02/15/04

INTERESTING FINITE AND INFINITE PRODUCTS FROM SIMPLE ALGEBRAIC IDENTITIES

Thomas J. Osler
Mathematics Department
Rowan University
Glassboro NJ 08028

Osler@rowan.edu

1. Introduction

The difference of two squares, $x^2 - y^2 = (x+y)(x-y)$ and its immediate generalizations $x^3 - y^3 = (x-y)(x^2 + xy + y^2)$, …, are among the most elementary identities. It is the purpose of this little note to show that these simple formulas are at the heart of a number of advanced products, some finite, and some infinite. The reader may find some of these surprising, especially those involving trigonometric functions.

Repeated use of the difference of two squares gives us

$$\begin{aligned} x - y &= \left(x^{1/2} - y^{1/2}\right)\left(x^{1/2} + y^{1/2}\right) \\ &= \left(x^{1/4} - y^{1/4}\right)\left(x^{1/2} + y^{1/2}\right)\left(x^{1/4} + y^{1/4}\right) \\ &= \left(x^{1/8} - y^{1/8}\right)\left(x^{1/2} + y^{1/2}\right)\left(x^{1/4} + y^{1/4}\right)\left(x^{1/8} + y^{1/8}\right) \end{aligned}$$

and so we have

$$x - y = \left(x^{1/2^n} - y^{1/2^n}\right)\prod_{k=1}^{n}\left(x^{1/2^k} + y^{1/2^k}\right). \tag{1}$$

Since we will be interested in infinite products, ($n \to \infty$), we need the factors of (1), $\left(x^{1/2^k} + y^{1/2^k}\right)$, adjusted so that they approach one. As k grows large, $\left(x^{1/2^k} + y^{1/2^k}\right)$ approaches 2, so the proper adjustment of (1) is

(2) $$x-y=2^n\left(x^{1/2^n}-y^{1/2^n}\right)\prod_{k=1}^{n}\left(\frac{x^{1/2^k}+y^{1/2^k}}{2}\right).$$

In a similar way we repeatedly use the identity

$$x-y=\left(x^{1/N}-y^{1/N}\right)\left(x^{(N-1)/N}+x^{(N-2)/N}y^{1/N}+x^{(N-3)/N}y^{2/N}+\cdots+y^{(N-1)/N}\right),\ N=2,\ 3,\ 4,\ \ldots,$$

to obtain the more general identity

(3) $$x-y=N^n\left(x^{1/N^n}-y^{1/N^n}\right)\prod_{k=1}^{n}\left(\frac{x^{(N-1)/N^k}+x^{(N-2)/N^k}y^{1/N}+\cdots+y^{(N-1)/N^k}}{N}\right).$$

It is the purpose of this note to show that several interesting known infinite products have their roots in this simple identity. These include Carlson's product for $\log z$, a generalization of Carlson's product by Levin, and Vieta's classical product for π. We also obtain closed form expressions for the corresponding finite products, some of which, might not have been noticed before.

2. Products for log *z* by Carlson and Levin.

In [2], Carlson found the interesting infinite product

(4) $$\log z=(z-1)\prod_{k=1}^{\infty}\frac{2}{1+z^{1/2^k}}.$$

Let $y=1$ and $x=z$ in (2) to get $z-1=2^n\left(z^{1/2^n}-1\right)\prod_{k=1}^{n}\left(\frac{z^{1/2^k}+1}{2}\right)$, and taking the reciprocal we have

(5) $$2^n\left(z^{1/2^n}-1\right)=(z-1)\prod_{k=1}^{n}\left(\frac{2}{1+z^{1/2^k}}\right).$$

As $n\to\infty$, the left-hand side of (4) can be evaluated by letting $x=1/2^n$ and using L'Hopital's rule. We have $\lim_{n\to\infty}2^n\left(z^{1/2^n}-1\right)=\lim_{x\to 0}\frac{z^x-1}{x}=\lim_{x\to 0}z^x\log z=\log z$. Thus we have derived Carlson's product (4), and also the related finite product (5). (In [2], Carlson used (1) in his derivation.

We can perform the same analysis starting with the more general identity (3) and get

$$N^n\left(z^{1/N^n}-1\right)=\left(z-1\right)\prod_{k=1}^{n}\left(\frac{N}{z^{(N-1)/N^k}+z^{(N-2)/N^k}+\cdots+1}\right). \tag{6}$$

As $n\to\infty$, the left-hand side of (6) approaches $\log z$ as before, and we get

$$\log z=\left(z-1\right)\prod_{k=1}^{\infty}\left(\frac{N}{z^{(N-1)/N^k}+z^{(N-2)/N^k}+\cdots+1}\right). \tag{7}$$

This generalization of Carlson's product (4) was derived by Levin in [4] using much more general infinite products as a starting point.

3. Vieta's product for pi.

Let $x=e^{i\theta}$ and $y=e^{-i\theta}$ in (2) to get

$$e^{i\theta}-e^{-i\theta}=2^n\left(e^{i\theta/2^n}-e^{-i\theta/2^n}\right)\prod_{k=1}^{n}\left(\frac{e^{i\theta/2^k}+e^{-i\theta/2^k}}{2}\right).$$

Divide both sides by $2i\theta$ and get

$$\frac{\sin\theta}{\theta}=\frac{\sin\left(\theta/2^n\right)}{\theta/2^n}\prod_{k=1}^{n}\cos\left(\theta/2^k\right). \tag{8}$$

This identity is almost always obtained by starting with $\sin\theta=2\sin\left(\theta/2\right)\cos\left(\theta/2\right)$, then using the same identity again and to get $\sin\theta=2^2\sin\left(\theta/2^2\right)\cos\left(\theta/2\right)\cos\left(\theta/2^2\right)$, and repeat. It is , perhaps, surprising to see (8) arise from repeated use of $x-y=(x^{1/2}+y^{1/2})(x^{1/2}-y^{1/2})$. Passing to the limit as $n\to\infty$, (8) becomes

$$\frac{\sin\theta}{\theta}=\prod_{k=1}^{\infty}\cos\left(\theta/2^k\right). \tag{9}$$

This last relation is frequently used to derive Vieta's original infinite product for π. (See [5], [6], [7] and [8]. Also, Levin's paper [4], generalizes this standard result.) By repeated use of $\cos(\theta/2)=\sqrt{\frac{1}{2}+\frac{1}{2}\cos\theta}$, valid for $-\pi\le\theta\le\pi$, we get

$$(10)\qquad \cos\left(\theta/2^k\right)=\underbrace{\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\cdots+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\cos\theta}}}}}_{\left|\left\langle ----\ k\ radicals\ ----\right\rangle\right|}.$$

Substituting (10) into (9) and letting $\theta=\pi/2$ we get Vieta's original product [1]

$$(11)\qquad \frac{2}{\pi}=\sqrt{\frac{1}{2}}\ \sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\ \sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdots.$$

In [5], (8) was used to show the connection between Vieta's product (11) and Wallis's product $\frac{2}{\pi}=\frac{1\cdot 3}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\frac{5\cdot 7}{6\cdot 6}\cdots$.

4. Another related algebraic identity.

Another simple algebraic identity can be obtained from the difference of two squares by repeated iteration. Notice that

$$x-y=\frac{x^2-y^2}{x+y}=\frac{x^4-y^4}{(x+y)\left(x^2+y^2\right)}=\frac{x^8-y^8}{(x+y)\left(x^2+y^2\right)\left(x^4+y^4\right)}$$

and in general we have

$$(12)\qquad x-y=\left(x^{2^n}-y^{2^n}\right)\prod_{k=0}^{n-1}\frac{1}{x^{2^k}+y^{2^k}}.$$

As before, repeated use of $x-y=\frac{x^3-y^3}{x^2+xy+y^2}$ and its generalizations gives us

$$(13)\qquad x-y=\left(x^{N^n}-y^{N^n}\right)\prod_{k=0}^{n-1}\frac{1}{x^{(N-1)N^k}+x^{(N-2)N^k}y^{N^k}+x^{(N-3)N^k}y^{2N^k}+\cdots+y^{(N-1)N^k}}.$$

We will not let $n\to\infty$ in (12) and (13) as the factors cannot be adjusted to approach one.

If we let $x=e^{i\theta}$ and $y=e^{-i\theta}$ in (12) we get after simplifying

$$(14)\qquad \frac{\sin\theta}{\theta}=\frac{\sin\left(2^n\theta\right)}{2^n\theta}\prod_{k=0}^{n-1}\frac{1}{\cos\left(2^k\theta\right)}.$$

It is interesting to compare (14) and (8). Again let $x=e^{i\theta}$ and $y=e^{-i\theta}$ in (13) and get after simplifying

$$(15)\qquad \frac{\sin\theta}{\theta}=\frac{\sin\left(N^n\theta\right)}{\theta}\prod_{k=0}^{n-1}\frac{1}{1+2\cos\left(2N^k\theta\right)+2\cos\left(4N^k\theta\right)+2\cos\left(6N^k\theta\right)+\cdots+2\cos\left((N-1)N^k\theta\right)}$$

for N odd, and

$$(16)\qquad \frac{\sin\theta}{\theta}=\frac{\sin\left(N^n\theta\right)}{\theta}\prod_{k=0}^{n-1}\frac{1}{2\cos\left(N^k\theta\right)+2\cos\left(3N^k\theta\right)+2\cos\left(5N^k\theta\right)+\cdots+2\cos\left((N-1)N^k\theta\right)}$$

for N even.

The product (14), as well as the special case of (15) in which $N=3$, can be found listed in Hansen's table [3]. By examining the examples in Hansen's table, and trying to derive them, we can find motivation for more identities like those shown above in (2), (3), (12), and (13).

References

[1] L. Berggren, J. Borwein and P. Borwein, *Pi, A Source Book,* Springer, New York, 1997, pp. 53-67.

[2] B. C. Carlson, *The logarithmic mean*, Amer. Math. Monthly, 79 (1972), pp. 615-618.

[3] E. R. Hansen, *A Table of Series and Products*, Prentice-Hall, Englewood Cliffs, N.J., 1975, pp. 503-504.

[4] A. Levin, *A new class of infinite products generalizing Viete's product formula for π,* preprint.

[5] T. J. Osler, *The union of Vieta's and Wallis's products for pi,* Amer. Math. Monthly, 106 (1999), pp. 774-776.

[6] T. J. Osler and M. Wilhelm, *Variations on Vieta's and Wallis's products for pi,* Mathematics and Computer Education, 35(2001), pp. 225-232.

[7] T. J. Osler, *The general Vieta-Wallis product for pi*, to appear in The Mathematical Gazette, (November, 2005.)

[8] K. B. Stolarsky, *Mapping properties, growth, uniqueness of Vieta (infinite cosine) products,* Pacific J. Math., 89 (1980), pp. 209-227.