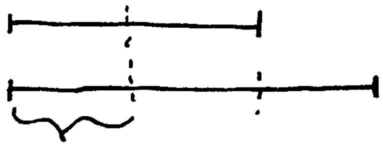
- Most important achievement of Pythagorean school in its influence on the number concept was the discovery of the "irrational"

- Intuitively they felt that every two line segments had a common measure,

ex

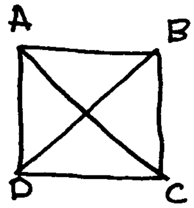


$\frac{2}{3}$

segment fits in both lengths a whole number of times

START 4

- Oldest known proof that $\sqrt{2}$ is irrational is in Euclid's Elements - book ten



Assume δ is common measure of AC and AB

$AC = m\delta$, $AB = n\delta$

m & n integers

$$\frac{AC}{AB} = \frac{m}{n}$$

Assume common factors of m & n cancelled

$$\frac{(AC)^2}{(AB)^2} = \frac{m^2}{n^2}$$

But $(AB)^2 + (AB)^2 = (AC)^2$

$$2(AB)^2 = (AC)^2$$

$$2 = \frac{(AC)^2}{(AB)^2}$$

Thus $\frac{m^2}{n^2} = 2$, $m^2 = 2n^2$, m is even

Thus $m = 2k$

$$m^2 = 2n^2$$
$$(2k)^2 = 2n^2$$
$$4k^2 = 2n^2$$
$$2k^2 = n^2$$

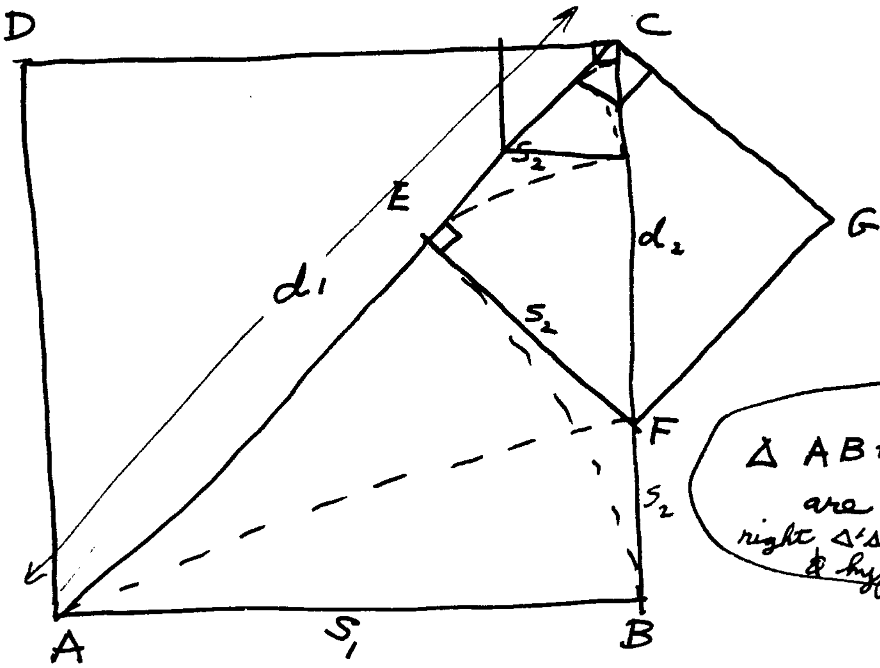
Thus n is even $\leftarrow$ contradiction since both m and n can't be even. (common factors were cancelled)

Babylonians had approximated $\sqrt{2}$ as

$$\sqrt{2} \cong 1;\ 24,\ 51,\ 10 = 1 + \frac{24}{60} + \frac{51}{60^2} + \frac{10}{60^3}$$
$$\cong 1.414213$$

This close to our value $\sqrt{2} = 1.414213562\ldots$

Older proof that diagonal & side of a square are incommensurable.



Δ ABF & ΔAEF are congruent right Δ's with same leg & hypotenus

$$s_2 = d_1 - s_1, \qquad d_2 = s_1 - \overline{FB}$$
$$d_2 = s_1 - s_2$$

$$s_n = d_{n-1} - s_{n-1}, \qquad d_n = s_{n-1} - s_n$$

Assume ~~Suppose~~ $s_1 = M_1\delta$, $d_1 = N_1\delta$

Where δ is the common measure.

$$s_2 = d_1 - s_1 = N_1\delta - M_1\delta = (N_1 - M_1)\delta = M_2\delta$$
$$d_2 = s_1 - s_2 = M_1\delta - M_2\delta = (M_1 - M_2)\delta = N_2\delta$$

$$M_1 > M_2 > M_3 > \cdots \geq 1$$
$$N_1 > N_2 > N_3 > \cdots \geq 1$$

these are impossible since there are only a finite no. of integers between M_1 and 1.

- irrationals challenged the philosophy that number is the essence of all things.

- EUDOXUS OF CNIDOS (408 – 355 BC) resolved the problem by ignoring the idea of number and using only geometric lengths.
  - Arithmetic theory of numbers is renounced.
  - Geometry is the more general science and is the basis of rigorous math for the next 2000 years.

HOMEWORK
Page 125; 8 and 17

THREE CONSTRUCTION PROBLEMS OF ANTIQUITY

(1) Squaring the circle
- Find a square whose area equals that of a given circle.

(2) Duplicating the cube.
- Find the side of a cube having double the volume of a given cube

(3) Trisecting the angle

(429 - 348 BC)

- Plato required "RULER & COMPASS ONLY"

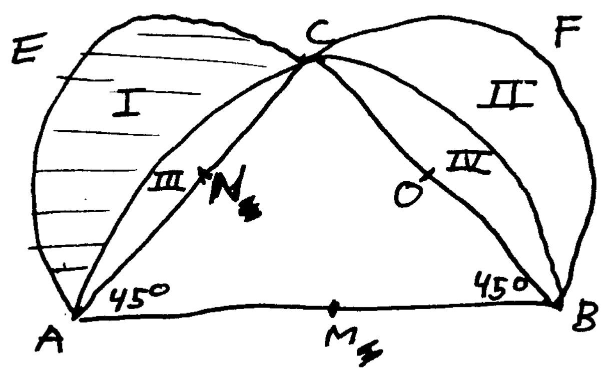
    1. With ruler a line can be drawn between two given points
    2. With compass, a circle with a given center and radius can be drawn.

- Not until the 19th century were these constructions proved to be impossible.

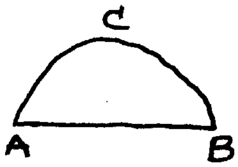
- Hippocrates of Chios (460 - 380 BC)
  (not father of medicine)

  Showed that certain plane regions with curved boundaries are squarable.

  - Consider the lune I



Arc ACB is a semi circle with center M
" AEC " " " N
" CFB O

Area 

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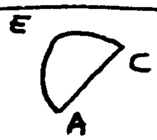
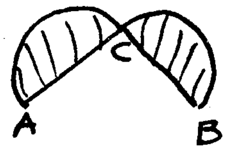
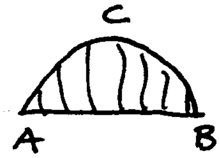
Area 

$$= \frac{\overline{AB}^2}{\overline{AC}^2} = 2$$

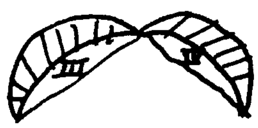
THUS



$=$



Now subtract area in regions III and IV



$=$

THUS $\quad 2\left(\text{Area of lune I}\right) = \frac{1}{2}(AC)^2$

or

$$\left(\text{Area of lune I}\right) = \left(\frac{AC}{2}\right)^2$$

And lune I has been squared!

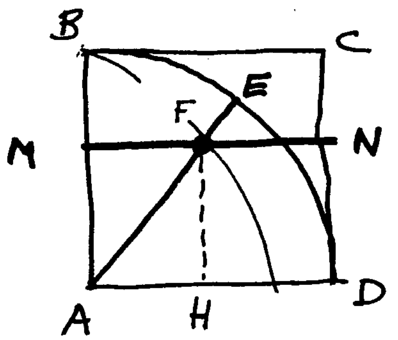
- For 2000 yrs no one could trisect the angle in 1837 Pierre Wantzel (1814–1848) gave the first rigorous proof that it's impossible.

THE QUADRATRIX OF HIPPIAS

Hippias of Elis (460 BC–)
invented a curve to trisect angles.

- he was a sophist
  - taught to argue well both sides of every argument
  - Plato made fun of him in one of his dialogues

QUADRATRIX



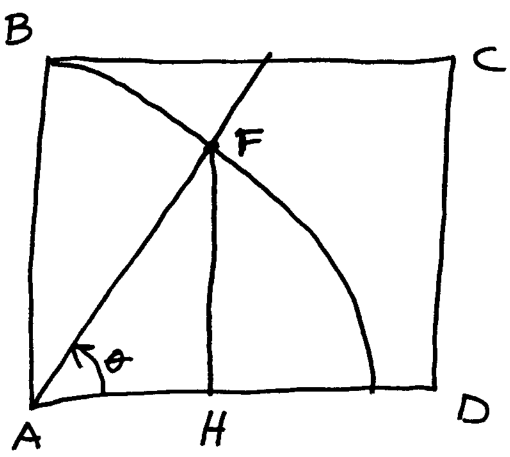
radius AE moves from AB to AD with constant velocity in time T

horizontal line MN moves from BC to AD with constant velocity in time T

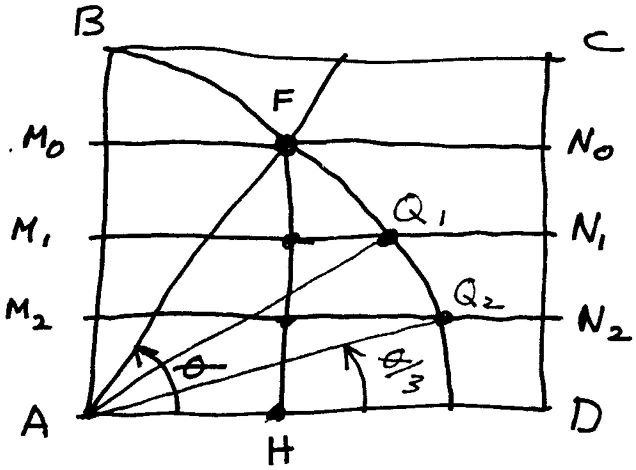
Locus of their intersections F is the desired curve.

START 5

GIVEN THE ANGLE θ TO TRISECT, LAY IT OUT ON THE QUADRATRIX AS $\angle DAF$



NEXT TRISECT $\overline{FH}$ AND DRAW HORRIZONALS THRU THESE POINTS $\overline{M_0 N_0}$, $\overline{M_1 N_1}$, $\overline{M_2 N_2}$



NEXT DRAW THE RADII $\overline{AQ_1}$ & $\overline{AQ_2}$ SHOWN

THE TIME FOR THE HORRIZONTAL $\overline{M_0 N_0}$ to drop to $\overline{M_1 N_1}$ = time to go on to $\overline{M_2 N_2}$ = time to reach bottom $\overline{AD}$, Also the time for radius $\overline{AF}$ to swing to $\overline{AQ_1}$ = time for it to go on to $\overline{AQ_2}$ = time to finish at $\overline{AD}$, SINCE THE RADIUS MOVIES AT CONSTANT ANGULAR VELOCITY, THE ANGLES

$\angle FAQ_1 = \angle Q_1 A Q_2 = \angle Q_2 A D$.

THUS θ HAS BEEN trisected

ABOUT 387 BC Plato founded an academy in the suburbs of Athens that would last 900 years

- intellectual center of Greece
- closed in 529 ad by Christian Emperor Justinian's orders
- over it's gates was a sign

"Let no man ignorant of geometry enter here"

- Plato gave math a favored place in the curriculum
- it is not known if he made any original contributions

- About 300 BC Ptolemy I founded a rival academy the "MUSEUM" in Alexandria
  - many math. went there

HOMEWORK

page 148; 2, 6, 7

Chapter 4

The First Alexandrian School: Euclid

Euclid & The Elements

- End of 4th cent. BC Math shifts from Greece to Egypt

- "Museum" is built at Alexandria
  - forerunner of modern university
  - fellows of Museum lived at King's expense

- In all history only span from 1600 - 1850 (Kepler to Gauss) compares with productivity of 300 - 100 BC.

- Great Alexandrian Library is established almost simultaneously with "Museum"

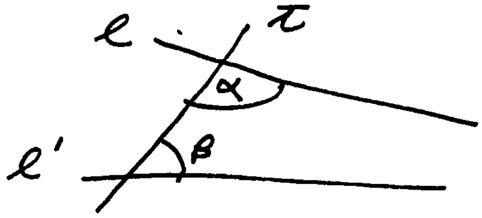
- A rival library was at Pergamon in western Asia Minor

- Euclid
  - unified collection of isolated discoveries
    - single deductive system
    - initial postulates
    - " definitions
    - " axioms
  - only the Bible has been more important in Western Thought & education than the Elements of Euclid
  - Euclid was author of at least ten other works

EUCLIDEAN GEOMETRY

~~The~~ Euclid's greatness
- not in original contributions
- in organizing vast body of facts
- introduced 5 postulates (or axioms)
- 23 definitions

<u>Postulate 5</u> (parallel postulate)



If $\alpha + \beta < 180°$
then l & l' intersect
on this side.

- this is one of the most famous and controversial statements in math history

- it was thought that Post. 5 should be provable

- Mathematicians tried to derive it from <u>the</u> first 4 ~~until~~ into the 19th century

    - these efforts led to discovery of "Non-Euclidean Geo."

- Numerous flaws have been found in the Elements

    - no undefined terms

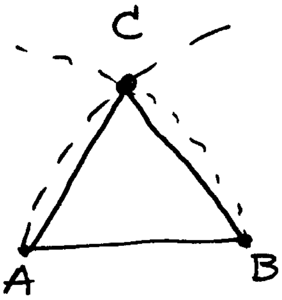
        - tried to define all technical vocabulary

    - axioms are woefully inadequate

Proposition 1

Given $\overline{AB}$, there is an equilateral triangle with $\overline{AB}$ as one side.

Proof:



Draw circles of radius $\overline{AB}$ intersecting at C

"How do we know that these circles will intersect?" This is not in the axioms!

David Hilbert (1862–1943)

- German
- Grundlagen der Geometrie
  Foundations of Geometry
  - most influential treatise on geo. in modern times
  - Euclidean geo
    - 6 undefined terms
    - 21 postulates

Supplement on "undefined terms"
-from "Patterns in the Sand"
by Bosstick & Cable - 1971 - Glencoe Press

- Sample Axiomatic System

1. There are at least two abas
2. For every two abas there is one and only one daba containing them.
3. Every daba contains at least one aba.
4. For every daba there is an aba not on that daba.

- here aba & daba are undefined terms
  - you can replace
    aba by airplane or point
    daba by runway or line

(airplane)

Theorem Every aba is on at least two dabas. (runways)

Proof

1. Let A denote an aba.
2. There is another aba, call it B – Axiom 1
3. There is a daba containing A & B – Axiom 2
4. There is another aba, call it C. – Axiom 4
5. There is another daba containing A & C – Axiom 2
6. Thus A is on two dabas. – END

Homework

Prove that there are at least 3 abas.

START CLASS #6

See p 580 at bottom & 581

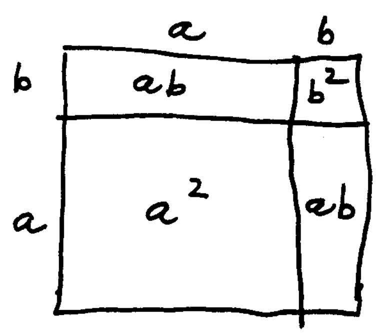
- KURT GÖDEL
  - Univ. of Vienna – 1931
  - in any system of axioms rich enough to cover arithmetic or Euclidean Geometry there would always be undecidable propositions
  - consistency of the axioms is one of the undecidable propositions!

THE QUADRATIC EQUATION

- In Elements algebraic problems are cast in geometric language,
- No adequate algebraic symbolism

We write $(a+b)^2 = a^2 + 2ab + b^2$

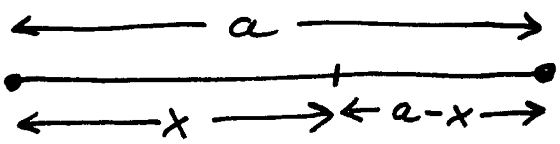
Euclid drew



$$(x+a)x = a^2$$

Kepler called this
"one of the two Jewels of Geometry"

-This divides the segment a into the
"golden section"



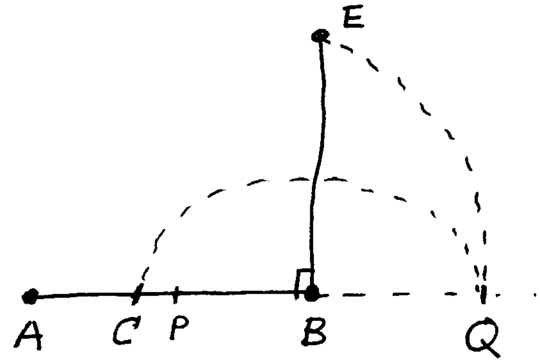
$$\frac{a}{x} = \frac{x}{a-x} \qquad x > a-x$$

$$x = \tfrac{1}{2}a(\sqrt{5}-1)$$

When $a=1$, $x = \frac{\sqrt{5}-1}{2}$ = reciprocal of the golden ratio

0.6180339

EUCLID'S CONSTRUCTION OF THE GOLDEN SECTION



(1) erect $\overline{BE} = \overline{AB}$

(2) FIND P = midpoint of AB

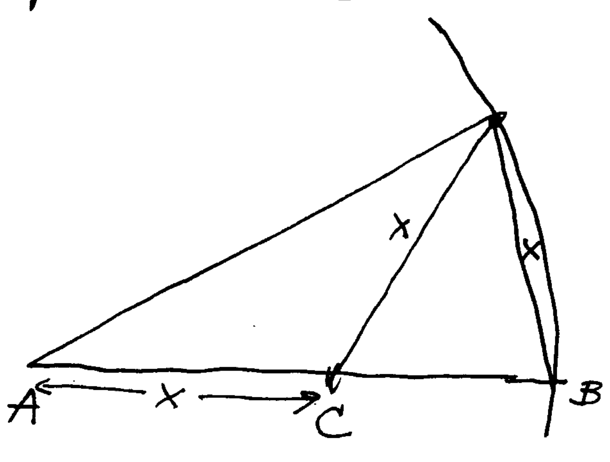
(3) WITH P AS CENTER DRAW ARC EQ, radius $\overline{PE}$

(4) WITH B AS CENTER, DRAW SEMICIRCLE OF RADIUS $\overline{BQ}$

(5) $\overline{BC}$ IS GOLDEN SECTION OF $\overline{AB}$

- Greeks were able to inscribe polygons in a circle of sides 3, 4, 5, 6, 8, 10, but not 7 sides.

- To find side length for 10 sided regular polygon, use golden section of the radius



- Western Europeans first learned algebra from works of

  Muhammed ibn Musa al-Khowarizmi (820 ad)

  - astronomer & math. from Bagdad

  - name algebra is from al-jabr, part of title of his treatise

    Hisab al-jabr w'al muqualbalah

    "The science of reunion and deduction"

  - In 12th century this book was translated into Latin

    "Algebrae et Almucabola"

$\boxed{\text{Homework} \quad \text{p. 186 \#3}}$

EUCLID'S NUMBER THEORY

Thm There are an infinity of primes.

Proof

Assume p is the largest prime.

Then form the number

$$N = \underbrace{(2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdots p)}_{\text{product of all primes}} + 1$$

Since $N > p$, N is not prime.

However no prime q divides N since

$$\frac{N}{q} = \underbrace{\frac{(2 \cdot 3 \cdot 5 \cdots \cancel{q} \cdots p)}{\cancel{q}}}_{\substack{\text{whole no.} \\ \text{since } q \text{ in} \\ \text{numerator}}} + \underbrace{\frac{1}{q}}_{\substack{\text{non zero} \\ \text{fraction}}}$$

Thus N is prime contradicting the assumption that p is the largest prime.

ERATOSTHENES (276–194 BC)

- studied at Plato's school in Athens
- called at age 30 to Alexandria to tutor Ptolemy II 's son
- became chief librarian at Museum last 40 years of his life
  - most prestigious position in Hellenistic world
- foremost scholar of his day
  - wrote on geography, philosophy, history, astronomy, math & literary criticism

Sieve of Eratosthenes (find all primes <100)

2 ~~3~~ 3 ~~4~~ 5 ~~6~~ 7 ~~8~~ ~~9~~ ~~10~~

11 ~~12~~ 13 ~~14~~ ~~15~~ ~~16~~ 17 ~~18~~ ~~19~~ ~~20~~

~~21~~ ~~22~~ 23 ~~24~~ ~~25~~ ~~26~~ ~~27~~ ~~28~~ ~~29~~ ~~30~~

⋮

~~100~~

First cross out all multiples of 2 (except 2)

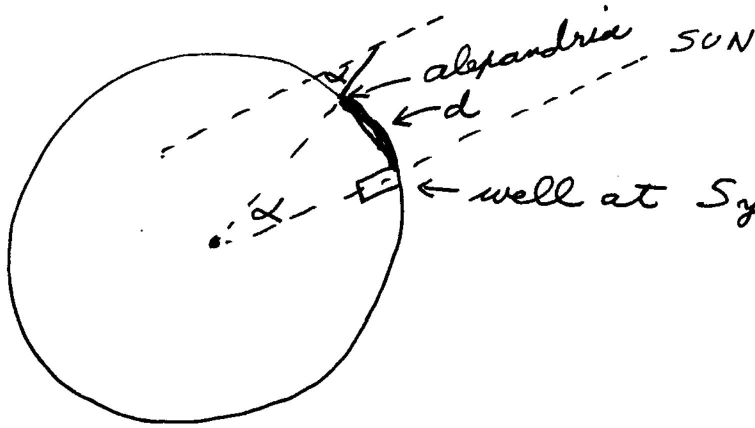
" 3 (" 3)

" 5 (" 5)

" 7 (" 7)

We can stop here since $11^2 > 100$

- best remembered for estimating earth's circumferen



$$\frac{\alpha}{360^\circ} = \frac{d}{\text{Circumference of earth}}$$

- got 24,662 miles (245 miles short)

Claudius Ptolemy (100 - 170 ad)

- did for astronomy what Euclid did for geo.
- wrote "Almagest"
    - supreme authority on astron. until Copernicus's "De Revolutionibus (1543)
    - chief flaw is earth centered