

2.4 The hyperbolic functions

In the real calculus, we encountered the definitions

$$\cosh x = \frac{e^x + e^{-x}}{2}, \text{ and}$$

$$\sinh x = \frac{e^x - e^{-x}}{2}.$$

There is, of course, no difficulty here in replacing real x by complex z since e^z and e^{-z} have already been defined. If we write the definitions we encountered in the previous section alongside of the definitions of the corresponding hyperbolic functions, we notice much similarity:

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \cosh z = \frac{e^z + e^{-z}}{2}$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \sinh z = \frac{e^z - e^{-z}}{2}.$$

Replacing z by iz in $\cos z$ we get

$$\cos(iz) = \frac{e^{-z} + e^z}{2}$$

$$(1) \quad \cos(iz) = \cosh z.$$

Since we already know how to compute the cosine function, (1) permits us to easily compute the hyperbolic cosine. In a similar manner, replacing z by iz in the $\sin z$ gives

$$\sin(iz) = \frac{e^{-z} - e^z}{2i} = i \frac{e^z - e^{-z}}{2}$$

$$(2) \quad \sin(iz) = i \sinh z .$$

Equation (2) allows us to compute $\sinh z$ when we know $\sin(iz)$.

Example

Find $\cosh(\pi i)$.

Solution

From (1) we see that $\cosh(\pi i) = \cos(-\pi) = -1$.

Example

Prove that $\cosh^2 z - \sinh^2 z = 1$.

Solution

From (1) and (2) we have

$$\begin{aligned} \cosh^2 z &= \cos^2(iz) \\ - \sinh^2 z &= \sin^2(iz) . \end{aligned}$$

Adding these last two relations we have $\cosh^2 z - \sinh^2 z = \cos^2(iz) + \sin^2(iz) = 1$.

The previous example illustrates that not only do the hyperbolic identities resemble the trigonometric identities, they are the very same identities in a new notation! When we first studied the hyperbolic functions in the real calculus, we were quite surprised to find that functions defined in terms of the exponential could have so many relations which resembled similar relations for the seemingly unrelated trigonometric functions. These relations included

$$(3) \quad \sinh 2z = 2 \sinh z \cosh z$$

$$(4) \quad 1 - \tanh^2 z = \operatorname{sech}^2 z$$

$$(5) \quad \cosh(z + w) = \cosh z \cosh w + \sinh z \sinh w .$$

Our recent study of these functions in the complex plane has removed all elements of surprise, and shows that such relations are indeed to be expected.

Problems:

21. Find (a) $\sinh(3\pi i)$, (b) $\cosh(3\pi i)$, (c) $\tanh(3\pi i)$, and (d) $\sinh(1-i)$.
22. Write definitions of the remaining hyperbolic functions in terms of the exponential function.
23. Using (1) and (2), and the appropriate trigonometric identities, derive (3), (4) and (5).

2.5 The function $w = \sqrt{z}$.

Let us now investigate the function $w = \sqrt{z}$. The analytic investigation of this function is easy when we replace z by its polar form $z = re^{i\theta}$.

$$w = \sqrt{z} = \sqrt{re^{i\theta}} = (re^{i\theta})^{1/2}$$

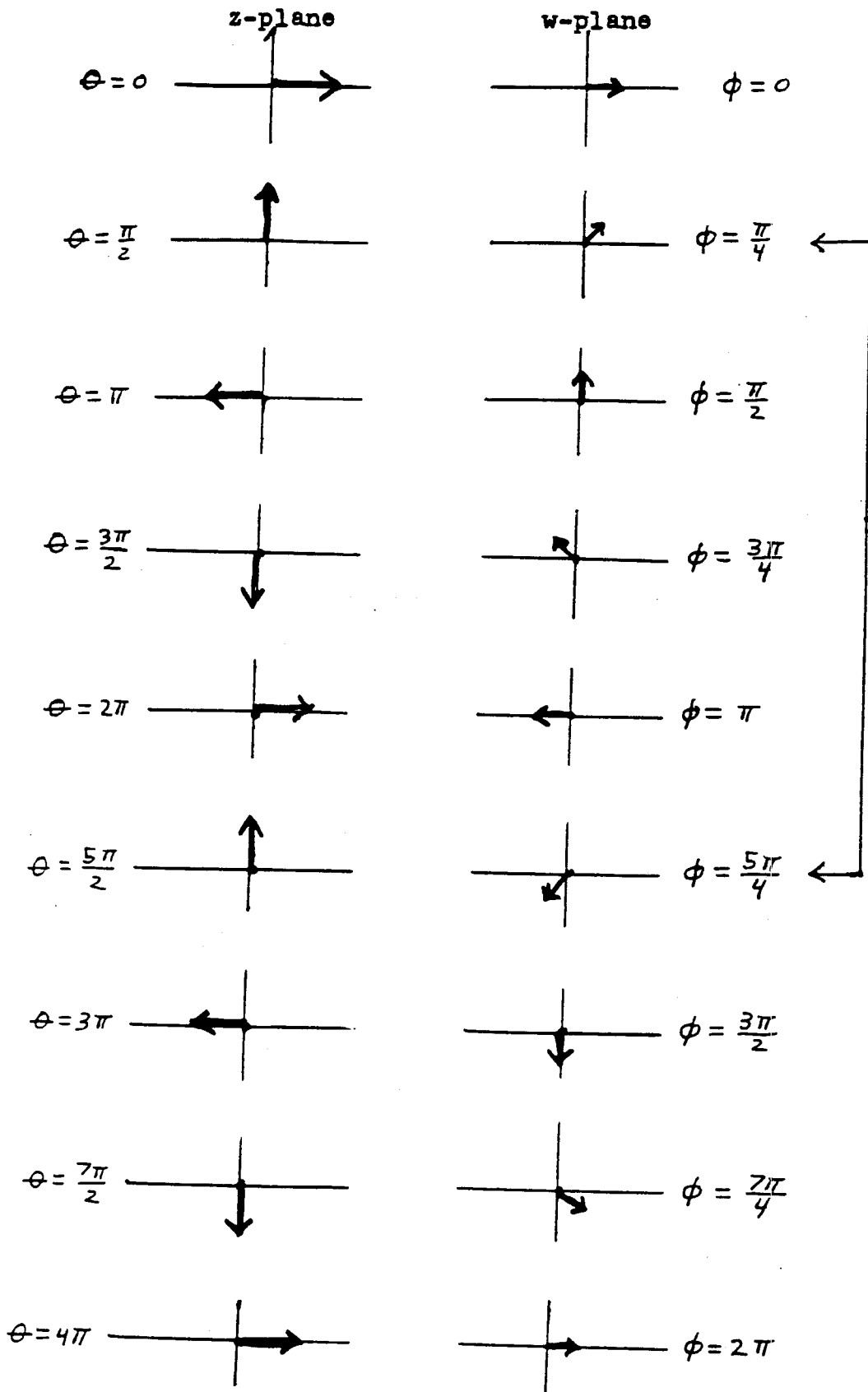
$$(1) \quad \sqrt{z} = r^{1/2} e^{i\theta/2}$$

To comprehend the significance of (1), let us imagine that the z -vector is like the hand of a rotating clock. Hold $|z|$ fixed, and let θ vary continuously starting from $\theta = 0$. Simultaneously, as the z -vector rotates, we watch the corresponding w -vector given by (1), $w = \sqrt{z}$. See Figure 2.9.

Figure 2.9

$w = \sqrt{z}$ (The z-vector must make two full revolutions before the w-vector makes one full revolution.)

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Notice here that the z-vectors are identical, but the w-vectors are distinct. The w-vectors are 180 degrees out of phase.

An examination of Figure 2.9 reveals the following important features:

(1) The z -vector must negotiate two full revolutions before the w -vector negotiates one full revolution.

(2) To each z -vector there corresponds two distinct w -vectors. (for example, the z -vectors at $\theta = \pi/2$ and $\theta = 5\pi/2$ are identical, while the w -vectors are out of phase by 180 degrees.) We say that the function $w = \sqrt{z}$ is "double valued".

Example

Find all values of $\sqrt{4i}$.

Solution

Write $4i = 4e^{i\pi/2}$ and $4e^{i5\pi/2}$. Now $\sqrt{4i} = (4e^{i\pi/2})^{1/2} = 2e^{i\pi/4}$, and $\sqrt{4i} = (4e^{i5\pi/2})^{1/2} = 2e^{i5\pi/4} = -2e^{i\pi/4}$. Thus $\sqrt{4i}$ equals $\sqrt{2} + \sqrt{2}i$ and $-\sqrt{2} - \sqrt{2}i$.

Problems

24. Find all values of $\sqrt{2 + 2\sqrt{3}i}$.

25. Consider the function $\sqrt[3]{z}$.

(a) Writing $z = re^{i\theta}$, determine a relation similar to (1) for $\sqrt[3]{z}$.

(b) Let the z -vector rotate and examine the corresponding w -vector as in Figure 2.10.

(c) How many revolutions must the z -vector make before the w -vector makes one full revolution?

(d) How many distinct w -vectors correspond to each z -vector? If you have one of these w -vectors, how can you immediately write down the others?

26. Consider the function $\sqrt[N]{z}$, where N is a positive integer. Answer the corresponding questions raised in the previous problem.

The function $w = \sqrt{z}$ is the first function we have encountered which is by nature "multiple-valued". It is convenient to introduce some artificial device which will remove this ambiguity. This idea is not unfamiliar. In the study of trigonometry we encountered the function $\sin^{-1}x$, which has infinitely many values for each x in the domain $-1 \leq x \leq 1$. We called $\text{Sin}^{-1}x$ the "principal value" of the inverse sine function and restricted it to the range $-\pi/2 \leq \text{Sin}^{-1}x \leq \pi/2$ thereby forcing an artificial restriction on the values this function could assume. In this way $\text{Sin}^{-1}x$ becomes single valued. We next introduce a means of removing the ambiguity inherent in our function $w = \sqrt{z}$.

We saw in Figure 2.9 that the z -vector must sweep over the z -plane twice before the w -vector can assume all its possible values. Why not then use two z -planes? Each z -plane can be used to correspond to "half" of the possible values of w . To be specific, let us assign to $z = re^{i\theta}$ values of θ in the range $-\pi < \theta \leq \pi$ on one plane (called branch one) and assign the values of θ to the range $\pi < \theta \leq 3\pi$ on the second plane (called branch two). Since z executes only one revolution on each of these sheets, only one value of w will correspond to each z , $w = r^{1/2}e^{i\theta/2}$. In Figure 2.10 we plot lines of constant modulus and lines of constant argument for the function $w = \sqrt{z}$ directly over both branches of the function. Notice that on each branch $w = \sqrt{z}$ is single valued. Let us arbitrarily define the "principal value" of \sqrt{z} to be the values obtained on branch one. For example, the principal value of $\sqrt{4i}$ is $2e^{i\pi/4} = \sqrt{2} + \sqrt{2}i$. We have therefore, by the introduction of the two "branches" effectively removed the ambiguity in the nature of \sqrt{z} .

This line of discontinuities is called a "branch line" or "branch cut" of the function. The two end points of the line are at $z = 0$ and $z = \infty$, and these points are called "branch points".

We also remark that the choice of the negative real axis as the branch cut was quite arbitrary. We could have chosen the positive real axis, or the positive y-axis, or indeed, any line from $z=0$ to $z = \infty$ as our branch cut. The resulting branches of \sqrt{z} would, of course be different, but the ambiguity in $w = \sqrt{z}$ would again have been removed. While the branch line is arbitrary, the branch points $z = 0$ and $z = \infty$ remain fixed.

Problems:

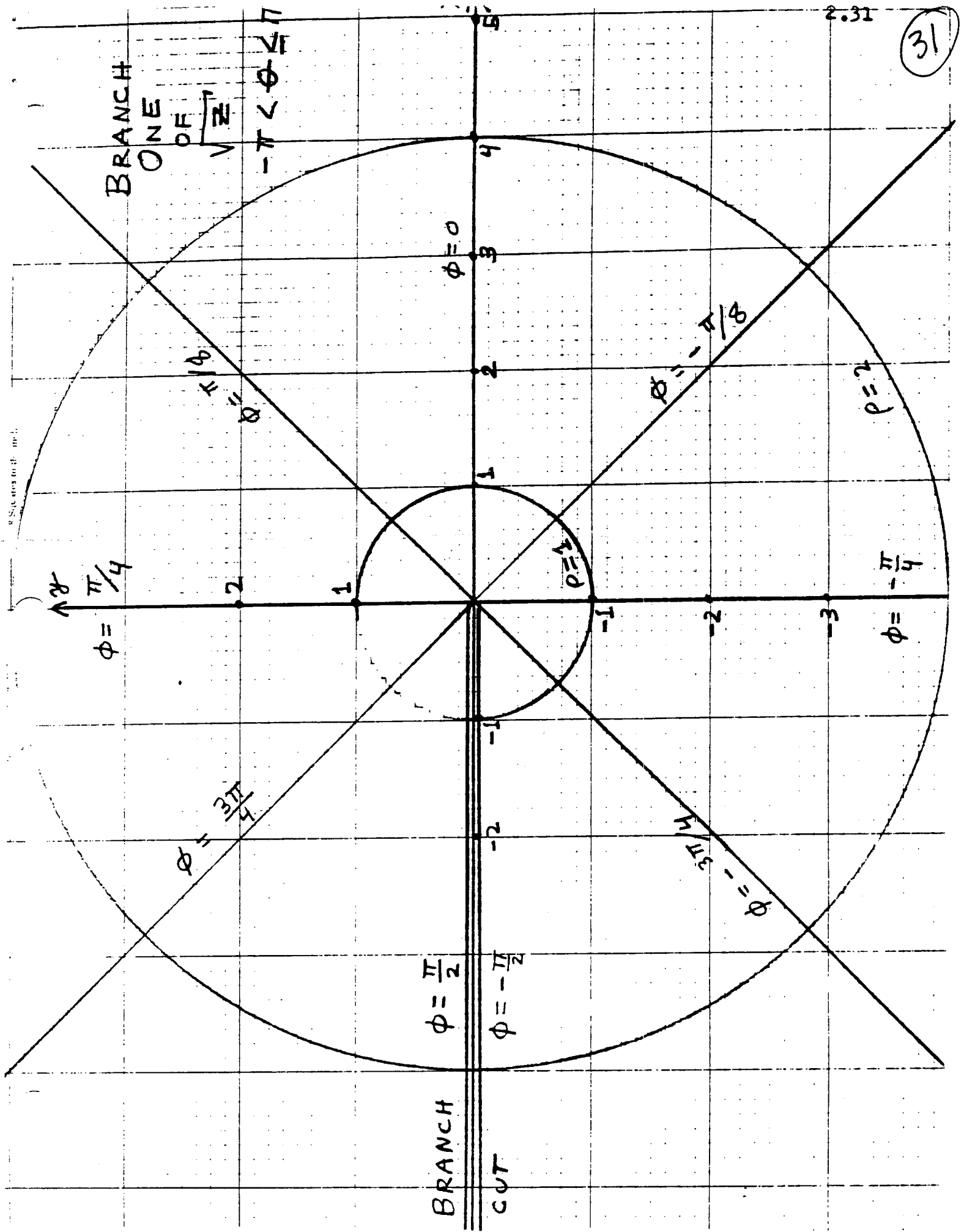
27. Introduce new branches for $w = \sqrt{z}$ by taking the negative y-axis as the branch cut. Make a diagram similar to Figure 2.10.

28. Introduce appropriate branches for the function $w = \sqrt[n]{z}$.

Select the negative real axis as the branch cut. Make a diagram similar to Figure 2.10.

29. Discuss the problem of introducing branches for the function $w = \sqrt[N]{z}$, where N is a positive integer. How many branches are necessary?

Besides the introduction of two branches for $w = \sqrt{z}$ to remove the confusion caused by its natural double-valuedness, we have yet another method. In this new method we replace the z-plane by a new surface which is called a "Riemann-surface". The Riemann surface is best understood by actually constructing it. Pages 2.31 and 2.32 show enlarged versions of branches one and two of \sqrt{z} originally described in Figure 2.10. Remove these pages, and actually cut with scissors along the branch line to the branch point at $z = 0$. Lay branch two on top of branch one. Notice that along the top part of the cut on branch one $\phi = \pi/2$,



BRANCH ONE OF \sqrt{z}
 $-\pi < \theta \leq \pi$

BRANCH CUT

$\phi = \pi/4$

$\phi = 3\pi/4$

$\phi = \pi/2$
 $\phi = -\pi/2$

$\phi = -\pi/8$

$\rho = 2$

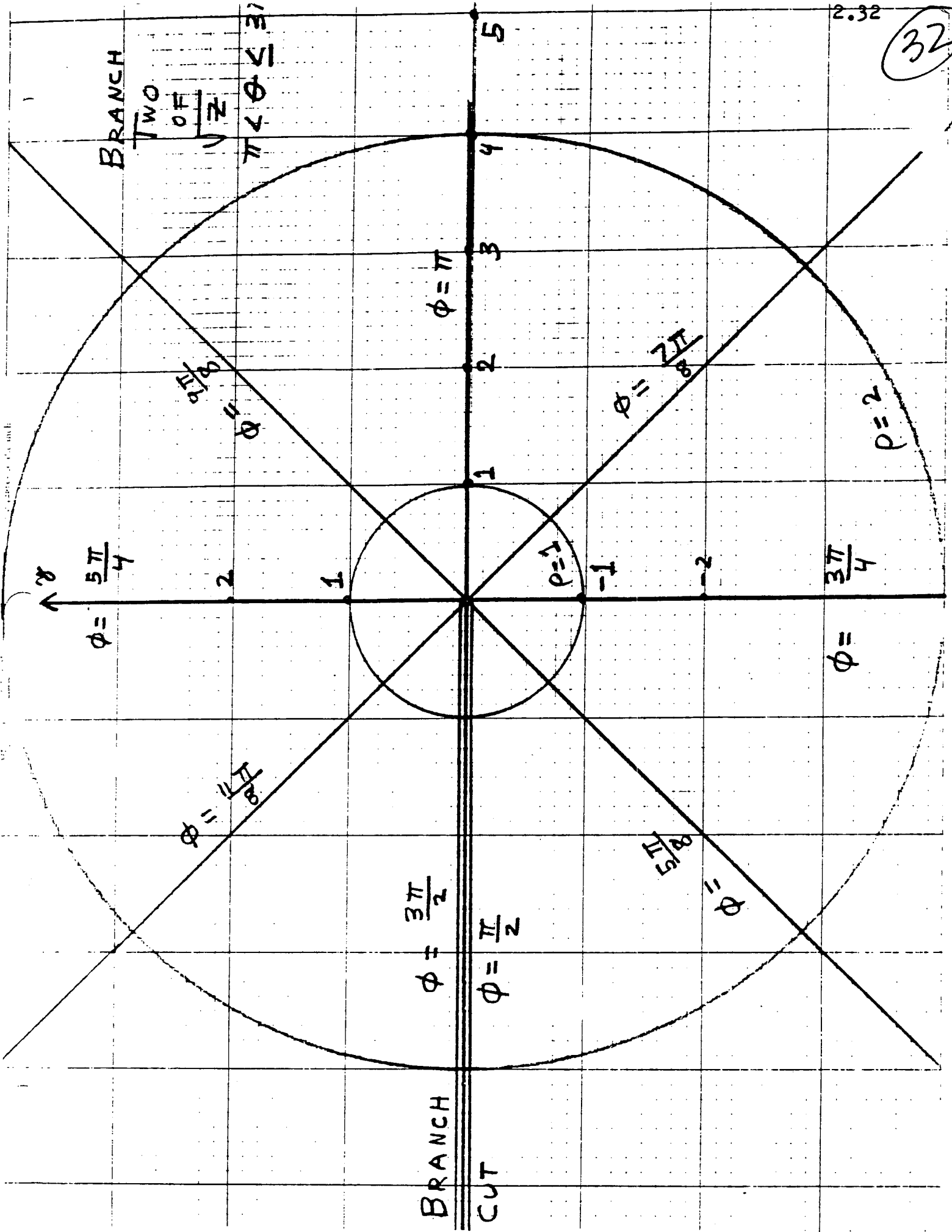
$\phi = -\pi/4$

$\phi = -3\pi/4$

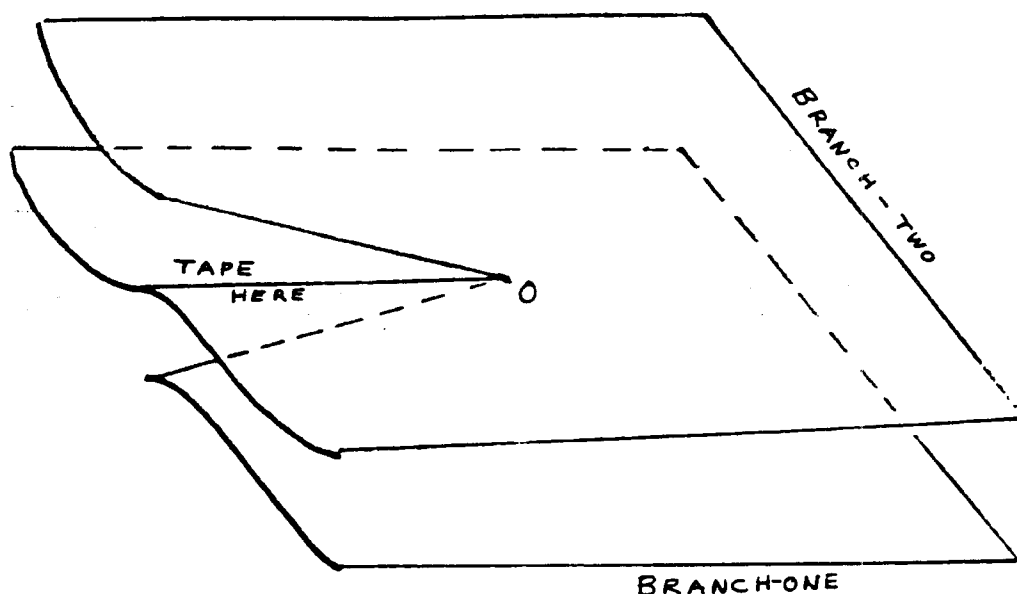
$\rho = 1$

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and $\phi = \pi/2$ also on the bottom part of the cut of branch two. Thus ϕ wants to continue from the lower sheet to the upper sheet along this cut. Now actually tape the top edge of the branch cut of the lower sheet to the bottom edge of the branch cut of the upper sheet.

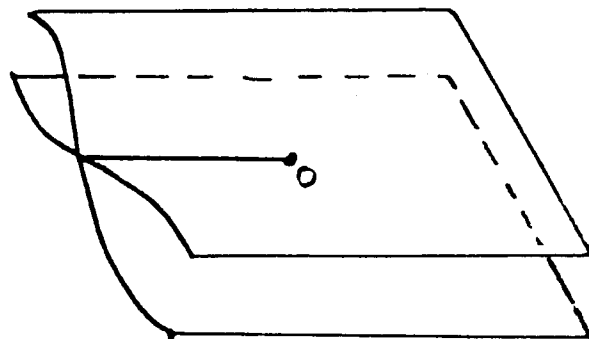


Along the remaining two edges of the branch cuts we have $\phi = -\pi/2$ (branch one) and $\phi = 3\pi/2$ (branch two). Since $e^{-\pi i} = e^{3\pi i}$ we see that the values of \sqrt{z} also join continuously across these edges. We might imagine that these edges are also taped together (although this is physically impossible). This new

surface is our

"Riemann-surface" for

the $w = \sqrt{z}$ function.



The advantage of the Riemann surface over the two separate branches considered previously is that the discontinuities previously seen along the branch cuts are now removed.

Problems:

30. Starting at $z = -1$ on the lower sheet of the Riemann surface for $w = \sqrt{z}$, use a pencil and trace the values of \sqrt{z} encountered as you move along the unit circle $|z| = 1$ in the counterclockwise sense. After making two full revolutions, you should imagine that the starting and ending points are joined together.
31. Make a Riemann surface for $w = \sqrt[3]{z}$.
32. Describe the Riemann surface for $w = \sqrt[N]{z}$, where N is a positive integer.

2.6 The natural logarithm

The next function that we wish to explore in the complex plane is the natural logarithm $w = \log z$. How are we to define this function? The notation $w = \log z$ implies that $z = e^w$. If we write $z = r e^{i\theta}$ and $w = u + iv$ we have

$$w = \log z$$

$$z = e^w$$

$$r e^{i\theta} = e^{u + iv}$$

$$r e^{i\theta} = e^u e^{iv}.$$

This last relation tells us that

$$(1) \quad r = e^u, \text{ and}$$

$$(2) \quad \theta = v.$$

From (1) we have $u = \text{Log } r$, (where we have introduced the notation $\text{Log } r$ to mean the familiar real logarithm defined for positive real values of r), Using (2) we get

$$w = \log z = u + iv$$

$$(3) \quad w = \text{Log } r + i\theta.$$

Relation (3) is our defining relation for the natural logarithm. Since the angle " θ " is not uniquely defined for each z , we see that $\log z$ is indeed a multiple valued function.

Example

Find all values of $\log i$.

Solution

Write i in polar form as $i = 1 e^{i(\pi/2 + 2\pi n)}$, where $n = 0, \pm 1, \pm 2, \dots$. Thus $r = 1$ and $\theta = \pi/2 + 2\pi n$, and (3) gives $\log i = \text{Log } 1 + i(\pi/2 + 2\pi n) = i(\pi/2 + 2\pi n)$.

Problem

33. Find all values of (a) $\log(-1)$, (b) $\log e$, (c) $\log(1+i)$, (d) $\log 0$.