Wallis product.

To derive the GVWP (1.6) we start by applying the double angle formula for the sine function p times to obtain

$$\sin\theta = 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}$$

$$= 2^2\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\sin\frac{\theta}{2^2}$$

$$= 2^3\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}\sin\frac{\theta}{2^3}$$

...

$$\sin\theta = 2^p\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}\cdots\cos\frac{\theta}{2^p}\sin\frac{\theta}{2^p} \tag{3.1}$$

Next we use the infinite product (1.5) for the sine function, (valid for all z), with $z = \theta/2^p$ to replace the last factor in (3.1). We get after dividing by θ

$$\frac{\sin\theta}{\theta} = \cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}\cdots\cos\frac{\theta}{2^p}\exp\left(\frac{\theta}{2^p\pi}\log\frac{M}{N}\right)\times$$
$$\prod_{k=0}^{\infty}\left(\frac{2^p\pi(kM+1)-\theta}{2^p\pi(kM+1)}\right)\left(\frac{2^p\pi(kM+2)-\theta}{2^p\pi(kM+2)}\right)\cdots\left(\frac{2^p\pi(kM+M)-\theta}{2^p\pi(kM+M)}\right)\times . \tag{3.2}$$
$$\left(\frac{2^p\pi(kN+1)+\theta}{2^p\pi(kN+1)}\right)\left(\frac{2^p\pi(kN+2)+\theta}{2^p\pi(kN+2)}\right)\cdots\left(\frac{2^p\pi(kN+N)+\theta}{2^p\pi(kN+N)}\right)$$

We evaluate each of the cosine factors in (3.2) in terms of $\cos\theta$ by repeated use of the half-angle formula for the cosine. (Here we will assume $-\pi/2 \le \theta \le \pi/2$ so that the cosines are never negative.)

$$\cos\frac{\theta}{2}=\sqrt{\frac{1}{2}+\frac{1}{2}\cos\theta}$$

$$\cos\frac{\theta}{2^2}=\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\cos\theta}}$$

...

(3.3)
$$\cos\frac{\theta}{2^p}=\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\cdots+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\cos\theta}}}}$$
$$(p\ \ radicals)$$

Combining (3.3) with (3.2) we obtain

(3.4)
$$\frac{\sin\theta}{\theta}\exp\left(\frac{\theta}{2^p\pi}\log\frac{N}{M}\right)=\prod_{n=1}^{p}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\cdots+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\cos\theta}}}}\times$$
$$<------(n\ \ radicals)------>$$
$$\prod_{k=0}^{\infty}\left(\frac{2^p\pi(kM+1)-\theta}{2^p\pi(kM+1)}\right)\left(\frac{2^p\pi(kM+2)-\theta}{2^p\pi(kM+2)}\right)\cdots\left(\frac{2^p\pi(kM+M)-\theta}{2^p\pi(kM+M)}\right)\times$$
$$\left(\frac{2^p\pi(kN+1)+\theta}{2^p\pi(kN+1)}\right)\left(\frac{2^p\pi(kN+2)+\theta}{2^p\pi(kN+2)}\right)\cdots\left(\frac{2^p\pi(kN+N)+\theta}{2^p\pi(kN+N)}\right)$$

If we set $\theta=\pi/2$ in (3.4), and $M=N=1$, and simplify we obtain the original VWP (1.3). If we let $\theta=\frac{r\pi}{s}$, then (3.4) becomes the new GVWP

(3.5)
$$\frac{s\sin(r\pi/s)}{r\pi}\exp\left(\frac{r}{2^p s}\log\frac{N}{M}\right)=\prod_{n=1}^{p}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\cdots+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\cos(r\pi/s)}}}}\times$$
$$<------(n\ \ radicals)------>$$
$$\prod_{k=0}^{\infty}\left(\frac{2^p s(kM+1)-r}{2^p s(kM+1)}\right)\left(\frac{2^p s(kM+2)-r}{2^p s(kM+2)}\right)\cdots\left(\frac{2^p s(kM+M)-r}{2^p s(kM+M)}\right)\times$$
$$\left(\frac{2^p s(kN+1)+r}{2^p s(kN+1)}\right)\left(\frac{2^p s(kN+2)+r}{2^p s(kN+2)}\right)\cdots\left(\frac{2^p s(kN+N)+r}{2^p s(kN+N)}\right).$$

4. Generalized Wallis like products, the case $p = 0$ in the GVWP.

If we set $p = 0$ in the GVWP (3.5) we get a product without radicals

$$\frac{s\sin(r\pi/s)}{r\pi}\exp\left(\frac{r}{s}\log\frac{N}{M}\right)= \prod_{k=0}^{\infty}\left(\frac{s(kM+1)-r}{s(kM+1)}\right)\left(\frac{s(kM+2)-r}{s(kM+2)}\right)\cdots\left(\frac{s(kM+M)-r}{s(kM+M)}\right)\times \left(\frac{s(kN+1)+r}{s(kN+1)}\right)\left(\frac{s(kN+2)+r}{s(kN+2)}\right)\cdots\left(\frac{s(kN+N)+r}{s(kN+N)}\right).$$

These are all variations of the classical Wallis's product (1.2). The results of selecting special values for M, N, r, and s are shown in the following table.

M	***N***	$\frac{r\pi}{s}$	**Wallis-like Infinite Products**
1	1	$\frac{\pi}{2}$	$\frac{2}{\pi}=\left(\frac{1\cdot 3}{2\cdot 2}\right)\left(\frac{3\cdot 5}{4\cdot 4}\right)\left(\frac{5\cdot 7}{6\cdot 6}\right)\left(\frac{7\cdot 9}{8\cdot 8}\right)\cdots$ (Original Wallis's product)
1	1	$\frac{\pi}{3}$	$\frac{3\sqrt{3}}{2\pi}=\left(\frac{2\cdot 4}{3\cdot 3}\right)\left(\frac{5\cdot 7}{6\cdot 6}\right)\left(\frac{8\cdot 10}{9\cdot 9}\right)\left(\frac{11\cdot 13}{12\cdot 12}\right)\cdots$
2	1	$\frac{\pi}{2}$	$\frac{\sqrt{2}}{\pi}=\left(\frac{1\cdot 3}{2\cdot 4}\cdot\frac{3}{2}\right)\left(\frac{5\cdot 7}{6\cdot 8}\cdot\frac{5}{4}\right)\left(\frac{9\cdot 11}{10\cdot 12}\cdot\frac{7}{6}\right)\left(\frac{13\cdot 15}{14\cdot 16}\cdot\frac{9}{8}\right)\cdots$
2	1	$\frac{\pi}{4}$	$\frac{2\sqrt[4]{2}}{\pi}=\left(\frac{3\cdot 7}{4\cdot 8}\cdot\frac{5}{4}\right)\left(\frac{11\cdot 15}{12\cdot 16}\cdot\frac{9}{8}\right)\left(\frac{19\cdot 23}{20\cdot 24}\cdot\frac{13}{12}\right)\left(\frac{27\cdot 31}{28\cdot 32}\cdot\frac{17}{16}\right)\cdots$
2	1	$\frac{\pi}{3}$	$\frac{3\sqrt{3}}{2\sqrt[3]{2}\,\pi}=\left(\frac{2\cdot 5}{3\cdot 6}\cdot\frac{4}{3}\right)\left(\frac{8\cdot 11}{9\cdot 12}\cdot\frac{7}{6}\right)\left(\frac{14\cdot 17}{15\cdot 18}\cdot\frac{10}{9}\right)\left(\frac{20\cdot 23}{21\cdot 24}\cdot\frac{13}{12}\right)\cdots$
2	1	$\frac{2\pi}{3}$	$\frac{3\sqrt{3}}{4\sqrt[3]{4}\,\pi}=\left(\frac{1\cdot 4}{3\cdot 6}\cdot\frac{5}{3}\right)\left(\frac{7\cdot 10}{9\cdot 12}\cdot\frac{8}{6}\right)\left(\frac{13\cdot 16}{15\cdot 18}\cdot\frac{11}{9}\right)\left(\frac{19\cdot 22}{21\cdot 24}\cdot\frac{14}{12}\right)\cdots$

2	1	$\frac{r\pi}{s}$	$\frac{s\sin(r\pi/s)}{2^{r/s}r\pi}=\left(\frac{s-r}{s}\cdot\frac{2s-r}{2s}\cdot\frac{s+r}{s}\right)\left(\frac{3s-r}{3s}\cdot\frac{4s-r}{4s}\cdot\frac{2s+r}{2s}\right)\left(\frac{5s-r}{5s}\cdot\frac{6s-r}{6s}\cdot\frac{3s+r}{3s}\right)\cdots$
3	1	$\frac{\pi}{2}$	$\frac{\sqrt{2}}{\sqrt{3}\,\pi}=\left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\frac{3}{2}\right)\left(\frac{7\cdot9\cdot11}{8\cdot10\cdot12}\cdot\frac{5}{4}\right)\left(\frac{13\cdot15\cdot17}{14\cdot16\cdot18}\cdot\frac{7}{6}\right)\left(\frac{19\cdot21\cdot23}{20\cdot22\cdot24}\cdot\frac{9}{8}\right)\cdots$
3	2	$\frac{\pi}{2}$	$\frac{2}{\sqrt{3}\,\pi}=\left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\frac{3\cdot5}{2\cdot4}\right)\left(\frac{7\cdot9\cdot11}{8\cdot10\cdot12}\cdot\frac{7\cdot9}{6\cdot8}\right)\left(\frac{13\cdot15\cdot17}{14\cdot16\cdot18}\cdot\frac{11\cdot13}{10\cdot12}\right)\cdots$

All of the above can also be obtained directly from the unusual product expansion for π (1.5) as was shown in [6].

5. Special cases of the GVWP with *M* = *N* = 1.

If we let $M = N = 1$ in (3.5) we get

(5.1)
$$\frac{s\sin(r\pi/s)}{r\pi}=\prod_{n=1}^{p}\underbrace{\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\cdots+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\cos(r\pi/s)}}}}}_{<------(n\ \ radicals)------>}\times \prod_{k=1}^{\infty}\left(\frac{2^p sk-r}{2^p sk}\right)\left(\frac{2^p sk+r}{2^p sk}\right).$$

Now we try various values of r and s in (5.1) to examine interesting special cases of the GVWP. Our results (which appeared earlier in [5]), are shown in the next table.

$\frac{r\pi}{s}$	**Vieta Wallis Product**
$\frac{\pi}{3}$	$\frac{3\sqrt{3}}{2\pi}=\prod_{n=1}^{p}\underbrace{\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\cdots+\frac{1}{2}\sqrt{\frac{1}{2}+\left(\frac{1}{2}\right)^2}}}}}_{\longleftarrow--\ n\,radicals\ ----\longrightarrow}\ \prod_{n=1}^{\infty}\left(\frac{3\cdot2^p n-1}{3\cdot2^p n}\cdot\frac{3\cdot2^p n+1}{3\cdot2^p n}\right)$

$\frac{\pi}{4}$	$\frac{2\sqrt{2}}{\pi}=\prod_{n=1}^{p}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\cdots+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\left(\frac{\sqrt{2}}{2}\right)}}}}\ \prod_{n=1}^{\infty}\left(\frac{2^{p+2}n-1}{2^{p+2}n}\cdot\frac{2^{p+2}n+1}{2^{p+2}n}\right)$ $\longleftarrow -- n\,radicals ---- \longrightarrow$
$\frac{\pi}{5}$	$\frac{5\sqrt{5-\sqrt{5}}}{2\sqrt{2}\pi}=\prod_{n=1}^{p}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\cdots+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{8}\left(1+\sqrt{5}\right)}}}}\ \prod_{n=1}^{\infty}\left(\frac{5\cdot 2^{p}n-1}{5\cdot 2^{p}n}\cdot\frac{5\cdot 2^{p}n+1}{5\cdot 2^{p}n}\right)$ $\longleftarrow -- n\,radicals ---- \longrightarrow$
$\frac{\pi}{6}$	$\frac{3}{\pi}=\prod_{n=1}^{p}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\cdots+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)}}}}\ \prod_{n=1}^{\infty}\left(\frac{3\cdot 2^{p+1}n-1}{3\cdot 2^{p+1}n}\cdot\frac{3\cdot 2^{p+1}n+1}{3\cdot 2^{p+1}n}\right)$ $\longleftarrow -- n\,radicals ---- \longrightarrow$
$\frac{\pi}{10}$	$\frac{5\left(\sqrt{5}-1\right)}{2\pi}=\prod_{n=1}^{p}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\cdots+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\left(5+\sqrt{5}\right)}}}}\ \prod_{n=1}^{\infty}\left(\frac{5\cdot 2^{p+1}n-1}{5\cdot 2^{p+1}n}\cdot\frac{5\cdot 2^{p+1}n+1}{5\cdot 2^{p+1}n}\right)$ $\longleftarrow -- n\,radicals ---- \longrightarrow$
$\frac{\pi}{12}$	$\frac{3\sqrt{2}\left(\sqrt{3}-1\right)}{\pi}=\prod_{n=1}^{p}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\cdots+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\left(\frac{1+\sqrt{3}}{2\sqrt{2}}\right)}}}}\ \prod_{n=1}^{\infty}\left(\frac{3\cdot 2^{p+2}n-1}{3\cdot 2^{p+2}n}\cdot\frac{3\cdot 2^{p+2}n+1}{3\cdot 2^{p+2}n}\right)$ $\longleftarrow -- n\,radicals ---- \longrightarrow$

6. New special cases of the GVWP.

In this final section we look at a few special cases of the GVWP of a general type that, to the best of our knowledge, have not been seen before.

Let us begin with the case for $M = 2$ and $N = 1$ in the GVWP (3.5). We have

(6.1)
$$\frac{s\sin(r\pi/s)}{2^{\left(\frac{r}{2^p s}\right)}r\pi}=\prod_{n=1}^{p}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\cdots+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\cos(r\pi/s)}}}}\times$$
$$<---------(n\ \ radicals)------->$$
$$\prod_{k=0}^{\infty}\left\{\left(\frac{2^p s(2k+1)-r}{2^p s(2k+1)}\right)\left(\frac{2^p s(2k+2)-r}{2^p s(2k+2)}\right)\right\}\left(\frac{2^p s(k+1)+r}{2^p s(k+1)}\right).$$

The special case of (6.1) using $M=2,\ N=1,\ r=1,\ s=2$ is

$$\frac{2}{2^{\left(2^{-p-1}\right)}\pi}=\prod_{n=1}^{p}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\cdots+\frac{1}{2}\sqrt{\frac{1}{2}}}}}\times$$
$$<----(n\ \ radicals)---->$$
$$\prod_{k=0}^{\infty}\left\{\left(\frac{2^{p+1}(2k+1)-1}{2^{p+1}(2k+1)}\right)\left(\frac{2^{p+1}(2k+2)-1}{2^{p+1}(2k+2)}\right)\right\}\left(\frac{2^{p+1}(k+1)+1}{2^{p+1}(k+1)}\right).$$

Next we examine the general case of our GVWP (3.5) with $M = 3$ and $N = 1$. We have

(6.2)
$$\frac{s\sin(r\pi/s)}{3^{\left(\frac{r}{2^p s}\right)}r\pi}=\prod_{n=1}^{p}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\cdots+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\cos(r\pi/s)}}}}\times$$
$$<---------(n\ \ radicals)------->$$
$$\prod_{k=0}^{\infty}\left\{\left(\frac{2^p s(3k+1)-r}{2^p s(3k+1)}\right)\left(\frac{2^p s(3k+2)-r}{2^p s(3k+2)}\right)\left(\frac{2^p s(3k+3)-r}{2^p s(3k+3)}\right)\right\}\left(\frac{2^p s(k+1)+r}{2^p s(k+1)}\right).$$

The special case $M=3,\ N=1,\ r=1,\ s=2$ gives us

$$\frac{2}{3^{\left(2^{-p-1}\right)}\pi}=\prod_{n=1}^{p}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\cdots+\frac{1}{2}\sqrt{\frac{1}{2}}}}}\times$$
$$<----(n\ \ radicals)---->$$
$$\prod_{k=0}^{\infty}\left\{\left(\frac{2^{p+1}(3k+1)-1}{2^{p+1}(3k+1)}\right)\left(\frac{2^{p+1}(3k+2)-1}{2^{p+1}(3k+2)}\right)\left(\frac{2^{p+1}(3k+3)-1}{2^{p+1}(3k+3)}\right)\right\}\left(\frac{2^{p+1}(k+1)+1}{2^{p+1}(k+1)}\right).$$

Finally we examine the general case of the GVWP for $M = 3, N = 2$:

$$\frac{s\sin(r\pi/s)}{(3/2)^{\left(\frac{r}{2^p s}\right)} r\pi} = \prod_{n=1}^{p} \sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\cdots+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\cos(r\pi/s)}}}} \times$$

$$<--------(n \ \ radicals)------->$$

$$\prod_{k=0}^{\infty}\left\{\left(\frac{2^p s(3k+1)-r}{2^p s(3k+1)}\right)\left(\frac{2^p s(3k+2)-r}{2^p s(3k+2)}\right)\left(\frac{2^p s(3k+3)-r}{2^p s(3k+3)}\right)\right\}\left(\frac{2^p s(2k+1)+r}{2^p s(2k+1)}\right)\left(\frac{2^p s(2k+2)+r}{2^p s(2k+2)}\right).$$

The special case case $M = 3$, $N = 2$, $r = 1$, $s = 2$ is

$$\frac{2}{(3/2)^{\left(2^{-p-1}\right)}\pi} = \prod_{n=1}^{p} \sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\cdots+\frac{1}{2}\sqrt{\frac{1}{2}}}}} \times$$

$$<----(n \ \ radicals)---->$$

$$\prod_{k=0}^{\infty}\left\{\left(\frac{2^{p+1}(3k+1)-1}{2^{p+1}(3k+1)}\right)\left(\frac{2^{p+1}(3k+2)-1}{2^{p+1}(3k+2)}\right)\left(\frac{2^{p+1}(3k+3)-1}{2^{p+1}(3k+3)}\right)\right\}\left(\frac{2^{p+1}(2k+1)+1}{2^{p+1}(2k+1)}\right)\left(\frac{2^{p+1}(2k+2)+1}{2^{p+1}(2k+2)}\right).$$

This completes our investigation of the general Vieta-Wallis product.

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