Part 2

Outline for the History of Mathematics

Dr Osler

Start 7 (PART II (NEW VERSION SPRING 2004))

ARCHIMEDES (287–212 BC)

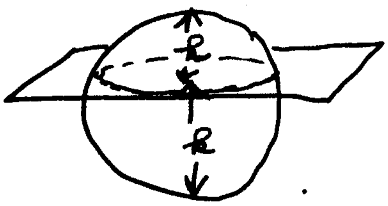
- greatest creative genius of ancient world
- lived in Syracuse
  - Greek settlement in Sicily
- worked under King Hieron II
- invented Archimedean Screw
  - simple water pump 
- was able to launch a ship of great weight using pulleys and leavers
- Plutarch wrote
  "he placed his whole ambition in those speculations whose beauty and subtlety are untainted by any admixture of the common needs of life"
- Solved the problem of determining if King Hieron's crown
  - was the gold dilated with silver
- Built machines to assit in the war defense of the city of Syracuse
  - terrified Roman Soldiers
- killed by common soldier sacking the city
- inscribed on his tomb

cylinder containing sphere

Sphere = $\frac{2}{3}$ cylinder

- Archimedes was led to a cubic eq from the problem:

  Cut a sphere by a plane into 2 volumes in a given ratio



r = radius

$$\frac{\text{Volume Top}}{\text{" Bottom}} = \frac{m}{n} \leftarrow \text{given ratio}$$

$$\frac{\pi h^2 (r - \frac{h}{3})}{\pi k^2 (r - \frac{k}{3})} = \frac{m}{n}$$

where $h + k = 2r$

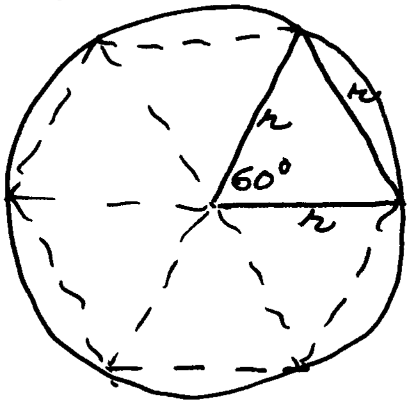
This reduces to

$(m+n)h^3 - 3r(m+n)h^2 + 4mr^3 = 0$

a cubic to solve for h

- Archimedes solved this as the solution of two simultaneous quadratic curves
- Cubics would not appear again in math history for over 2000 yrs

- To approximate π
  - he inscribed and circumscribed polygons of sides 6, 12, 24, 48, 96
  - to inscribe hexagon (6 sides) simply use radius as chord



Let p_n = perimeter of reg. polygon on n sides inscribed

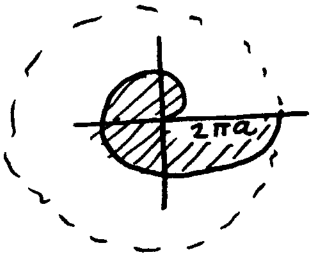
P_n = " " "
" circumscribed

$$p_6 < p_{12} < p_{24} < \cdots p_n < \overset{\substack{\text{circumferense}\\ \downarrow}}{C} < P_n \cdots < P_{24} < P_{12} < P_6$$

- Archimedes estimated the number of grains of sand needed to fill the universe
  - invented an exponential number notation to achieve this

- Archimedean Spiral

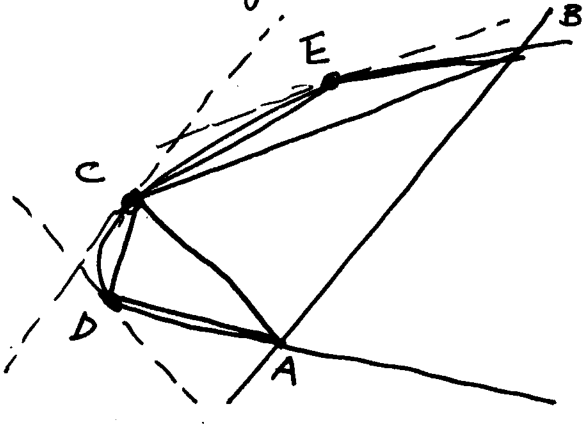
$$r = a\theta$$



showed that area = $\frac{1}{3}\pi(2\pi a)^2$

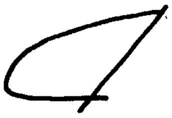
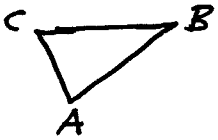
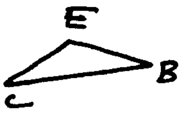
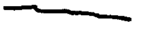
Homework
p. 227; 4, 5, 10

- Archimedes found area of parabola cut by a chord



area of ABC = Δ
" " CEB = $\frac{1}{8}\Delta$
" " CDA = $\frac{1}{8}\Delta$

Area of

 =  + 2  + 4  + ...

$$= \Delta + 2\left(\frac{1}{8}\Delta\right) + 4\left(\frac{1}{8^2}\Delta\right) + 8\left(\frac{1}{8^3}\Delta\right) + \cdots$$

$$= \Delta\left[1 + \frac{1}{4} + 2^2\left(\frac{1}{2^6}\right) + 2^3\left(\frac{1}{2^9}\right) + \cdots\right]$$

$$= \Delta\left[1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots\right]$$

$$= \Delta \frac{1}{1 - 1/4} = \frac{4}{3}\Delta$$

CHAPTER 5

THE SECOND ALEXANDRIAN SCHOOL; DIOPHANTUS

THE DECLINE OF ALEXANDRIAN MATH

146 BC Ptolemy VII
- epiled scholars not loyal to him
- Alexandria no longer leading center of research
- refugee scholars bring knowledge to other lands

- from 750 BC to 450AD
  - there is no Roman mathematician of note
  - Romans interested in practical applications

- BY 4th century ad
  - great days of Greek math had passed
  - intellectual energy turns to questions of theology

- Western Europe
  - blanketed with barbarism & illiteracy

- Eastern Europe (Byzantine Empire)
  - preserved copies of ancient works
  - did little original work

THE Arithmetica

- Diophantus
  - lived in Alexandria (about 250ad)
  - wrote "Arithmetica"
    - earliest treatise on algebra
    - introduced symbols for unknowns

DIOPHANTINE EQS

- letter from Archimedes to Eratosthenes has famous "cattle problem"

W = no. of bulls that are white
w = " cows " "
X = " bulls " black
x = " cows " "
Y = " bulls " spotted
y = " cows " "
Z = " bulls " brown
z = " cows " "

$$W = \frac{5}{6}X + Z, \quad X = \frac{9}{20}Y + Z, \quad Y = \frac{13}{42}W + Z$$

$$w = \frac{7}{12}(X + x), \quad x = \frac{9}{20}(Y + y), \quad z = \frac{13}{42}(W + w)$$

$W + X$ = square no. $\quad Y + Z$ = triangular no.

This all reduces to

$$x^2 - 4{,}729{,}494\, y^2 = 1$$

where y is a multiple of 9314

- leads to $x^2 - a y^2 = 1$
  Pell's eq.
- many tried to solve cattle prob. but were discouraged by large nos.
- 1880 Amthor
  no. of cattle = 776 ... in
  206,542 digits
- 1965 . solved on computer giving all 206,545 digits

- Alexander's invasion of India
  - stimulates math
- from 400 to 1200 Indians develop a math system superior to Greeks in all but geometry
  - Aryabhata
    summed arith. & geo. series
    table of sines
    $\pi = 3.1416$
  - Brahmagupta (century after Aryabhata)
    - $\pi = \sqrt{10}$
    - negative nos.
    - had both sols of quadratic
    - solved eqs for whole nos.
    - studied indeterminate eqs.

THE LATER COMMENTATORS

- time of Diophantus.
  - final stages of Greek math
  - most later writers are commentators on older works
  - Pappus is an exception

Hypatia (370 - 415)
- 1st prominent woman math.
- killed by mob of religious zealots in streets

Roman period
- devoid of interest in theoretical math
- Boethius
  - early Roman commentator
  - condensed versions of
    - Euclid,
    - Nicomachus
  - middle ages learned arith thru him

CHAPTER 6

THE FIRST AWAKENING: FIBONACCI

In western Europe the dark ages (5th to 11th centuries) is low ebb of math,

Arabs who overran southern & eastern shores of Mediterranean

- bring no original scholarship
- collect old manuscripts from lands they conquer
- these they translate into Arabic
- by 10th century nearly all surviving Greek texts are available in Arabic
- Arabs prevented many Greek texts from becomming lost
  - Islam's great contribution to the advancement of knowledge

- Arabs added significant material from Persia and India to the Greek
    - Hindus had interest in arethmatic & algebra
    - Most significant idea borrowed from east:
        - Arabic numerals
            - 9 digits & zero
    - Also developed trigonometry

- As soon as it was known that masterpieces of antiquity were in Arabic, many scholar's undertook to translate them into Latin

- Gerard of Cremona 1114-1187
    - drawn to Toledo
        - Translated Almagest into Latin
        - also 90 other Arabic texts including
          Archimedes
          apollonius

Abelard of Bath (1090-1150)

- traveled widely seeking knowledge
  - Spain, Italy, Sicily, Greece, Syria, Palestine
  - disguised as a Mohammedan
  - translated Euclid's Elements to Latin

- these struggling translators received little or no remuneration
  - devoted to truth & knowledge

about start of class #9

"Liber Abaci" & "Liber Quadratorum"

Leonardo of Pisa
- Fibonacci (born 1175)
- greatest math. of middle ages
- wrote Liber Abaci in 1202 (Book of Counting)
  - explained arabic number system
    - chief way in which Europe learned of these numbers

- math masterwork of middle ages

- Origins of Hindu - Arabic number sys.
  - obscure & disputed
    - originated in India ? 3rd Century
    - carried to Bagdad 8th Century
    - transmitted to western Europe by way of Moorish Spain

- Resistance to spread of new numerals
  - 1299 Florence
    - ordinance forbidding merchants to record in these
    - due to great variety of shapes of digits

      ex O looks like 6 or 9

- After printing (1450)
  - digits stabilized
  - appear as today

Liber Quadratorum (1225)

- by Fibonacci
  - pioneer in revival of math
  - gave many original proofs and results
  - does not surpass work of his Arab predecessors
  - notes that Euclid's Book X classification of irrationals is not complete
- given challenge problem

$$x^3 + 2x^2 + 10x = 20$$

  - first cubic in Europe since time of Greeks

$$10\left(x + \frac{x^2}{5} + \frac{x^3}{10}\right) = 20$$

$$x + \frac{x^2}{5} + \frac{x^3}{10} = 2$$

Fibonacci showed that x cannot be rational!

assume $x = \frac{a}{b}$ where a & b are integers with no common factors.

$$\frac{a}{b} + \frac{a^2}{5b^2} + \frac{a^3}{10b^3} = \frac{a\,(10b^2+2ab+a^2)}{10b^3}$$

$$= 2$$

This will not be an integer unless $10b^3$ (and certainly b) divides $10b^2+2ab+a^2$. Since b divides $10b^2+2ab$, then b must divide a^2. This violates the assumption that $\frac{a}{b}$ is reduced!

Thus x is not rational!

Fibonacci next demonstrated that x cannot be any of the Euclidean irrationals:

$a \pm \sqrt{b}$, $\sqrt{a} \pm \sqrt{b}$, $\sqrt{a \pm \sqrt{b}}$,
$\sqrt{\sqrt{a} \pm \sqrt{b}}$, a & b rational

Hence it cannot be constructed with straightedge & compass only.

- First indication that Greek ~~algebra~~ geometric algebra is insufficient to describe all numbers

- gave a remarkably accurate estimate of real root:

$1,3688081075$

- Same problem appears in algebra of great Persian Poet Omar Khayyam (1050-1130)
  - solved by intersecting circle & hyperbola

START CLASS #10

THE FIBONACCI SEQUENCE

Let $F_1 = F_2 = 1$

$$F_n = F_{n-1} + F_{n-2} \qquad n \geq 3$$

$$F_3 = F_2 + F_1 = 1+1 = 2$$

$$F_4 = F_3 + F_2 = 2+1 = 3$$

$$F_5 = F_4 + F_3 = 3+2 = 5$$

8

13

21

34

55

89

144

THM No two consecutive Fibonacci numbers have a factor $d > 1$ in common.

Proof

(1) Suppose $d > 1$ divides F_n & F_{n+1}

(2) $F_{n+1} - F_n = F_{n-1}$ is also divisible by d

(3) $F_n - F_{n-1} = F_{n-2}$ "

(4) $F_{n-1} - F_{n-2} = F_{n-3}$ "

$\vdots$

$= F_1$ "

But $F_1 = 1$ which cannot be divisible by $d > 1$. Thus a contradiction

PROVE THAT $\sum_{1}^{n} F_k = F_{n+2} - 1$

PROOF

$$
\begin{aligned}
F_1 &= \cancel{F_3} - F_2 \\
F_2 &= \cancel{F_4} - \cancel{F_3} \\
F_3 &= \cancel{F_5} - \cancel{F_4} \\
&\equiv \\
F_n &= F_{n+2} - \cancel{F_{n+1}}
\end{aligned}
$$

$$\sum_{1}^{n} F_k = F_{n+2} - F_2$$ ← adding

$$= F_{n+2} - 1$$

PROVE THAT $F_n^{\,2} = F_{n-1}\, F_{n+1} + (-1)^{n-1}$

PROOF

$$
\begin{aligned}
F_n^{\,2} - F_{n-1}\, F_{n+1} &= F_n\, F_n - F_{n-1}\, F_{n+1} \\
&= F_n\left(F_{n-1} + F_{n-2}\right) - F_{n-1}\, F_{n+1} \\
&= F_n\, F_{n-1} + F_n\, F_{n-2} - F_{n-1}\, F_{n+1} \\
&= F_n\, F_{n-1} - F_{n-1}\, F_{n+1} + F_n\, F_{n-2} \\
&= F_{n-1}\underbrace{\left(F_n - F_{n+1}\right)}_{-F_{n-1}} + F_n\, F_{n-2} \\
&= -F_{n-1}^{\,2} + F_n\, F_{n-2}
\end{aligned}
$$

$$F_n^2 - F_{n-1}F_{n+1} = (-1)\left(F_{n-1}^2 - F_n F_{n-2}\right)$$

But is the same as with subscripts diminished by one.

$$\begin{aligned}
&= (-1)^2\left(F_{n-2}^2 - F_{n-1}F_{n-3}\right)\\
&= (-1)^3\left(F_{n-3} - F_{n-2}F_{n-4}\right)\\
&\ \ \vdots\\
&= (-1)^{n-2}\left(F_2 - F_3 F_1\right)\\
&= (-1)^{n-2}\left(1 - 2\cdot 1\right)\\
&= (-1)^{n-2}(-1)\\
&= (-1)^{n-1}
\end{aligned}$$

qed

A GEOMETRIC DECEPTION

Use $n = 2k$ in the above to get

$$F_{2k}^2 = F_{2k-1}\underbrace{F_{2k+1}}_{(F_{2k}+F_{2k-1})} - 1$$

square of side $F_{2k-1} + F_{2k-2}$

Rectangle of sides F_{2k-1} and $F_{2k} + F_{2k-1}$

deceptive little -1 which shows that the square & rectangle are close but not exact

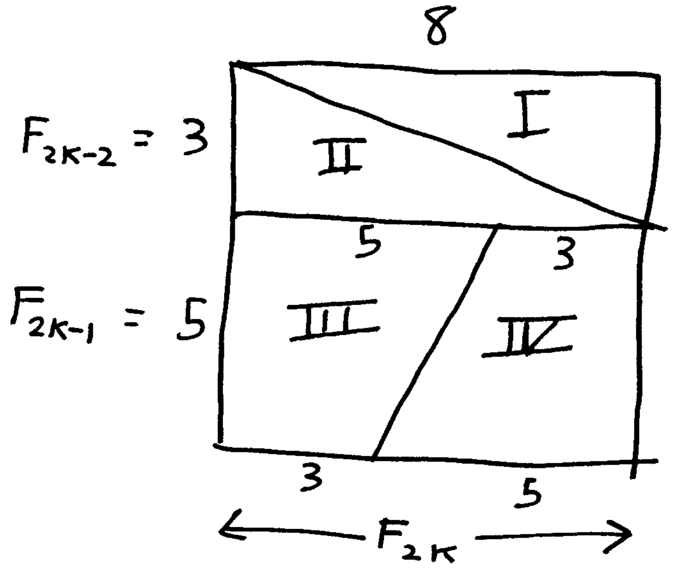
Ex. Take $k = 3$

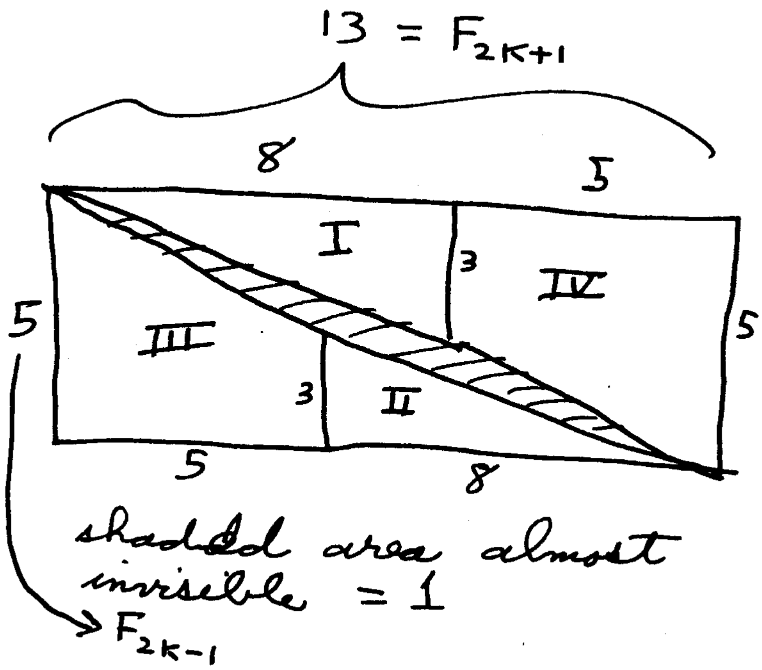
$F_{2k} = F_6 = 8$

$F_{2k-1} = 5$

$F_{2k-2} = 3$

$F_{2k+1} = 13$





shaded area almost invisible = 1

Connection between F_n and golden ratio.

$$u_n = \frac{F_{n+1}}{F_n}$$

$u_1 = \frac{1}{1} = 1$

$u_2 = \frac{2}{1} = 2$

$u_3 = \frac{3}{2} = 1.5$

$u_4 = \frac{5}{3} = 1.66\ldots$

$u_5 = \frac{8}{5} = 1.60$

$u_6 = \frac{13}{8} = 1.625$

$u_7 = \frac{21}{13} = 1.615\ldots$

$u_8 = \frac{34}{21} = 1.619\ldots$

$$\frac{F_{n+1}}{F_n} = \frac{F_n + F_{n-1}}{F_n} = 1 + \frac{F_{n-1}}{F_n}$$

$$= 1 + \frac{1}{\frac{F_n}{F_{n-1}}}$$

Let $\lim_{n \to \infty} u_n = \alpha$

$$\alpha = 1 + \frac{1}{\alpha}$$

$$\alpha^2 = \alpha + 1$$

$$\alpha^2 - \alpha - 1 = 0$$

$$\alpha = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$$\frac{1+\sqrt{5}}{2} = \text{golden ratio}$$

$$= 1.618033987\ldots$$

<u>Homework</u>

P. 282, 2, 4, 8

CHAP 7

THE CUBIC CONTROVERSY!

CARDAN & TARTAGLIA

EUROPE IN 14TH & 15TH CENTURIES

- IF 13th cent. is high pt. of medieval Europe, then 14th is lowest

- 13th cent. gave abundant promise for future

- 14th cent.
    - famine, plague, war
    - opened with heavy rainfalls
    - turned significantly colder
        - Little Ice Age
        - crop failure
        - famine
    - Black Death
        - bubonic plague
    - smoke of war hung over whole sad century
        - 100 yrs war
            - England vs France

- By 1450 , war, plague & famine had tapered off
    - population increased
    - remarkable rebirth
        - called the Renaissance

- Printed Books
    - First were little concerned with math
    - "Treviso Arithmetic"
        - 1478
        - at Treviso (North of Venice)
        - 1st popular textbook
            - list of calculating rules
    - Before 1500 over 200 math books had been printed in Italy
        - Euclid's Elements
            - 1482

- Regiomontanus (Johannes Müller)
  - 1436-1476
  - one of earliest European scholars to use original Greek texts
  - actively translated original classical texts
    - Ptolemy's Almagest
  - wrote "De Triangulis Omnimodis" (On Triangles of all kinds)
    - used trigonometric fctns

- Fra Luca di Borga (Luca Pacioli 1445-1514)
  - Franciscan Friar
  - wrote "Summa de Arithmetica Geometria Proportioni et Proportionalita"
    - Venice 1494
    - most influential math book of its time
- Peter Abelard (1079-1142)
  see p. 295 at bottom
  helped found University of Paris

- Growth of Universities
    - in 12th & 13th centuries
    - older cathedral & monastery schools could not handle the numbers of students
    - Paris & Bologna
        - great "mother" universities
- cult of classics
    - belief that antiquity, both Greek & Latin offered a model of perfection
    - its literature could provide solutions to political, social & ethical problems.
- Renaissance
    - made little progress in science
    - opened the way for Scientific revolution of 1600's by recovering more of ancient learning

THE BATTLE OF THE SCHOLARS

- RENAISSANCE
  - great achievements in literature, painting, architecture
  - small " in math & science
  - humanist passion for the discovery, translation & circulation of ancient Greek texts
    - benifited math
      - manuscript collectors assemble in Italy an almost complete collection of Greek math writings
- In 1500's
  - advances in algebra
    - solve cubic
    - better symbolism
      - Francois Vieta 1540-1603
        - vowels for unknowns
        - consonants " knowns

$$3x^2+5x+10=0$$

becomes

$$ax^2+bx+c=0$$

  - advances in arithmetic
  - advances in trigonometry
    - Rhaeticus (Georg Joachim 1514-1576)
      - worked out table of sines to 15 dec. places for every 10 sec of arc.

- Italian math of 1500's
  - del Ferro
  - Tartaglia
  - Cardan
  - Ferrari
  - Bombelli

  solve cubic & quartic
  - greatest algebraic feat since Babylonians 4000 yrs earlier

- Scipione del Ferro 1465–1526
  - University of Bologna
    solved $x^3 + px = q$

- math discoveries kept secret
  - to solve contest problems

- Nicolo Tartaglia (the stammerer) 1500–1557
  - sabre cut that cleft his jaw as boy
    - stammered all his life

- in 1535 Fiore (student of del Ferro) challenged Tartaglia
  - each posses 30 problems for the other
    - within 2 hrs Tartaglia solved all 30
    - Fiore solved none

Girolamo Cardan (1501-1576)

- famous physician
- involved in many scandals
- astrologer to papal court
- begged Tartaglia for solution to cubic
  - given under oath that it be kept secret
  - Cardan published it in his "Ars Magna" 1545

CARDAN'S Ars Magna (1545)

- text on algebraic eqs
  - takes notice of negative roots
- Bombelli
  - in his "Algebra" 1572 accepted existence of imaginary numbers and developed skill in using them when solving cubic's

04/11/02

AN EASY LOOK AT THE CUBIC FORMULA

Introduction

All students learn the quadratic formula for finding the roots of a quadratic equation. The cubic formula for solving cubic polynomials is seldom used, even though it has been known since 1545 when Girolamo Cardano published his *Ars Magna* [2]. This cubic formula, like the quadratic formula, gives the exact answer in closed form. Fifty years ago, when this author was a schoolboy, algebra text books frequently included a detailed discussion of the cubic formula. Precalculus texts of today rarely consider the subject. Why? Because the cubic formula, unlike the quadratic formula, frequently involves awkward cube roots of complex numbers. Besides, excellent numerical methods are available, such as Newton's iterative method, which converge very rapidly to approximations with many accurate digits. However, there are cases where the exact closed form answer is appealing, and where the effort involved in using the cubic formula is not overwhelming.

It is the purpose of this brief note to show how the cubic formula can be presented easily at the precalculus level. While none of this material is new, the selection of items and their presentation is designed to avoid the difficulties mentioned above. We give a nice canonical form for the cubic formula that is relatively easy to remember. We show how to verify that the formula is correct, and we identify when it is profitable to use it.

The cubic formula in simplest form

To solve the cubic equation

$$y^3 + py^2 + qy + r = 0 \tag{1}$$

we must first remove the quadratic term. This is always achieved with the substitution

(2) $$y = x - \frac{p}{3}.$$

Substituting (2) into (1) we get

(3) $$x^3 - 3cx - 2a = 0,$$

where

(4) $$c = \frac{p^2}{9} - \frac{q}{3} \text{ and } a = \frac{pq}{6} - \frac{p^3}{27} - \frac{r}{2}.$$

(Equation (3) is known as the *reduced cubic.*) Now we can write our cubic formula for the real root of (3).

Theorem:

Let

(5) $$b = a^2 - c^3 \geq 0.$$

Then a real root of $x^3 - 3cx - 2a = 0$ *is*

(6) $$x = \sqrt[3]{a + \sqrt{b}} + \sqrt[3]{a - \sqrt{b}}\,. \text{ (Cubic formula)}$$

We interpret all the roots in (6) as real numbers. (Actually, with proper interpretation of the radicals involved, formula (6) can give all three roots of (3) regardless of the values of the coefficients c and a. This does get messy when $b < 0$, and we will not consider that case here.)

Proof:

First notice that

(7) $$\sqrt[3]{a + \sqrt{b}}\,\sqrt[3]{a - \sqrt{b}} = \sqrt[3]{a^2 - b} = \sqrt[3]{a^2 - (a^2 - c^3)} = c.$$

Cubing both sides of (6) we get

$$x^3 = \left(\sqrt[3]{a+\sqrt{b}} + \sqrt[3]{a-\sqrt{b}}\right)^3$$

$$= \left(\sqrt[3]{a+\sqrt{b}}\right)^3 + 3\left(\sqrt[3]{a+\sqrt{b}}\right)^2 \sqrt[3]{a-\sqrt{b}} + 3\sqrt[3]{a+\sqrt{b}}\left(\sqrt[3]{a-\sqrt{b}}\right)^2 + \left(\sqrt[3]{a-\sqrt{b}}\right)^3$$

$$= a+\sqrt{b} + 3\sqrt[3]{a+\sqrt{b}}\sqrt[3]{a-\sqrt{b}}\left(\sqrt[3]{a+\sqrt{b}} + \sqrt[3]{a-\sqrt{b}}\right) + a - \sqrt{b}$$

Using (6) and (7) we can rewrite this last expression as

$$x^3 = 3cx + 2a\,.$$

Thus we have verified that (6) is a root of (3) and the theorem is proved.

Examples

Example 1: Find a real root of $y^3 + 3y^2 + 6y + 2 = 0$.

Solution: Comparing our problem with (1), we see that $p = 3$ so we begin by making the substitution (2) $y = x - p/3 = x - 1$. This converts the original problem to $x^3 + 3x - 2 = 0$, in which the quadratic term does not appear. Comparing this with (3) we see that $c = -1$ and $a = 1$. From (5) we get $b = a^2 - c^3 = 2$. Since $b > 0$ our theorem says that (6) gives us a real root $x = \sqrt[3]{1+\sqrt{2}} + \sqrt[3]{1-\sqrt{2}}$. Since the second cube root is negative, it is best written as $x = \sqrt[3]{1+\sqrt{2}} - \sqrt[3]{\sqrt{2}-1}$. Finally, a root of our original cubic is given by $y = x - 1 = \sqrt[3]{1+\sqrt{2}} - \sqrt[3]{\sqrt{2}-1} - 1$.

Example 2: Find a real root of $y^3 - 7y^2 + 14y - 20 = 0$.

Solution: We compare our problem with (1) and see that $p = -7$. We start with the substitution (2) $y = x - \dfrac{p}{3} = x + \dfrac{7}{3}$. Our original equation is now reduced to

$x^3-3cx-2a=x^3-\frac{7}{3}x-\frac{344}{27}=0$ in which the quadratic term has been removed.

Comparing this last equation with (3) we see that $c=\frac{7}{9}$ and $a=\frac{172}{27}$. Calculating b from (5) we get $b=a^2-c^3=\left(\frac{171}{27}\right)^2$. Since b is positive, a real root of our cubic is given by (6) as $x=\sqrt[3]{\frac{172}{27}+\frac{171}{27}}+\sqrt[3]{\frac{172}{27}-\frac{171}{27}}=\frac{7}{3}+\frac{1}{3}=\frac{8}{3}$. Finally, a root of our original cubic is given by $y=x+\frac{7}{3}=\frac{8}{3}+\frac{7}{3}=5$. Since this root is an integer, it is easy to find the other two roots by dividing our original cubic $y^3-7y^2+14y-20$ by $y-5$. This gives us the quadratic $y^2-2y+4=0$ which has roots $y=1\pm\sqrt{3}i$.

Example 3: Find a real root of $x^3-x-1=0$.

Solution: This problem comes from the interesting article [6] in which the "plastic number" is defined as the root of our equation. Our equation has no quadratic term, so there is no need to use the linear substitution. Comparing our equation with (3) we see that $c=\frac{1}{3}$ and $a=\frac{1}{2}$. Using (4) we get $b=a^2-c^3=\frac{1}{4}-\frac{1}{27}=\frac{23}{108}$. Since b is positive, we can use (6) to get the root

$$x=\sqrt[3]{\frac{1}{2}+\sqrt{\frac{23}{108}}}+\sqrt[3]{\frac{1}{2}-\sqrt{\frac{23}{108}}}=\frac{1}{6}\left(\sqrt[3]{108+12\sqrt{69}}+\sqrt[3]{108-12\sqrt{69}}\right).$$

Example 4: It is clear that $x=1$ is a root of the cubic $x^3+3x-4=0$. Use the cubic formula to obtain a surprising expression for this root.

Solution: Comparing our cubic with (3) we see at once that $c=-1$ and $a=2$. Calculating b we get $b=a^2-c^3=5$. Our cubic formula now gives us

$x=\sqrt[3]{2+\sqrt{5}}+\sqrt[3]{2-\sqrt{5}}=\sqrt[3]{2+\sqrt{5}}-\sqrt[3]{\sqrt{5}-2}$. This does not look like $x = 1$, but a quick check with a calculator helps to convince us that it probably is 1. The reader might try to simplify this difference of two cube roots into the number 1, but all attempts to do this simply lead back to the original cubic $x^3+3x-4=0$. The paper [4] discusses how to recognize when radical expressions of the form $\sqrt[n]{a+\sqrt{b}}+\sqrt[n]{a-\sqrt{b}}$, for $n = 2, 3, 4, \ldots$, reduce to simple numbers like integers or rational values.

Final remarks

There is much more that could be said about the cubic formula. How do you find the two complex roots when $b>0$, and how do we find any roots when $b<0$? To answer these questions requires a quantum leap in the difficulty of our presentation. This is not our purpose. We hope that we have shown that there is partial information about the cubic formula that is both interesting and useful. The reader can find complete presentations of this subject in many algebra text books dating from before 1960 such as [5] and nice summaries in handbooks such as [3].

References

[1] Atkinson, Kendall, *Elementary Numerical Analysis,* John Wiley, New York, 1993, pp. 68-77.

[2] Cardano, Girolamo, (translated by T. Richard Witmer), *Ars Magna or the Rules of Algebra,* Dover, New York, NY, 1993.

[3] Korn, Granino A. and Korn, Theresa M., *Mathematical Handbook for Scientists and Engineers,*, Dover, New York, NY, 1968, p. 23.

[4] Osler, Thomas J., . *Cardan polynomials and the reduction of radicals,* Mathematics Magazine, Vol 47, No. 1, (2001), pp. 26-32.

[5] Rosenbach, Joseph B. and Whitman, Edwin A., *College Algebra,* 3rd Ed., Ginn and Company, New York, 1949, pp. 325-330.

[6] Stewart, Ian, *Tales of a Neglected Number,* Scientific American, June 1996, pp. 102-3.

THE SOLUTION OF THE GENERAL CUBIC

$$X^3 + bX^2 + cX + d = 0$$

IS X_0, X_1, X_2 WHERE

$$x_n =$$

$$e^{\frac{2\pi i n}{3}} \sqrt[3]{\frac{1}{2}\left(\frac{1}{3}bc - \frac{2}{27}b^3 - d\right) + \sqrt{\frac{1}{27}\left(c - \frac{1}{3}b^2\right)^3 + \frac{1}{4}\left(d - \frac{1}{3}bc + \frac{2}{27}b^3\right)^2}}$$

$$+ e^{-\frac{2\pi i n}{3}} \sqrt[3]{\frac{1}{2}\left(\frac{1}{3}bc - \frac{2}{27}b^3 - d\right) - \sqrt{\frac{1}{27}\left(c - \frac{1}{3}b^2\right)^3 + \frac{1}{4}\left(d - \frac{1}{3}bc + \frac{2}{27}b^3\right)^2}}$$

$$-\frac{1}{3}b \qquad (n = 0, 1, 2)$$

FERRARI'S SOLUTION OF QUARTIC

- Solution to Quartic
  - discovered during work on this prob. proposed by Cardan in 1540

"Divide 10 into 3 proportional parts so that the product of the 1st & 2nd parts is 6"

The numbers

$$\frac{6}{x}, \quad x \quad \text{and} \quad \frac{x^3}{6}$$

satisfy the conditions

(1) they are proportional parts

$$\frac{1^{st}}{2^{nd}} = \frac{2^{nd}}{3^{rd}}$$

$$\frac{\frac{6}{x}}{x} = \frac{x}{\frac{x^3}{6}}$$

(2) product of 1st & 2nd is 6

(3) sum is 10

$$\frac{6}{x} + x + \frac{x^3}{6} = 10$$

$$6 + x^2 + \frac{x^4}{6} = 10x$$

$$\boxed{x^4 + 6x^2 - 60x + 36 = 0}$$

Solved by

Ludovico Ferrari (1522-1565)

- Ferrari championed Cardan in his arguments over priority with Tartaglia
- Ferrari defeated Tartaglia in a math contest in Milan in 1548

- Paolo Ruffini (1765-1822)
  - Italian physician
  - proof that general 5th deg. eq is unsolvable in 1799
    - sound in general outline
    - faulty in some details
- Niels Henrik Abel (1802-1829)
  - Norwegian genious
  - proved unsolvability of quintic 1824
    - Gauss
      - "Here is another of those monstrosities"

- Evariste Galois (1811-1832)
  - completed Abel's research
  - founded group theory

- 1st submitted results to Academy of Sciences in May 1829 (17 yrs old)
  - referee - Cauchy
    - lost
- 2nd submitted new version Feb 1830
  - referee - Fourier
    - referee dies
- 3rd submission Jan 1831
  - rejected by Poision

May 1832 - died in duel

- Liouville in 1843
  - studies Galois papers and is convinced of their importance
  - has them published in 1846
- Richard Dedekind
  - 1st Galois lectures 1856-7

HISTORY OF MATH P. 71-2
T J OSLER

TO SOLVE

(1) $x^4 + bx^3 + cx^2 + dx + e = 0$

(2) LET $x = y - \frac{b}{4}$

AND GET

$$y^4 + \left(c - \frac{3}{8}b^2\right)y^2 + \left(\frac{b^3}{8} - \frac{bc}{2} + d\right)y$$
$$+ \left(\frac{b^2c}{16\cancel{8}} - \frac{bd}{4} + e - \frac{3b^4}{256}\right) = 0$$

or

(3) $y^4 + py^2 + qy + r = 0$

SUPPOSE (3) FACTORED AS

$$(y^2 + Ay + B)(y^2 + Cy + D) = 0$$

(4) $y^4 + (A+C)y^3 + (B + D + AC)y^2 + (AD + BC)y + BD = 0$

TO REMOVE THE y^3 TERM SET $C = -A$ AND GET

(5) $y^4 + (B + D - A^2)y^2 + A(D - B)y + BD = 0$

COMPARING (5) AND (3) WE GET

(6) $p = B + D - A^2$
(7) $q = A(D - B)$
(8) $r = BD$

FROM (8) WE SOLVE FOR D

(9) $D = \frac{r}{B}$

NOW USE (9) TO REMOVE D FROM (6) \$ (7)

(10) $p = B + \frac{r}{B} - A^2$

(11) $q = A\left(\frac{r}{B} - B\right)$

MULTIPLY (10) BY A ;

(12) $Ap = AB + \frac{Ar}{B} - A^3$

SUBTRACT (11) FROM (12)

(13) $Ap - q = 2AB - A^3$

SOLVING (13) FOR 2AB

*(14) $2AB = A^3 + Ap - q$

NEXT MULTIPLY (11) BY B

(15) $qB = Ar - AB^2$

REWRITE (15) AS A QUADRATIC IN B AND SOLVE FOR B

$$AB^2 + qB - Ar = 0$$

$$B = \frac{-q \pm \sqrt{q^2 + 4A^2 r}}{2A}$$ or

*(16) $2AB = -q \pm \sqrt{q^2 + 4A^2 r}$

NEXT COMPARE (14) & (16)

$$A^3 + Ap = \pm\sqrt{q^2+4A^2r}$$

SQUARE

$$A^6 + 2A^4p + A^2p^2 = q^2 + 4A^2r$$

$$\Rightarrow (17) \quad A^6 + 2pA^4 + (p^2-4r)A^2 - q^2 = 0$$

NOTE THAT (17) IS A CUBIC IN A^2
(RESOLVENT CUBIC)
SOLVE (17) FOR $A^2 \rightarrow A$
KNOWING A WE CAN USE (14) TO GET B

$$\Rightarrow (18) \quad B = \frac{A^3 + pA - q}{2A}$$

KNOWING B WE USE (9) TO FIND D

$$\Rightarrow (19) \quad D = \frac{r}{B}$$

$\Rightarrow$ WE ALSO RECALL $C = -A$

NOW OUR QUARTIC (3) HAS FACTORED INTO TWO KNOWN QUADRATICS

$$(y^2 + Ay + B)(y^2 + Cy + D) = 0$$

AND WE CAN NOW SOLVE FOR y.

EXAMPLE

Solve $x^4 - 8x^3 + 21x^2 - 14x - 10 = 0$

$x = y - \left(\frac{-8}{4}\right) = y + 2$

yields $$y^4 \underset{p}{-3}y^2 \underset{q}{+6}y \underset{r}{-2} = 0$$

Next the resolvent cubic is

$$A^6 - 6A^4 + 17A^2 - 36 = 0$$

Try $\boxed{A^2 = 4}$

$\frac{\begin{matrix}16\\6\end{matrix}}{6}$

~~$64 - 24 + 68 - 36$~~

~~$8 - 12 + 34 - 36$~~

$64 - 96 + 68 - 36 = 132 - 132 = 0$

Thus $A = 2$ is a sol of the resolvent cubic,

$$B = \frac{A^3 + Ap - q}{2A} = \frac{8 + 2(-3) - 6}{2(2)} = \frac{-4}{4} = -1$$

$$D = \frac{r}{B} = \frac{-2}{-1} = 2$$

$(y^2 + Ay + B)(y^2 - Ay + D) = y^4 - 3y^2 + 6y - 2 = 0$

$(y^2 + 2y - 1)(y^2 - 2y + 2) = 0$

roots are $y = -1 \pm \sqrt{2}$ and $y = 1 \pm i$

hence

$x = 1 \pm \sqrt{2}$ and $3 \pm i$

SOME HW PROBLEMS ON QUARTICS

① $x^4 + x^2 + 14x - 6 = 0$

$x = -1 \pm \sqrt{2},\ 1 \pm i\sqrt{5}$

② $x^4 - 3x^2 + 6x - 2 = 0$

$x = -1 \pm \sqrt{2},\ 1 \pm i$

③ $x^4 - 4x^2 + 8x + 35 = 0$

$x = -2 \pm i,\ 2 \pm i\sqrt{3}$

④ $x^4 - 4x^3 - 4x^2 - 4x - 5 = 0$

$x = -1,\ 5,\ \pm i$

⑤ $2x^4 + 8x^3 - 9x^2 + 4x - 5 = 0$

$x = -5,\ 1,\ \pm \frac{1}{2} i\sqrt{2}$

EXTRA CREDIT PROBLEM

WRITE OUT THE SOLUTION OF

$x^4 + bx^3 + cx^2 + dx + e = 0$

IN TERMS OF b, c, d, e

MAKE COPIES FOR THE CLASS