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GEOMETRIC CONSTRUCTION OF PYTHAGOREAN TRIANGLES

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A right triangle with legs *x* and *y* and hypotenuse *z* in which *x, y* and *z* are all positive integers is called a *Pythagorean triangle* (PT) and the triple denoted by $[x, y, z]$ is a *Pythagorean triple.* If *x, y* and *z* are all relatively prime, (gcd is 1), then we call the triangle a *primitive Pythagorean triangle* (PPT) and the triple a *primitive Pythagorean triple.* We assume it known [1], that for any two positive integers *m* and *n*, with $n < m$, the three numbers $x = 2mn$, $y = m^2 - n^2$, and $z = m^2 + n^2$ form a Pythagorean triple. We say that the Pythagorean triple $[x, y, z]$ is generated by the parameters *m* and *n*. For example, when $m = 2$ and $n = 1$ we get the triple $[4,3,5]$, and when $m = 3$ and $n = 2$ we get the triple $[12,5,13]$. The triple is not always primitive, for when $m = 3$ and $n = 1$, we get $[6,8,10]$ with the common factor 2. The triple is primitive if *m* and *n* are relatively prime and not both odd. All PPTs can be generated this way.

It is the purpose of this short note to show how the PT can be constructed from the parameters *m* and *n*. We refer to Figure 1. First we construct the horizontal line *AB* of length $m + n$. Next we erect a perpendicular line *BD*. Locate the point *I* at the corner of the square of side *n* as shown. The line *AI* makes angle $\alpha/2$ with *AB*. Next double the angle $\alpha/2$ forming the line *AC*.

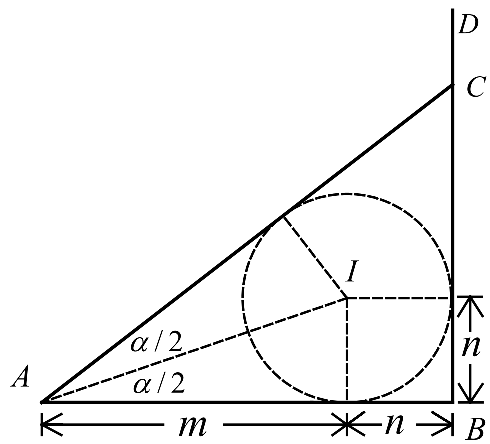

Figure 1

Let us now examine the lengths of the sides of the triangle *ABC*. Side *AB* has length.

(1) $$AB = m + n\,.$$

We calculate the length of *BC* from $BC = (m+n)\tan\alpha = (m+n)\dfrac{2\tan\alpha/2}{1-\tan^2\alpha/2}$. But from Figure 1 we see that $\tan\alpha/2 = \dfrac{n}{m}$, so $BC = (m+n)\dfrac{2mn}{m^2-n^2}$ and we get

(2) $$BC = \frac{2mn}{m-n}\,.$$

Using (1), (2) and $AC^2 = AB^2 + BC^2$, we easily calculate

(3) $$AC = \frac{m^2 + n^2}{m - n}.$$

Thus if *m* and *n* are positive integers, then the triangle *ABC* has rational sides. Multiplying each of the sides by the factor $m - n$ produces a new similar Pythagorean triangle with sides $m^2 - n^2$, $2mn$ and $m^2 + n^2$ as required.

We get more from our construction. The *incircle* of a triangle is that circle which just touches all three sides of the triangle. Its center (*incenter*) is at the point where the angle bisectors of the triangle meet. Its radius is called the *inradius.* The circle shown in Figure 1 is the incircle of the triangle, the point *I* is the incenter and its inradius is *n*. After magnifying the triangle in Figure 1 by the factor $m - n$ to get the PT, we see that the inradius becomes

(4) $$r = n(m - n).$$

From (4) we see that the inradius of a PT is always an integer.

Another fact concerning PTs is evident from the construction. Notice that $\tan\alpha / 2 = n / m$, and thus the tangents of the half angles of a PT are always rational numbers.

Reference

[1] Ore, Oystein, *Number Theory and Its History*, (originally published by McGraw-Hill in 1948), Dover Publication, New York, NY, (1976), pp. 165-170.

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