Fibonacci next demonstrated that x cannot be any of the Euclidean irrationals:

$$a \pm \sqrt{b}, \quad \sqrt{a} \pm \sqrt{b}, \quad \sqrt{a \pm \sqrt{b}},$$
$$\sqrt{\sqrt{a} \pm \sqrt{b}}, \quad a \,\&\, b \text{ rational}$$

Hence it cannot be constructed with straightedge & compass only.

- First indication that Greek ~~algebraic~~ geometric algebra is insufficient to describe all numbers

- gave a remarkably accurate estimate of real root:

1,3688081075

- Same problem appears in algebra of great Persian Poet Omar Khayyam (1050-1130)
  - solved by intersecting circle & hyperbola

START CLASS #10

THE FIBONACCI SEQUENCE

Let $F_1 = F_2 = 1$

$$F_n = F_{n-1} + F_{n-2} \qquad n \geq 3$$

$$F_3 = F_2 + F_1 = 1+1 = 2$$
$$F_4 = F_3 + F_2 = 2+1 = 3$$
$$F_5 = F_4 + F_3 = 3+2 = 5$$

8

13

21

34

55

89

144

THM No two consecutive Fibonacci numbers have a factor $d > 1$ in common.

Proof

(1) Suppose $d > 1$ divides F_n & F_{n+1}

(2) $F_{n+1} - F_n = F_{n-1}$ is also divisible by d

(3) $F_n - F_{n-1} = F_{n-2}$ ״

(4) $F_{n-1} - F_{n-2} = F_{n-3}$ ״

$\vdots$

$= F_1$ ״

But $F_1 = 1$ which cannot be divisible by $d > 1$. Thus a contradiction

PROVE THAT $\displaystyle\sum_{1}^{n} F_k = F_{n+2} - 1$

PROOF

$$
\begin{aligned}
F_1 &= \cancel{F_3} - F_2 \\
F_2 &= \cancel{F_4} - \cancel{F_3} \\
F_3 &= \cancel{F_5} - \cancel{F_4} \\
&\vdots \\
F_n &= F_{n+2} - \cancel{F_{n+1}}
\end{aligned}
$$

$$\sum_{1}^{n} F_k = F_{n+2} - F_2 \qquad \leftarrow \text{adding}$$

$$= F_{n+2} - 1$$

PROVE THAT $F_n^{\,2} = F_{n-1}\, F_{n+1} + (-1)^{n-1}$

PROOF

$$
\begin{aligned}
F_n^{\,2} - F_{n-1} F_{n+1} &= F_n\, F_n - F_{n-1} F_{n+1} \\
&= F_n \left(F_{n-1} + F_{n-2}\right) - F_{n-1} F_{n+1} \\
&= F_n F_{n-1} + F_n F_{n-2} - F_{n-1} F_{n+1} \\
&= F_n F_{n-1} - F_{n-1} F_{n+1} + F_n F_{n-2} \\
&= F_{n-1} \underbrace{\left(F_n - F_{n+1}\right)}_{-F_{n-1}} + F_n F_{n-2} \\
&= -F_{n-1}^{\,2} + F_n\, F_{n-2}
\end{aligned}
$$

$$F_n^2 - F_{n-1} F_{n+1} = (-1)\left(F_{n-1}^2 - F_n F_{n-2}\right)$$

But is the same as with subscripts diminished by one.

$$= (-1)^2 \left(F_{n-2}^2 - F_{n-1} F_{n-3}\right)$$

$$= (-1)^3 \left(F_{n-3} - F_{n-2} F_{n-4}\right)$$

$$\vdots$$

$$= (-1)^{n-2} \left(F_2 - F_3 F_1\right)$$

$$= (-1)^{n-2} \left(1 - 2 \cdot 1\right)$$

$$= (-1)^{n-2} (-1)$$

$$= (-1)^{n-1}$$

qed

A GEOMETRIC DECEPTION

Use $n = 2k$ in the above to get

$$F_{2k}^2 = F_{2k-1} \underbrace{F_{2k+1}}_{(F_{2k} + F_{2k-1})} - 1$$

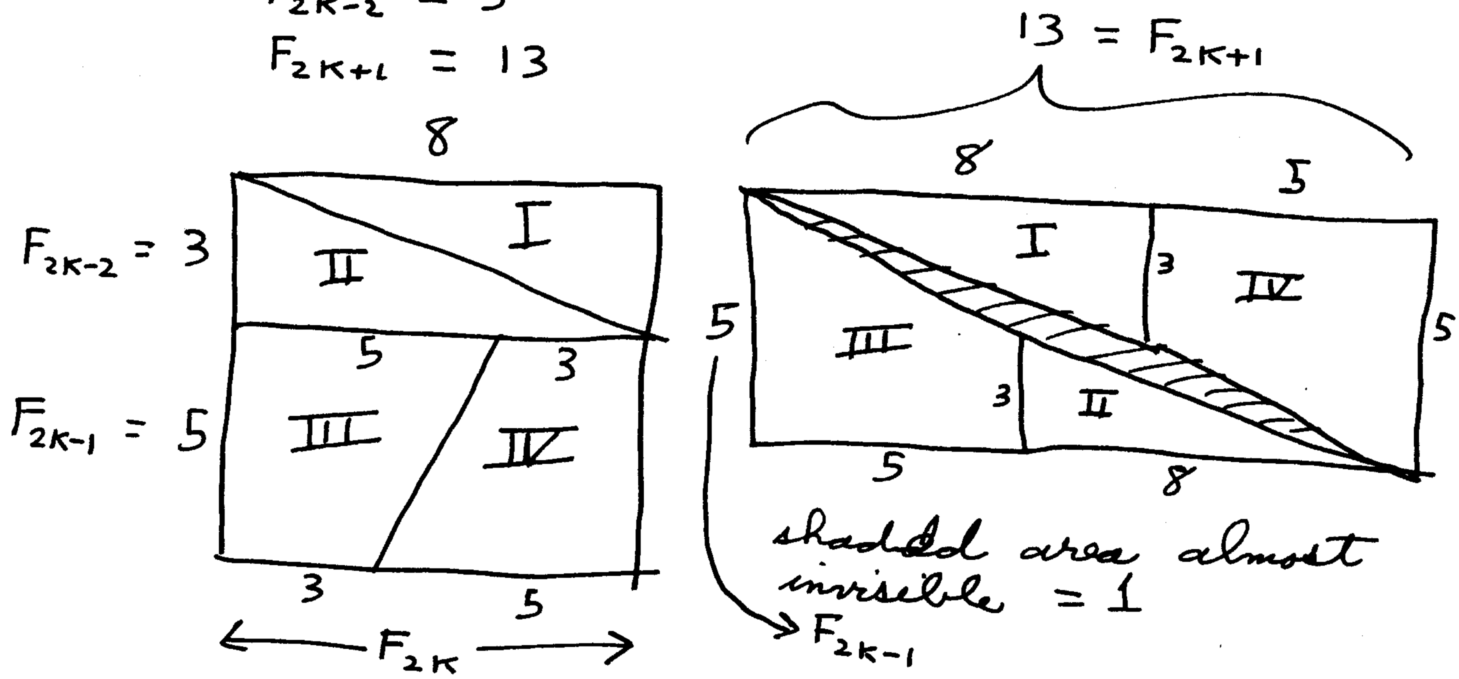
square of side $F_{2k-1} + F_{2k-2}$

Rectangle of sides F_{2k-1} and $F_{2k} + F_{2k-1}$

deceptive little -1 which shows that the square & rectangle are close but not exact

Ex. Take $k = 3$

$$F_{2k} = F_6 = 8$$
$$F_{2k-1} = 5$$
$$F_{2k-2} = 3$$
$$F_{2k+1} = 13$$



Connection between F_n and golden ratio.

$$u_n = \frac{F_{n+1}}{F_n}$$

$$u_1 = \frac{1}{1} = 1$$
$$u_2 = \frac{2}{1} = 2$$
$$u_3 = \frac{3}{2} = 1,5$$
$$u_4 = \frac{5}{3} = 1,66\ldots$$
$$u_5 = \frac{8}{5} = 1,60$$
$$u_6 = \frac{13}{8} = 1,625$$
$$u_7 = \frac{21}{13} = 1,615\ldots$$
$$u_8 = \frac{34}{21} = 1,619\ldots$$

$$\frac{F_{n+1}}{F_n} = \frac{F_n + F_{n-1}}{F_n} = 1 + \frac{F_{n-1}}{F_n}$$

$$= 1 + \frac{1}{\frac{F_n}{F_{n-1}}}$$

Let $\lim_{n \to \infty} u_n = \alpha$

$$\alpha = 1 + \frac{1}{\alpha}$$

$$\alpha^2 = \alpha + 1$$

$$\alpha^2 - \alpha - 1 = 0$$

$$\alpha = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$$\frac{1+\sqrt{5}}{2} = \text{golden ratio}$$

$$= 1.618033987\ldots$$

<u>Homework</u>

P. 282, 2, 4, 8

CHAP 7

THE CUBIC CONTROVERSY!

CARDAN & TARTAGLIA

EUROPE IN 14TH & 15TH CENTURIES

- IF 13th cent. is high pt. of medieval Europe, then 14th is lowest

- 13th cent. gave abundant promise for future

- 14th cent.
  - famine, plague, war
  - opened with heavy rainfalls
  - turned significantly colder
    - Little Ice Age
    - crop failure
    - famine
  - Black Death
    - bubonic plague
  - smoke of war hung over whole sad century
    - 100 yrs war
      - England vs France

- By 1450, war, plague & famine had tapered off
  - population increased
  - remarkable rebirth
    - called the Renaissance

- Printed Books
  - First were little concerned with math
  - "Treviso Arithmetic"
    - 1478
    - at Treviso (North of Venice)
    - 1st popular textbook
      - list of calculating rules
  - Before 1500 over 200 math books had been printed in Italy
    - Euclid's Elements
      - 1482

- Regiomontanus (Johannes Müller)
  - 1436 - 1476
  - one of earliest European scholars to use original Greek texts
  - actively translated original classical texts
    - Ptolemy's Almagest
  - wrote "De Triangulis Omnimodis" (On Triangles of all kinds)
    - used trigonometric fctns

- Fra Luca di Borga (Luca Pacioli 1445-1514)
  - Franciscan Friar
  - wrote "Summa de Arithmetica Geometria Proportioni et Proportionalita"
    - Venice 1494
    - most influential math book of its time
- Peter Abelard (1079-1142)
  see p. 295 at bottom
  helped found University of Paris

- Growth of Universities
  - in 12th & 13th centuries
  - older cathedral & monastery schools could not handle the numbers of students
  - Paris & Bologna
    - great "mother" universities

- cult of classics
  - belief that antiquity, both Greek & Latin offered a model of perfection
  - its literature could provide solutions to political, social & ethical problems.

- Renaissance
  - made little progress in science
  - opened the way for Scientific revolution of 1600's by recovering more of ancient learning

THE BATTLE OF THE SCHOLARS

- RENAISSANCE
  - great achievements in literature, painting, architecture
  - small " in math & science
  - humanists passion for the discovery, translation & circulation of ancient Greek texts
    - benefited math
      - manuscript collectors assemble in Italy an almost complete collection of Greek math writings

- In 1500's
  - advances in algebra
    - solve cubic
    - better symbolism
      - Francois Vieta 1540–1603
        - vowels for unknowns
        - consonants " knowns

$$3x^2 + 5x + 10 = 0$$

becomes

$$ax^2 + bx + c = 0$$

  - advances in arithmetic
  - advances in trigonometry
    - Rhaeticus (Georg Joachim 1514–1576)
      - worked out table of sines to 15 dec. places for every 10 sec of arc.

- Italian math of 1500's
  - del Ferro
  - Tartaglia
  - Cardan
  - Ferrari
  - Bombelli

  solve cubic & quartic
  - greatest algebraic feat since Babylonians 4000 yrs earlier

- Scipione del Ferro 1465–1526
  - University of Bologna
    solved $x^3 + px = q$

- math discoveries kept secret
  - to solve contest problems

- Nicolo Tartaglia (the stammerer) 1500–1557
  - sabre cut that cleft his jaw as boy
    - stammered all his life

- in 1535 Fiore (student of del Ferro) challenged Tartaglia
  - each posses 30 problems for the other
    - within 2 hrs Tartaglia solved all 30
    - Fiore solved none

Girolamo Cardan (1501-1576)

- famous physician
- involved in many scandals
- astrologer to papal court
- begged Tartaglia for solution to cubic
    - given under oath that it be kept secret
    - Cardan published it in his "Ars Magna" 1545

CARDAN'S Ars Magna (1545)

- text on algebraic eqs
    - takes notice of negative roots
- Bombelli
    - in his "Algebra" 1572 accepted existence of imaginary numbers and developed skill in using them when solving cubics

04/11/02

AN EASY LOOK AT THE CUBIC FORMULA

Introduction

All students learn the quadratic formula for finding the roots of a quadratic equation. The cubic formula for solving cubic polynomials is seldom used, even though it has been known since 1545 when Girolamo Cardano published his *Ars Magna* [2]. This cubic formula, like the quadratic formula, gives the exact answer in closed form. Fifty years ago, when this author was a schoolboy, algebra text books frequently included a detailed discussion of the cubic formula. Precalculus texts of today rarely consider the subject. Why? Because the cubic formula, unlike the quadratic formula, frequently involves awkward cube roots of complex numbers. Besides, excellent numerical methods are available, such as Newton's iterative method, which converge very rapidly to approximations with many accurate digits. However, there are cases where the exact closed form answer is appealing, and where the effort involved in using the cubic formula is not overwhelming.

It is the purpose of this brief note to show how the cubic formula can be presented easily at the precalculus level. While none of this material is new, the selection of items and their presentation is designed to avoid the difficulties mentioned above. We give a nice canonical form for the cubic formula that is relatively easy to remember. We show how to verify that the formula is correct, and we identify when it is profitable to use it.

The cubic formula in simplest form

To solve the cubic equation

(1) $$y^3 + py^2 + qy + r = 0$$

we must first remove the quadratic term. This is always achieved with the substitution

(2) $$y = x - \frac{p}{3}.$$

Substituting (2) into (1) we get

(3) $$x^3 - 3cx - 2a = 0,$$

where

(4) $$c = \frac{p^2}{9} - \frac{q}{3} \text{ and } a = \frac{pq}{6} - \frac{p^3}{27} - \frac{r}{2}.$$

(Equation (3) is known as the *reduced cubic.*) Now we can write our cubic formula for the real root of (3).

Theorem:

Let

(5) $$b = a^2 - c^3 \geq 0.$$

Then a real root of $x^3 - 3cx - 2a = 0$ *is*

(6) $$x = \sqrt[3]{a + \sqrt{b}} + \sqrt[3]{a - \sqrt{b}}\,. \text{ (Cubic formula)}$$

We interpret all the roots in (6) as real numbers. (Actually, with proper interpretation of the radicals involved, formula (6) can give all three roots of (3) regardless of the values of the coefficients c and a. This does get messy when $b < 0$, and we will not consider that case here.)

Proof:

First notice that

(7) $$\sqrt[3]{a + \sqrt{b}}\,\sqrt[3]{a - \sqrt{b}} = \sqrt[3]{a^2 - b} = \sqrt[3]{a^2 - (a^2 - c^3)} = c.$$

Cubing both sides of (6) we get

$$x^3 = \left(\sqrt[3]{a+\sqrt{b}} + \sqrt[3]{a-\sqrt{b}}\right)^3$$

$$= \left(\sqrt[3]{a+\sqrt{b}}\right)^3 + 3\left(\sqrt[3]{a+\sqrt{b}}\right)^2 \sqrt[3]{a-\sqrt{b}} + 3\sqrt[3]{a+\sqrt{b}}\left(\sqrt[3]{a-\sqrt{b}}\right)^2 + \left(\sqrt[3]{a-\sqrt{b}}\right)^3$$

$$= a+\sqrt{b} + 3\sqrt[3]{a+\sqrt{b}}\sqrt[3]{a-\sqrt{b}}\left(\sqrt[3]{a+\sqrt{b}} + \sqrt[3]{a-\sqrt{b}}\right) + a - \sqrt{b}$$

Using (6) and (7) we can rewrite this last expression as

$$x^3 = 3cx + 2a.$$

Thus we have verified that (6) is a root of (3) and the theorem is proved.

Examples

Example 1: Find a real root of $y^3 + 3y^2 + 6y + 2 = 0$.

Solution: Comparing our problem with (1), we see that $p = 3$ so we begin by making the substitution (2) $y = x - p/3 = x - 1$. This converts the original problem to $x^3 + 3x - 2 = 0$, in which the quadratic term does not appear. Comparing this with (3) we see that $c = -1$ and $a = 1$. From (5) we get $b = a^2 - c^3 = 2$. Since $b > 0$ our theorem says that (6) gives us a real root $x = \sqrt[3]{1+\sqrt{2}} + \sqrt[3]{1-\sqrt{2}}$. Since the second cube root is negative, it is best written as $x = \sqrt[3]{1+\sqrt{2}} - \sqrt[3]{\sqrt{2}-1}$. Finally, a root of our original cubic is given by $y = x - 1 = \sqrt[3]{1+\sqrt{2}} - \sqrt[3]{\sqrt{2}-1} - 1$.

Example 2: Find a real root of $y^3 - 7y^2 + 14y - 20 = 0$.

Solution: We compare our problem with (1) and see that $p = -7$. We start with the substitution (2) $y = x - \frac{p}{3} = x + \frac{7}{3}$. Our original equation is now reduced to

$x^3-3cx-2a=x^3-\frac{7}{3}x-\frac{344}{27}=0$ in which the quadratic term has been removed.

Comparing this last equation with (3) we see that $c=\frac{7}{9}$ and $a=\frac{172}{27}$. Calculating b from (5) we get $b=a^2-c^3=\left(\frac{171}{27}\right)^2$. Since b is positive, a real root of our cubic is given by (6) as $x=\sqrt[3]{\frac{172}{27}+\frac{171}{27}}+\sqrt[3]{\frac{172}{27}-\frac{171}{27}}=\frac{7}{3}+\frac{1}{3}=\frac{8}{3}$. Finally, a root of our original cubic is given by $y=x+\frac{7}{3}=\frac{8}{3}+\frac{7}{3}=5$. Since this root is an integer, it is easy to find the other two roots by dividing our original cubic $y^3-7y^2+14y-20$ by $y-5$. This gives us the quadratic $y^2-2y+4=0$ which has roots $y=1\pm\sqrt{3}i$.

Example 3: Find a real root of $x^3-x-1=0$.

Solution: This problem comes from the interesting article [6] in which the "plastic number" is defined as the root of our equation. Our equation has no quadratic term, so there is no need to use the linear substitution. Comparing our equation with (3) we see that $c=\frac{1}{3}$ and $a=\frac{1}{2}$. Using (4) we get $b=a^2-c^3=\frac{1}{4}-\frac{1}{27}=\frac{23}{108}$. Since b is positive, we can use (6) to get the root

$$x=\sqrt[3]{\frac{1}{2}+\sqrt{\frac{23}{108}}}+\sqrt[3]{\frac{1}{2}-\sqrt{\frac{23}{108}}}=\frac{1}{6}\left(\sqrt[3]{108+12\sqrt{69}}+\sqrt[3]{108-12\sqrt{69}}\right).$$

Example 4: It is clear that $x = 1$ is a root of the cubic $x^3+3x-4=0$. Use the cubic formula to obtain a surprising expression for this root.

Solution: Comparing our cubic with (3) we see at once that $c=-1$ and $a=2$. Calculating b we get $b=a^2-c^3=5$. Our cubic formula now gives us

$x = \sqrt[3]{2+\sqrt{5}} + \sqrt[3]{2-\sqrt{5}} = \sqrt[3]{2+\sqrt{5}} - \sqrt[3]{\sqrt{5}-2}$. This does not look like $x = 1$, but a quick check with a calculator helps to convince us that it probably is 1. The reader might try to simplify this difference of two cube roots into the number 1, but all attempts to do this simply lead back to the original cubic $x^3+3x-4=0$. The paper [4] discusses how to recognize when radical expressions of the form $\sqrt[n]{a+\sqrt{b}} + \sqrt[n]{a-\sqrt{b}}$, for $n = 2, 3, 4, \ldots$, reduce to simple numbers like integers or rational values.

Final remarks

There is much more that could be said about the cubic formula. How do you find the two complex roots when $b > 0$, and how do we find any roots when $b < 0$? To answer these questions requires a quantum leap in the difficulty of our presentation. This is not our purpose. We hope that we have shown that there is partial information about the cubic formula that is both interesting and useful. The reader can find complete presentations of this subject in many algebra text books dating from before 1960 such as [5] and nice summaries in handbooks such as [3].

References

[1] Atkinson, Kendall, *Elementary Numerical Analysis,* John Wiley, New York, 1993, pp. 68-77.
[2] Cardano, Girolamo, (translated by T. Richard Witmer), *Ars Magna or the Rules of Algebra*, Dover, New York, NY, 1993.
[3] Korn, Granino A. and Korn, Theresa M., *Mathematical Handbook for Scientists and Engineers*, , Dover, New York, NY, 1968, p. 23.
[4] Osler, Thomas J., . *Cardan polynomials and the reduction of radicals,* Mathematics Magazine, Vol 47, No. 1, (2001), pp. 26-32.
[5] Rosenbach, Joseph B. and Whitman, Edwin A., *College Algebra*, 3rd Ed., Ginn and Company, New York, 1949, pp. 325-330.
[6] Stewart, Ian, *Tales of a Neglected Number*, Scientific American, June 1996, pp. 102-3.