AN INTUITIVE INTRODUCTION TO COMPLEX ANALYSIS

Thomas J Osler
Mathematics Department
Rowan University
Glassboro NJ 08028

osler@rowan.edu

CHAPTER 4

POWER SERIES EXPANSIONS

4.1 The power series as an extension of the decimal number

Ask the man in the street to divide 3/4 by 1/2 . Chances are that he will be incapable of solving it, even if he is a college graduate ! But, give him pencil and paper and ask him to divide 0.75 by 0.5, and he will easily do it. Why ? Arithmetic with fractions requires special rules which few use in everyday life. Invert, find the lowest common denominator, etc. . But arithmetic with decimals is exactly like arithmetic with whole numbers (except for positioning the decimal point). The invention of decimal numbers brings a remarkable simplicity to arithmetic.

We forget, perhapse, that there was a price for this simplicity. We had to grow accustomed to infinite representations such as 1/3 = 0.3333..., and even more disturbing, 0.999... = 1 . How did we cope with the infinite at so tender an age ? We simply said "throw away all but the first so many digits, and the resultant quantity is close enough for our purpose".

If decimals can simplify arithmetic so very much, is there not an analogous concept in higher mathematics which can simplify our manipulations with functions ?

The decimal expansion for 92. 14 ... is really a series in powers of 1/10 :

$$9(1/10)^{-1} + 2(1/10)^0 + 1(1/10)^1 + 4(1/10)^2 + \dots .$$

It is therefore, not unreasonable to replace 1/10 by the algebraic variable x and obtain the expression:

$$9x^{-1} + 2 + x + 4x^2 + \dots .$$

This "power series" is the desired extension of the numerical decimal. We know from the calculus that there are many such power series. Indeed, tables of functions are often computed using them. We saw in the calculus such power series expansions as

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots .$$

Early mathematicians expanded their functions in power series to enable them to compute results which were otherwise too difficult. For example

$$\int_0^x \frac{dx}{1+x^2} = \int_0^x 1 - x^2 + x^4 - x^6 + \dots \; dx$$

$$= \quad x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \quad .$$

The deep theoretical problem born of the fact that these series are infinite was disposed of by simply saying "throw away all but the first few terms, since the accuracy so obtained is good enough" . While this attitude can certainly not be considered today as precise mathematical thinking, it nevertheless has a useful purpose. Much important information can be obtained through the uncritical manipulation of power series.

We shall see that the power series expansion for a function is a very "natural" expression. Indeed, if $f(z)$ is regular at the point $z = z_0$, then $f(z)$ can be expanded in a convergent series of the form

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots .$$

4.2 Formal manipulations with power series

The term "formal", when applied to manipulations with series, implies that operations are performed without consideration being given to the question of the convergence of the series. Let us now consider, formally, the algebraic manipulations of addition, subtraction, multiplication, division and composition for power series.

Given the two power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad \text{and} \qquad g(z) = \sum_{n=0}^{\infty} b_n z^n ,$$

it is only natural to expect that the following manipulations are valid:

Addition

(A) $$f(z) + g(z) = \sum_{n=0}^{\infty} (a_n + b_n) z^n$$

Subtraction

(B) $$f(z) - g(z) = \sum_{n=0}^{\infty} (a_n - b_n) z^n$$

Multiplication

$$
\begin{array}{l}
b_0 + b_1 z + b_2 z^2 + \dots \\
\underline{a_0 + a_1 z + a_2 z^2 + \dots} \\
a_0 b_0 + a_0 b_1 z + a_0 b_2 z^2 + \dots \\
\qquad a_1 b_0 z + a_1 b_1 z^2 + \dots \\
\qquad\qquad a_2 b_0 z^2 + \dots \\
\hline
a_0 b_0 + (a_0 b_1 + a_1 b_0) z + (a_0 b_2 + a_1 b_1 + a_2 b_0) z^2 + \dots
\end{array}
$$

(C) $$f(z)\, g(z) = \sum_{n=0}^{\infty} \left(\sum_{r=0}^{n} a_r b_{n-r} \right) z^n$$

Division

$$
\begin{array}{r|l}
 & \dfrac{a_0}{b_0} + \dfrac{a_1 b_0 - b_1}{b_0} z + \dots \\
\hline
b_0 + b_1 z + \dots & a_0 + a_1 z + \dots \\
 & \underline{a_0 + b_1/b_0 z + \dots} \\
 & \dfrac{a_1 b_0 - b_1}{b_0} z + \dots \\
 & \underline{\dfrac{a_1 b_0 - b_1}{b_0} z + \dots} \\
 & + \dots
\end{array}
$$

(D) $$\frac{f(z)}{g(z)} = \frac{a_0}{b_0} + \frac{a_1 b_0 - b_1}{b_0} z + \dots$$

Composite functions

(E) $$f(h(z)) = \sum_{n=0}^{\infty} a_n (h(z))^n$$

In order to illustrate these formal rules, we accept for the moment, the following power series expansions:

(1) $$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + \dots \quad \text{(geometric series)}$$

(2) $$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + \frac{z}{1} + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$$

(3) $$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots$$

(4) $$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots$$

At this time we ignore the question of the convergence of these series and concentrate our attention on the formal manipulations.

Example 1

Verify (1) by long division.

Solution

$$\begin{array}{r|l} & 1 + z + z^2 + \dots \\ \hline 1 - z & 1 \\ & \underline{1 - z} \\ & \quad z \\ & \quad \underline{z - z^2} \\ & \qquad z^2 \\ & \qquad \underline{z^2 - z^3} \\ & \qquad\quad z^3 \\ & \qquad\quad \dots \end{array}$$

Example 2

Using (1) and (E), show that $\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n$.

Solution

Replacing z by -z in (1) we get

$$\frac{1}{1-(-z)} = \sum_{n=0}^{\infty} (-z)^n$$

$$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n \quad .$$

Example 3

Show that $\frac{1}{3+z} = \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{3^{n+1}}$.

Solution

We manipulate the expression $\frac{1}{3+z}$ until we get a suitable expression of the form $\frac{1}{1-x}$, then we apply (1).

$$\frac{1}{3+z} = \frac{1}{3(1+\frac{z}{3})} = \frac{1}{3}\left[\frac{1}{1-(-\frac{z}{3})}\right] = \frac{1}{3}\sum_{n=0}^{\infty}\left(-\frac{z}{3}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{3^{n+1}} \; .$$

Problems

1. Show that $\frac{1}{1-z^2} = \sum_{n=0}^{\infty} z^{2n}$.

2. Show that $\frac{1}{a-z} = \sum_{n=0}^{\infty} \frac{z^n}{a^{n+1}}$.

3. Show that $\frac{1}{2 - z} = \sum_{n=0}^{\infty} (z-1)^n$.

4. Show that $\frac{1}{4 + z} = \sum_{n=0}^{\infty} \frac{(-1)^n (z+1)^n}{3^{n+1}}$.

5. Show that $\frac{1}{a - z} = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(a - z_0)^{n+1}}$.

Example 4

Using (3) and (4) verify the identity $2 \sin z \cos z = \sin 2z$.

Solution

$$\sin z = z - \frac{z^3}{6} + \frac{z^5}{120} - \dots$$

$$\cos z = 1 - \frac{z^2}{2} + \frac{z^4}{24} - \dots$$

$$\begin{array}{rrrl} z & - \frac{z^3}{6} & + \frac{z^5}{120} & - \dots \\ & - \frac{z^3}{2} & + \frac{z^5}{12} & - \dots \\ & & \frac{z^5}{24} & - \dots \end{array}$$

$$\sin z \cos z = z - \frac{4z^3}{6} + \frac{16z^5}{120} - \dots$$

$$= z - \frac{2^2 z^3}{3!} + \frac{2^4 z^5}{5!} - \dots$$

$$= \frac{1}{2}\left((2z) - \frac{(2z)^3}{3!} + \frac{(2z)^5}{5!} - \dots \right)$$

$$= \frac{1}{2} \sin 2z \quad .$$

Problems

6. Using (3) and (4), verify that $\sin^2 z + \cos^2 z = 1$.

7. Using (2) and (4) verify that $e^{iz} + e^{-iz} = 2 \cos z$.

8. Using power series, verify that $e^{iz} - e^{-iz} = 2i \sin z$.

9. Obtain the power series expansion of $\cosh z$ using (2) and the analytic definition of $\cosh z$ from section 2.4 .

10. Obtain the first three terms in the power series expansion of $\tan z$ by dividing (4) into (3) .

Two formal manipulations of great utility arise from differentiation and integration of the power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

term by term.

Differentiation term by term

$$\frac{d\, f(z)}{dz} = \sum_{n=0}^{\infty} a_n \frac{d\, z^n}{dz}$$

$$\text{(F)} \qquad \frac{d\, f(z)}{dz} = \sum_{n=0}^{\infty} n\, a_n\, z^{n-1}$$

Integration term by term

$$\int_0^z f(t)dt = \sum_{n=0}^{\infty} a_n \int_0^z t^n\, dt$$

$$\text{(G)} \qquad \int_0^z f(t)dt = \sum_{n=0}^{\infty} a_n \frac{z^{n+1}}{n+1}$$

We have not yet given any careful thought as to the meaning of differentiation and integration of functions of a complex variable. Nevertheless, it is not unreasonable to expect that the result of differentiating or integrating a familiar function of a complex variable will be the formula familiar from the real calculus. Thus we would expect that

$$\frac{d\, z^p}{dz} = p\, z^{p-1}\,, \qquad \int z^p\, dz = \frac{z^{p+1}}{p+1} + C$$

$$\frac{d\, e^z}{dz} = e^z\,, \qquad \int e^z\, dz = e^z + C$$

$$\frac{d \log z}{dz} = \frac{1}{z}\,, \qquad \int \frac{dz}{z} = \log z + C$$

$$\frac{d \sin z}{dz} = \cos z \qquad \int \cos z\, dz = \sin z + C$$

$$\ldots \qquad \ldots$$

We saw in Chapter 2 that because our definitions for z^p, e^z, $\log z$, $\sin z$, ..., for complex numbers z were "natural", identities familiar from the study of real variables carried over in the very same form for complex variables. Therefore the above differentiation formulas are to be expected. In the next chapters we will take a much closer look at these operations.

In addition, we note that just because we can always differentiate and integrate a finite sum of terms

$$\frac{d(\ f_1(z) + f_2(z) + \ldots + f_N(z)\ )}{dz} = f_1'(z) + f_2'(z) + \ldots + f_N'(z)$$

$$\int_a^b (\ f_1(z) + \ldots + f_N(z)\ )dz = \int_a^b f_1(z)dz + \ldots + \int_a^b f_N(z)dz$$

we cannot always do the same for an infinite series of terms

$$\frac{d}{dz} \sum_{n=0}^{\infty} f_n(z) = \sum_{n=0}^{\infty} \frac{d\ f_n(z)}{dz}$$ and

$$\int_a^b \sum_{n=0}^{\infty} f_n(z)\ dz = \sum_{n=0}^{\infty} \int_a^b f_n(z)\ dz\ .$$

While these last two formulas are not always true, quite fortunately they are often true. We will examine more closely the validity of (F) and (G) later. For now, we ignore the deeper question of the appropriateness of term by term differentiation and integration and we simply operate formally.

Example 5

Derive (4) from (3).

Solution

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots$$

$$\frac{d \sin z}{dz} = 1 - \frac{3z^2}{3!} + \frac{5z^4}{5!} - \ldots$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots$$

Example 6

Using the formula $\arctan z = \int_0^z \frac{dt}{1+t^2}$, familiar from calculus, find the power series expansion for $\arctan z$.

Solution

Replacing z by $-t^2$ in (1) we get

$$\frac{1}{1+t^2} = \sum_{n=0}^{\infty} (-1)^n t^{2n} .$$

Integrating term by term we get

$$\int_0^z \frac{dt}{1+t^2} = \sum_{n=0}^{\infty} (-1)^n \int_0^z t^{2n}\, dt$$

$$\arctan z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{2n+1} .$$

Problems

11. Since $\frac{d \cosh z}{dz} = \sinh z$, use the result of problem 9 to get the power series for $\sinh z$.

12. We know from the calculus that $\int_0^z \frac{dt}{1+t} = \mathrm{Log}(z+1)$.

Obtain a power series for $\mathrm{Log}(z+1)$ in powers of z .

13. Using the result of problem 12, obtain the power series expansion of $\mathrm{Log}\, z$ in powers of $(z-1)$.

14. By differentiating (1), obtain the power series for $(1-z)^{-2}$.

15. Obtain the power series expansion for $(1-z)^{-3}$.

As a final formal consideration, we obtain a method that determines the numbers a_n in the series

(6) $$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + a_3(z - z_0)^3 + \dots ,$$

when the function $f(z)$ has been given.

Step 1: Set $z = z_0$ in (6) .

$$f(z_0) = a_0 + 0 + 0 + 0 + \dots$$

Thus we have found the first number

$$\boxed{a_0 = f(z_0)} .$$

Step 2: Differentiate (6) and then set $z = z_0$.

(7) $$f^{(1)}(z) = a_1 + 2a_2(z - z_0) + 3a_3(z - z_0)^2 + 4a_4(z - z_0)^3 + \dots$$

$$f^{(1)}(z_0) = a_1 + 0 + 0 + 0 + \dots$$

Thus we have

$$\boxed{a_1 = f^{(1)}(z_0)} .$$

Step 3: Differentiate (7) and then set $z = z_0$.

(8) $$f^{(2)}(z) = 2a_2 + 3 \cdot 2\, a_3(z - z_0) + 4 \cdot 3\, a_4(z - z_0)^2 + \dots$$

$$f^{(2)}(z_0) = 2\, a_2 + 0 + 0 + \dots$$

Thus we have

$$\boxed{a_2 = \frac{f^{(2)}(z_0)}{2}} \, .$$

<u>Step 4:</u> Differentiate (8) and set $z = z_0$.

$$f^{(3)}(z) = 3 \cdot 2\, a_3 + 4 \cdot 3 \cdot 2\, a_4(z - z_0) + 5 \cdot 4 \cdot 3\, a_5(z - z_0)^2 + \dots$$

$$f^{(3)}(z_0) = 3 \cdot 2\, a_3 + 0 + 0 + 0 + \dots$$

Thus we have

$$\boxed{a_3 = \frac{f^{(3)}(z_0)}{3!}} \, .$$

Continuing in this way we get

$$\boxed{a_n = \frac{f^{(n)}(z_0)}{n!}} \, .$$

Substituting this last relation into (6) we have

<u>Taylor's formula</u>

$$\text{(H)} \quad f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

Series of the form (H) are often called Taylor's series. All the series examined in this section were Taylor's series. The special case of (H) in which $z_0 = 0$ is sometimes called a Maclaurin series. Series (1) through (4) are Maclaurin series.

Our derivation of (H) was again purely formal. We did not consider the convergence of the infinite series at all. Even if the infinite series does converge, how do we know that it converges to the original function $f(z)$? In spite of the fact that we have left these deeper questions unanswered, we can anticipate that for analytic functions (that is "natural functions") the expression (H) should make some sense. We will return to this question in the next sectionS .

Example 7

In Example 1 we derived (1) be long division. Now derive (1) using the Taylor's formula (H).

Solution

We select $z_0 = 0$, since the formula we wish to verify (1) is a Maclaurin series. We must find a formula for the n^{th} derivative of $(1-z)^{-1}$ evaluated at $z = 0$.

$D^0(1-z)^{-1} = (1-z)^{-1}$	and setting $z = 0$ we get	1
$D^1(1-z)^{-1} = (1-z)^{-2}$	" " " "	1
$D^2(1-z)^{-1} = 2(1-z)^{-3}$	" " " "	2
$D^3(1-z)^{-1} = 2\cdot 3\ (1-z)^{-4}$	" " " "	3!
...		
$D^N(1-z)^{-1} = n!\ (1-z)^{-n-1}$	" " " "	n!

Since $D^n(1-z)^{-1}\Big|_{z=0} = n!$, we see at once from (H) that

$$(1-z)^{-1} = \sum_{n=0}^{\infty} z^n .$$

Problems:

16. Derive (2) using Taylor's formula (H).

17. Derive a Taylor's series for e^z in powers of $z-z_0$. Use two different methods. First assume (2) is true and use (E) with $e^z = \exp(z_0)\exp(z-z_0)$, and second apply Taylor's formula (H).

18. Derive the series considered in problem 5 using Taylor's formula (H).

19. Derive the series considered in problem 13 using Taylor's formula (H).

All the manipulations performed in this section were natural extensions of manipulations the reader has performed before with polynomials $w = a_0 + a_1 z + \dots + a_n z^n$. We have viewed the Taylor's series simply as a very "long" polynomial. This is not unreasonable. In addition, several of the Taylor's series were derived formally by entirely different methods, and in each case, the same series was obtained. This evidence adds weight to our suspicion that Taylor's series are "natural" expressions for describing our analytic (natural) functions.

4.3 Numerical computations

An important use of power series is for the numerical computation of tables of frequently used functions.

Example 1

Using the power series

(1) $$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

compute the number e .

Solution

It is helpful to employ the following table of reciprocal factorials rounded off to seven decimal places.

$1/2! = 0.5000000$
$1/3! = 0.1666667$
$1/4! = 0.0416667$
$1/5! = 0.0083333$
$1/6! = 0.0013889$
$1/7! = 0.0001984$
$1/8! = 0.0000248$
$1/9! = 0.0000028$
$1/10! = 0.0000003$

Using (1) with $z = 1$ we have

(2) $$e = 1 + 1 + 1/2! + 1/3! + \ldots + 1/10! + \ldots .$$

Adding the first eleven terms we get $e = 2.7182819$ which is accurate only to the first six decimal places. Compare this result with the more accurate value $e = 2.71828\ 18284\ 59045$.

Example 2

Use the Maclaurin series

(3) $$(1-z)^{-1} = 1 + z + z^2 + z^3 + z^4 + z^5 + \ldots$$

to compute $(1-z)^{-1}$ when $z = -1$.

Solution

Substituting $z = -1$ in (3) we get

(4) $$0.5 = 1 - 1 + 1 - 1 + 1 - 1 + \dots .$$

The series just obtained makes no sense. Unlike the series used in Example 1 which gave us e correct to 6 decimal places using eleven terms, this series does not approach the value 0.5 regardless of the number of terms employed. We say that the series (4) diverges, while the series (2) converges. We have not defined these terms precisely, and for the moment, we will simply use our mathematical instinct to help us decide if a given series converges or diverges.

Problems:

Use appropriate series from the previous section. Decide if the series converge or diverge.

20. Show that $\sin i = 1.1752\, i$.

21. Show that $\operatorname{Log}(1.1) = 0.0953$.

22. Try to compute $\operatorname{Log}(-9)$ using the Taylor's series from the previous problem.

23. Compute $\cos\left(\frac{1+i}{\sqrt{2}}\right)$.

4.4 The region of convergence

We saw in the previous section that power series were sometimes quite useful for computing the values of a function, but at other times they gave no meaningful information at all. For what values of z will a given power series converge ? We discover the answer to this question now.

We begin by examining the Maclaurin series for $(1-z)^{-1}$ by long division.

$$
\begin{array}{r|l}
 & 1 + z + z^2 + \dots + z^N \\
\hline
1 - z & 1 \\
 & \underline{1 - z} \\
 & \quad z \\
 & \quad \underline{z - z^2} \\
 & \qquad z^2 \\
 & \qquad \underline{z^2 - z^3} \\
 & \qquad\quad z^3 \\
 & \qquad\quad \underline{\dots} \\
 & \qquad\qquad z^{N+1}
\end{array}
$$

Therefore

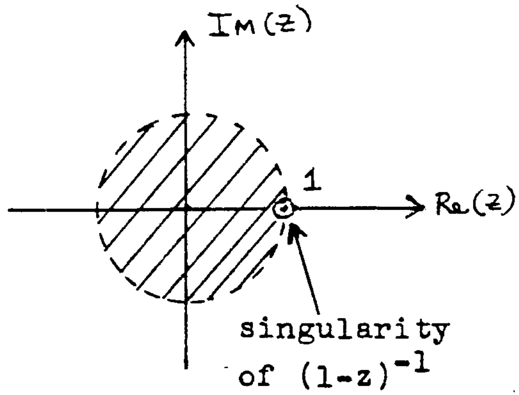
$$
\underbrace{\frac{1}{1-z}}_{\text{given function}} = \underbrace{1 + z + z^2 + \dots + z^N}_{\text{approximation to the given function using } N+1 \text{ terms of the Taylor's series}} + \underbrace{\frac{z^{N+1}}{1-z}}_{\text{error resulting from the use of the terminated Taylor's series}}
$$

The question: "Does the power series converge ? " reduces to "Does the error term approach zero as N approaches infinity ?" . Replacing z by $re^{i\theta}$ we see that the modulus of the error term becomes

$$
\left|\frac{z^{N+1}}{1-z}\right| = \left|\frac{r^{N+1}\, e^{i\theta(N+1)}}{1 - re^{i\theta}}\right| = \left|\frac{r^{N+1}}{1 - re^{i\theta}}\right| .
$$

Hold r fixed and let N tend to infinity. If $0 \leq r < 1$, then r^{N+1} approaches zero as N approaches infinity and thus the series will converge. However, if $r \geq 1$, then r^{N+1} does not approach zero, and the series will diverge. Thus we see that the region of convergence for the series

(1) $$(1 - z)^{-1} = \sum_{n=0}^{\infty} z^n \quad \text{for} \quad |z| < 1$$



is the open circle with center at $z = 0$ and radius one.

Example 1

We saw in the previous section that

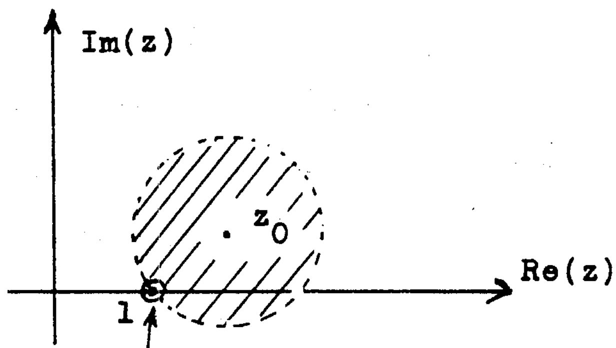
(2) $$(1-z)^{-1} = (1-z_0)^{-1} \sum_{n=0}^{\infty} \left\{ \frac{z - z_0}{1 - z_0} \right\}^n .$$

Determine the region of convergence.

Solution

If we replace z in the infinite series (1) by $\frac{z - z_0}{1 - z_0}$ we get the infinite series in (2). Since (1) converges for $|z| < 1$, then (2) converges for

$$\left| \frac{z - z_0}{1 - z_0} \right| < 1 .$$



This last inequality is $|z - z_0| < |1 - z_0|$ which is the interior of the circle with center at z_0 with radius $|1 - z_0|$.

Problem

24. In problem 5 we showed that $(a-z)^{-1} = \sum_{n=0}^{\infty} (z-z_0)^n/(a-z_0)^{n+1}$

Determine the region of convergence of this series and show the region on the complex z-plane.

Conjecture 4.1

Based on the experience gained above, what might we conjecture as to the shape, position, and size of the region of convergence of the Taylor's series

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$$

when $f(z)$ is some explicitly given analytic function. (See Appendix II.)

Example 2

Determine the region of convergence of the series

$$\text{Log } z = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(z-1)^n}{n} .$$

Solution

The function Log z has a singularity (branch point) at $z=0$. Since the Taylor's series converges inside the largest circle with center at $z=1$ not containing any singularities of Log z, we know that the circle of convergence must be $|z-1| < 1$.

Problem

25. Determine the circle of convergence for every power series generated in problems 1 through 15.

Conjecture 4.2

Conjecture the region in the z-plane over which the formal manipulations of the addition (A), subtraction (B), and the multiplication (C) of Taylor's series discussed in section 4.2 are valid.

Conjecture 4.3

In expression (D) of section 4.2 we formally divided the Taylor's series for $f(z)$ by the Taylor's series for $g(z)$. Conjecture the region in the z-plane in which this division is valid. What assumption must be made concerning $g(z)$?

Conjecture 4.4

In expression (E) of section 4.2 we discussed the formal manipulation of replacing z by $h(z)$ in the power series expansion of the function $f(z)$. Determine the region in the z-plane over which this substitution is valid.

Conjecture 4.5

Conjecture the region over which a Taylor's series may be differentiated or integrated term by term. See relations (F) and (G) of section 4.2 .

We were able to determine the circle of convergence of the power series expansion of the analytic function $f(z)$ from the position of the singularities of $f(z)$. Suppose now that the power series itself is given, but the function which generated it is unknown. How are we to determine the circle of convergence now ? For this purpose, a criterion known as the "ratio test" is often helpful. This is the same ratio test used in the real calculus, and its proof for complex series is almost identical to the proof for real series. We will not consider the proof here, but we will use the ratio test to determine the size of the circle of convergence.

Ratio test

Let $\sum_{n=0}^{\infty} c_n$ be a series of complex numbers. Suppose

$$\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| = L .$$

Then the series converges if $L < 1$, diverges if $L > 1$, and might converge or diverge if $L = 1$.

Example 3

We know from example 2 that $\sum_{n=1}^{\infty} (-1)^{n+1}(z-1)^n/n$ converges for $|z-1| < 1$. Use the ratio test to double check this conclusion.

Solution

We must examine the ratio

$$\left|\frac{\dfrac{(-1)^{n+2}(z-1)^{n+1}}{n+1}}{\dfrac{(-1)^{n+1}(z-1)^n}{n}}\right| = \left|\frac{n(z-1)}{n+1}\right| = \frac{|z-1|}{1+\frac{1}{n}}$$

as n approaches infinity. Since

$$\lim_{n\to\infty} \left|\frac{z-1}{1+\frac{1}{n}}\right| = |z-1| ,$$

we see that the series converges for all z such that $|z-1| < 1$ and diverges for $|z-1| > 1$.

Problems

26. Use the ratio test to find the region of convergence of the geometric series $\sum_{n=0}^{\infty} z^n$ which we have shown before converges for $|z| < 1$.

27. Find the region of convergence of the hypergeometric series

$$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n z^n}{(c)_n \, n!} ,$$

where $(a)_n = a(a+1)(a+2) \ldots (a+n-1)$. (We define $(a)_0 = 1$.)

28. Find the region of convergence of the confluent hypergeometric series

$$\sum_{n=0}^{\infty} \frac{(a)_n \, z^n}{(b)_n \, n!} .$$

4.5 The binomial series

One of the most important expansions is the binomial series

(1) $$(1 + z)^p = 1 + \frac{p}{1}z + \frac{p}{1}\frac{p-1}{2} z^2 + \frac{p}{1}\frac{p-1}{2}\frac{p-2}{3} z^3 + \ldots .$$

The history of this series shows how general results in mathematics are often obtained only after many years of experience are gained with special cases. We now take a brief look at this history.

As early as the thirteenth century, the following powers of the binomial $1+z$ were known to Arab algebraists

$(1 + z)^0 = 1$

$(1 + z)^1 = 1 + z$

$(1 + z)^2 = 1 + 2z + z^2$

$(1 + z)^3 = 1 + 3z + 3z^2 + z^3$

$(1 + z)^4 = 1 + 4z + 6z^2 + 4z^3 + z^4$

$(1 + z)^5 = 1 + 5z + 10z^2 + 10z^3 + 5z^4 + z^5$

...

About 1544 the German Michael Stifel introduced the term "binomial coefficients" for the coefficients of the the powers of z in the above series. He knew that when these numbers are arranged in a triangle

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

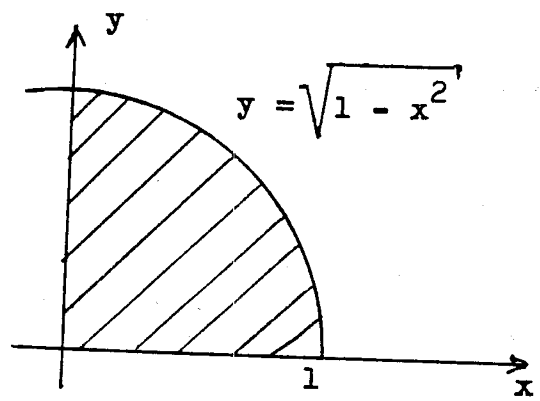
any number is the sum of the two numbers just above it to the left and right. For example, the next row in the triangle is

$$1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \ .$$

In this way he was able to get the expansion for $(1+z)^N$ knowing the expansion for $(1+z)^{N-1}$.

This triangular array of numbers was used by Pascal in 1654, and although it was known to many mathematicians before him, it has become known as "Pascal's triangle".

Isaac Newton made a significant advance in the use of the binomial theorem before 1669. Newton was trying to use his newly discovered integral calculus to compute the area under the circle of radius one in the first quadrant. This area is given by the integral



$$\frac{\pi}{4} = \int_0^1 \sqrt{1 - x^2}\ dx \ ,$$

Today, we would evaluate this integral by making a trigonometric substitution $x = \sin\theta$, but in Newton's day replacing $\sqrt{1 - x^2}$ by its power series was a much more familiar technique. Therefore the expansion of $(1 - x^2)^p$ with $p = 1/2$ is needed. The expansion for $p = 0, 1, 2, \ldots$ is easily obtained from Pascal's triangle. If a formula for the terms in Pascal's triangle could be obtained that did not depend on knowing previous coefficients, that formula might be helpful. Newton discovered that the coefficient of z^n in the expansion of $(1+z)^p$ for integral p is given by

(2) $$\binom{p}{n} = \frac{p}{1}\,\frac{p-1}{2}\,\frac{p-2}{3}\cdots\frac{p-n+1}{n} .$$

For example, the coefficient of z^2 in the expansion of $(1+z)^5$ is

$$\binom{5}{2} = \frac{5}{1}\,\frac{5-1}{2} = 10 .$$

Newton did not know that the expression (2) was valid for $p = 1/2$, but it seemed reasonable to him that it should be. Therefore, replacing p by $1/2$ in (2) he obtained for $(1+z)^{1/2}$

$$(1+z)^{1/2} = 1 + \frac{1}{2}z + \frac{1/2}{1}\,\frac{1/2 - 1}{2}\,z^2 + \frac{1/2}{1}\,\frac{1/2 - 1}{2}\,\frac{1/2 - 2}{3}\,z^3 + \ldots$$

(3) $$(1+z)^{1/2} = 1 + \frac{1}{2}z - \frac{1}{8}z^2 + \frac{1}{16}z^3 + \ldots$$

Now the series is infinite. Newton had no proof for this series, but he did convince himself for its validity in a formal way. He multiplied the series by itself and obtained $1+z$ as follows:

$$
\begin{array}{rrl}
(1+z)^{1/2} & = & 1 + \frac{1}{2} z - \frac{1}{8} z^2 + \frac{1}{16} z^3 + \dots \\
\text{"} & = & 1 + \frac{1}{2} z - \frac{1}{8} z^2 + \frac{1}{16} z^3 + \dots \\
\hline
 & & 1 + \frac{1}{2} z - \frac{1}{8} z^2 + \frac{1}{16} z^3 + \dots \\
 & & \quad\; \frac{1}{2} z + \frac{1}{4} z^2 - \frac{1}{16} z^3 + \dots \\
 & & \qquad\quad - \frac{1}{8} z^2 - \frac{1}{16} z^3 + \dots \\
 & & \qquad\qquad\quad \frac{1}{16} z^3 + \dots \\
\hline
 & & 1 + z + 0 + 0 + \dots
\end{array}
$$

Returning now to the integral of $(1 - x^2)^{1/2}$ we replace z by $-x^2$ in (3) and obtain

$$\frac{\pi}{4} = \int_0^1 (1 - x^2)^{1/2} dx$$

$$= \int_0^1 1 - \frac{1}{2} x^2 - \frac{1}{8} x^4 - \frac{1}{16} x^6 + \dots dx$$

$$= 1 - \frac{1}{6} - \frac{1}{40} - \frac{1}{112} + \dots .$$

Thus a series for $\pi/4$ is obtained. We see that formal manipulations were used to advantage even though the precise foundation for them was missing. The mathematicians of the seventeenth and eighteenth centuries used infinite series without having a rigorous theory which stated when manipulations were valid and when they were invalid. The concept of convergence itself was only understood intuitively.

It was not until the early years of the nineteenth century that the Frenchman Augustin-Louis Cauchy gave the first significant rigorous treatment of infinite series. In 1826 Niels Henrik Abel gave the first mathematically rigorous discussion of the binomial series (1). Abel showed that the series is valid for all complex p when $|z| < 1$.

Example

Obtain a Maclaurin series for $(a+z)^p$ and determine the region of convergence.

Solution

$$(a+z)^p = a^p \left(1 + \frac{z}{a} \right)^p = a^p \sum_{n=0}^{\infty} \binom{p}{n} \left(\frac{z}{a}\right)^n .$$

When p is not a positive integer, $(a+z)^p$ has a singularity at $z = -a$, and thus the series converges for $|z| < |a|$. When p is a positive integer, the series has only a finite number of terms and thus it is valid for all z .

Problems

29. Expand $(a+z)^{1/2}$ in a Taylor's series about z_0 and determine the circle of convergence.

30. Since

$$\arcsin x = \int_0^x \frac{dx}{\sqrt{1-x^2}} ,$$

obtain a Maclaurin series for arcsin x, and determine the region of convergence.

31. Derive (1) using Taylor's formula (H) of section 4.2 .

4.6 Laurent series

In the previous sections we studied series employing positive integral powers of $z-z_0$

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots$$

known as Taylor's series. Now we study series having negative integral or both negative and positive integral powers of $z-z_0$

$$\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n = \dots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + \dots$$

known as a Laurent series.

Example 1

Expand $(1-z)^{-1}$ in a Laurent series in powers of z.

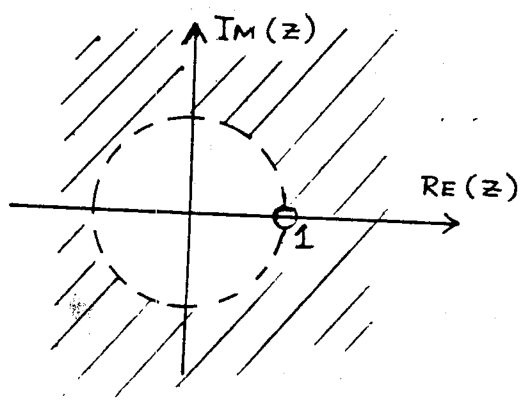
Solution

$$\frac{1}{1-z} = \frac{1}{z(\frac{1}{z}-1)} = -\frac{1}{z}\left[\frac{1}{1-\frac{1}{z}}\right].$$

This last factor $[\dots]$ can be expanded in the geometric series $(1-w)^{-1} = 1 + w + w^2 + \dots$ by replacing w with $1/z$.

$$\frac{1}{1-z} = -\frac{1}{z}\sum_{n=0}^{\infty}(1/z)^n$$

$$= \sum_{n=0}^{\infty}\frac{-1}{z^{n+1}}.$$



Since the geometric series converges for $|w| < 1$, this Laurent series converges for $|1/z| < 1$ which is $1 < |z|$.

Example 2

Expand $e^z(z-2)^{-3}$ in a Laurent series in powers of $z-2$.

Solution

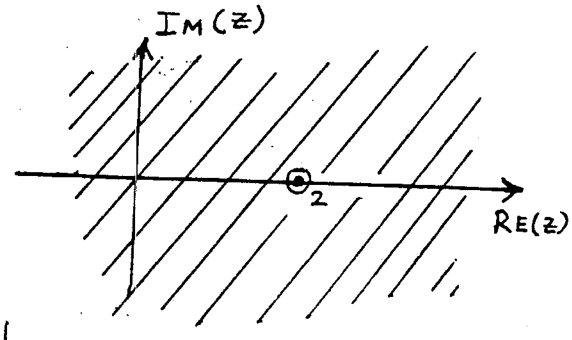
$$e^z(z-2)^{-3} = e^{2+z-2}(z-2)^{-3} = e^2 e^{z-2} (z-2)^{-3} .$$

Since $e^w = \sum_{n=0}^{\infty} \frac{w^n}{n!}$ for $|w| < \infty$, we can replace w by $z-2$

and get

$$e^z(z-2)^{-3} = e^2(z-2)^{-3} \sum_{n=0}^{\infty} \frac{(z-2)^n}{n!} = \sum_{n=0}^{\infty} \frac{e^2(z-2)^{n-3}}{n!}$$

$$= \sum_{k=-3}^{\infty} \frac{e^2(z-2)^k}{(k+3)!} .$$



The region of convergence for this Laurent series is the entire z-plane with the point $z=2$ removed (because of the negative powers of $z-2$). The series converges for $0 < |z-2|$.

Example 3

Expand $(z-1)^{1/2} z^{-1/2}$ in a Laurent series in powers of z .

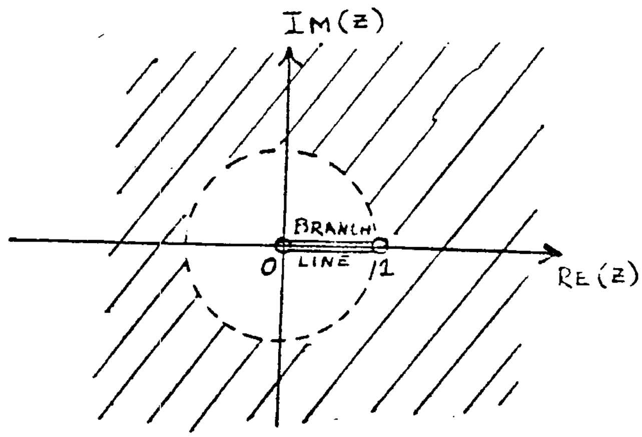
Solution

$$(z-1)^{1/2} z^{-1/2} = (1 - z^{-1})^{1/2}$$

Using the binomial series we get

$$(z-1)^{1/2} z^{-1/2} = \pm \sum_{n=0}^{\infty} \binom{1/2}{n} (-1)^n z^{-n} \quad \text{for } |z| > 1 .$$

The $\pm$ signs have been used to indicate the two distinct values of the square root. The function $(z-1)^{1/2}z^{-1/2}$ has branch points at $z=0$ and at $z=1$. The branch line joining these two points prevents us from obtaining a series in integral powers of z for $|z| < 1$.



Problems

32. Expand $(a + z)^{-1}$ in a Laurent series in powers of $z-z_0$ and determine the region of convergence.

33. Expand $z^5 e^{1/z}$ in a Laurent series in powers of z and determine the region of convergence.

34. Expand $\text{Log}\,\frac{z-1}{z}$ in a Laurent series in powers of z and determine the region of convergence. Where are the branch points and where is the branch line ?

Example 4

Expand $(z^2 - 3z + 2)^{-1}$ in powers of z in as many ways as are possible.

Solution

We first expand $(z^2 - 3z + 2)^{-1}$ in "partial fractions".

$$\frac{1}{z^2 - 3z + 2} = \frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

We can solve for B by multiplying by $z-2$ and obtaining

$$\frac{1}{z-1} = \frac{A(z-2)}{z-1} + B .$$

Now set $z=2$ and get $B = 1$. In a similar way, we solve for A by multiplying by $z-1$ and getting

$$\frac{1}{z-2} = A + \frac{B(z-1)}{z-2} .$$

Set $z=1$ and get $A = -1$. Thus we have

$$\frac{1}{z^2 - 3z + 2} = \frac{-1}{z-1} + \frac{1}{z-2}$$

(1) $$= \frac{1}{1-z} - \frac{1}{2}\left[\frac{1}{1-\frac{z}{2}}\right]$$

From our previous experience with the geometric series and from example 1 we have two expansions for each of these terms :

(A) $$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad \text{for } |z| < 1$$

(B) $$\frac{1}{1-z} = -\sum_{n=0}^{\infty} z^{-n-1} \qquad \text{for } |z| > 1$$

(C) $$\frac{1}{1-\frac{z}{2}} = \sum_{n=0}^{\infty} \frac{z^n}{2^n} \qquad \text{for } |z| < 2$$

(D) $$\frac{1}{1-\frac{z}{2}} = -\sum_{n=0}^{\infty} \frac{2^{n+1}}{z^{n+1}} \qquad \text{for } |z| > 2$$

There are three possible combinations of these four series.

(i) A and C convergent for $|z| < 1$

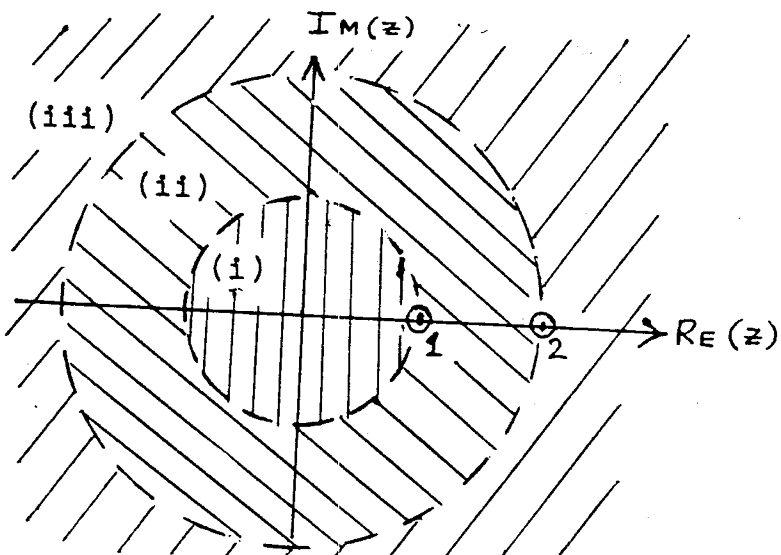
(ii) B and C " " $1 < |z| < 2$

(iii) B and D " " $2 < |z|$.

Any other combination will have an empty region of convergence. Using these series in (1) we get

$$(z^2 - 3z + 2)^{-1} = \begin{cases} \text{(i)} \quad \sum_{n=0}^{\infty} (1 - 2^{-n-1})\, z^n & \text{for } |z| < 1 \\ \text{(ii)} \quad -\sum_{n=0}^{\infty} z^{-n-1} - \sum_{n=0}^{\infty} 2^{-n-1} z^n & \text{for } 1 < |z| < 2 \\ \text{(iii)} \quad \sum_{n=0}^{\infty} \dfrac{2^n - 1}{z^{n+1}} & \text{for } 2 < |z| \end{cases}$$

The regions of convergence for each of these series are shown. Notice that the boundaries are circles with center at z=0 and circumference passing through a singularity of the original function $(z^2 - 3z + 2)^{-1}$.



The series (i) is a Taylor's series, while both (ii) and (iii) are Laurent series.

<u>Problems</u>

35. Expand $(z^2 + z - 6)^{-1}$ in both Taylor and Laurent series in

powers of z and determine the appropriate regions of convergence.

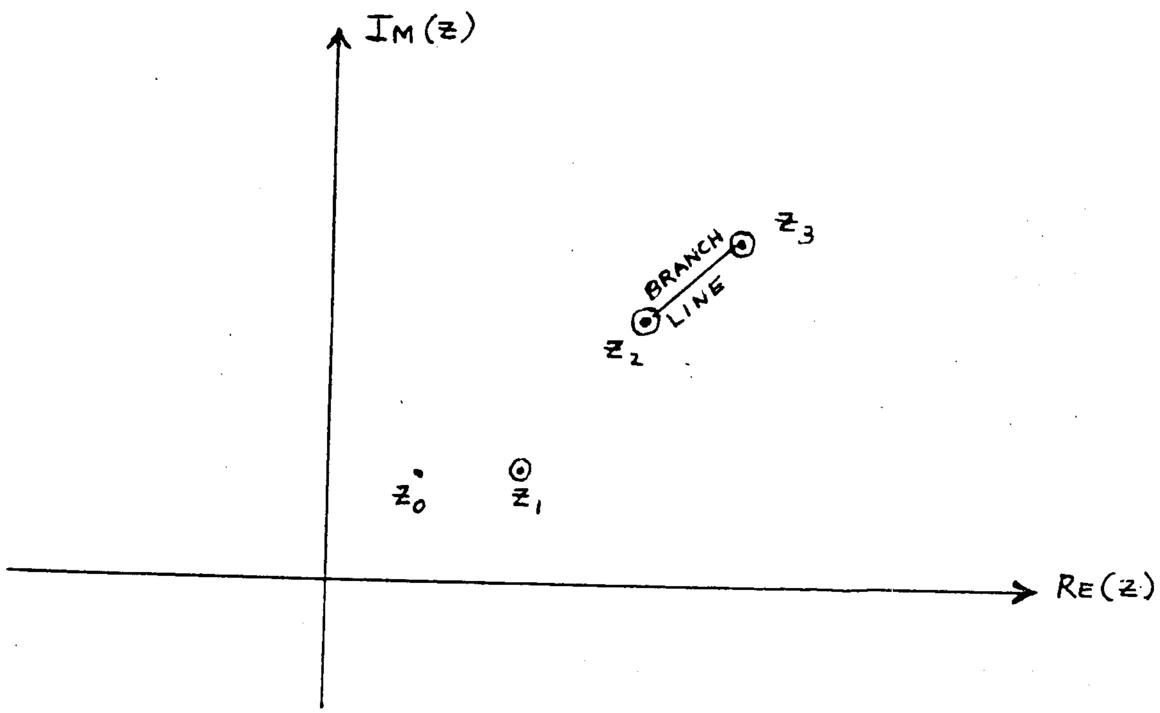
36. Expand the function considered in the previous problem in both Taylor and Laurent series with center at $z = 1$, and determine the appropriate regions of convergence.

Conjecture 4.6

Suppose $f(z)$ is an analytic function having only an isolated singularity at the point z_1 and branch points at z_2 and z_3 as shown. If $f(z)$ is expanded in several series of the form

$$f(z) = \sum_{n=0}^{\infty} b_n (z-z_0)^{-n-1} + \sum_{n=0}^{\infty} a_n (z-z_0)^n ,$$

sketch the regions in which each of these series will converge. Is there any region for which all the $b_n = 0$? Can all the $a_n = 0$ in some region ?



4.7 Isolated singularities revisited

In Chapter 3 we discussed isolated singularities of an analytic function and divided them into two types:

(i) POLES: near which the modulus of the function approaches infinity only

(ii) ESSENTIAL SINGULARITIES: near which the function assumes every value except possibly one

Now that we have studied Laurent series, we can give a simpler characterization of these singularities:

The point z_0 is an isolated singularity of the analytic function $f(z)$ if and only if $f(z)$ can be expanded in a Laurent series in powers of $z-z_0$ convergent for some annulus $0 < |z-z_0| < r$. (This annulus is the interior of the circle $|z-z_0| = r$ with only the center point removed.) If this Laurent series has only finitely many terms with negative powers of $z-z_0$, z_0 is called a pole. If this Laurent series has infinitely many terms with negative powers of $z-z_0$ we call z_0 an essential singularity.

Example 1

Determine the nature of the singularities of $(z-2)^{-3} e^z$ in the finite z-plane.

Solution

In example 2 of section 4.6 we saw that

$$(z-2)^{-3} e^z = \sum_{k=-3}^{\infty} \frac{e^2 (z-2)^k}{(k+3)!}$$

which has only three negative powers of $z-2$ and thus $z=2$ is a pole of order three.

Example 2

Determine the nature of the singularities of $\sqrt{z}\,\sin\frac{1}{\sqrt{z}}$ in the extended z-plane.

Solution

We might at first think that $z=0$ is a branch point because of $\sqrt{z}$, but this is not the case. Since

$$\sin w = w - \frac{w^3}{3!} + \frac{w^5}{5!} - \dots \qquad \text{for} \quad |w| < \infty ,$$

we can replace w by $\frac{1}{\sqrt{z}}$ and get

$$\sqrt{z}\,\sin\frac{1}{\sqrt{z}} = \sqrt{z}\left(\frac{1}{\sqrt{z}} - \frac{1}{3!\,z\sqrt{z}} + \frac{1}{5!\,z^2\sqrt{z}} - \dots\right)$$

$$= 1 - \frac{1}{3!\,z} + \frac{1}{5!\,z^2} - \dots \qquad \text{for} \quad |z| > 0.$$

Thus we see that $\sqrt{z}\,\sin\frac{1}{\sqrt{z}}$ has an essential singularity at $z=0$. To determine the nature of the function at infinity we can set $z = 1/\zeta$ in this last series because the resulting series in ζ will converge near $\zeta = 0$. We see then that z equal to infinity is a regular point of the function.

Problem

37. Determine the singularities of the following functions in the extended z-plane.

(a) $\frac{\sin z}{z}$, (b) $\sqrt{z}\,\sinh\sqrt{z}$, (c) $\frac{1-\cos z}{z^2}$, (d) $\frac{e^z - 1}{z^2}$.

4.8 A precise definition of analyticity

Up to this point, we have said that a function is analytic if its definition evolved in some "natural" way. The analytic function is singular at a point z_0 if it behaves in a "peculiar" way at this point, otherwise we say it is regular.

These concepts, though vague, have been of service. As we grow more familiar with these functions, we discover properties which could themselves be candidates for a more precise definition of analyticity . The Taylor's series itself is such an item. Our previous experience has shown that if z_0 is a regular point (analytic point) of the function $f(z)$, then we can write

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z-z_0)^n$$

where the series converges to $f(z)$ inside some circle $|z-z_0| < r$ of non-zero radius. Why not use this fact as a definition ?

Definition of analyticity

We say that the function $f(z)$ is "regular" or "analytic" at the point z_0 if $f(z)$ can be expanded in a Taylor's series about z_0 which converges to $f(z)$ inside some circle $|z-z_0| < r$ of non-zero radius. We say that $f(z)$ is analytic on some given open set S if it is analytic at each point of the set S . If a point is not regular, we say it is singular.

Thus the totality of all convergent Taylor's series is the collection of all analytic functions. A rigorous mathematical theory of analytic functions can be constructed upon this definition.

4.9 Intuitive consequences of Taylor's formula

We now know that if $f(z)$ is analytic at the point $z = z_0$, then it can be expanded in a convergent Taylor's series

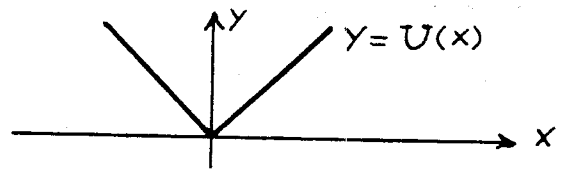
$$(1) \quad f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z-z_0)^n \qquad \text{convergent for } |z-z_0| < r.$$

The series can be used to compute $f(z)$ at each point z inside the circle of convergence. We now explore some consequences of (1).

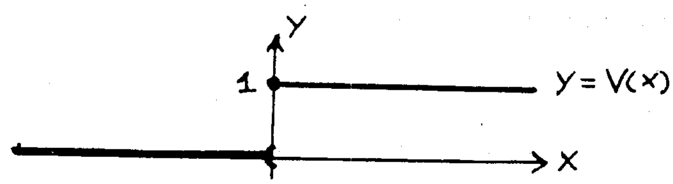
A. Comparison of analytic functions of a complex variable with the functions studied in real analysis

In the real calculus we studied polynomials, rational functions, trigonometric functions, exponential functions, etc. all of which have been extended into the complex plane in Chapter 2. In addition however, there are other functions such as

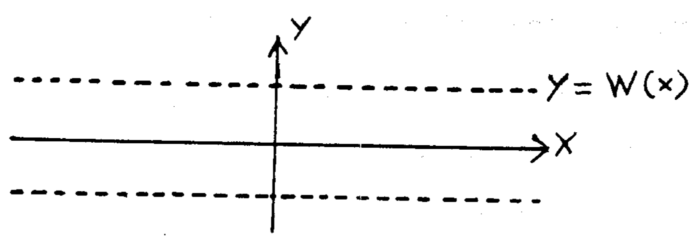
$U(x) = |x|$



$$V(x) = \begin{cases} 1 & \text{for } 0 \leq x \\ 0 & \text{for } x < 0 \end{cases}$$



$$W(x) = \begin{cases} 1 & \text{for rational } x \\ -1 & \text{for irrational } x \end{cases}$$



We did not try to extend the functions $U(x)$, $V(x)$, or $W(x)$ into the complex plane, even though such functions were studied in the real calculus. We somehow sense that these functions are "unnatural".

We note that :

1. The function $U(x)$ is continuous for all x, but not differentiable at $x = 0$.
2. The function $V(x)$ is not continuous at $x = 0$ and therefore not differentiable at $x=0$.
3. The function $W(x)$ is not continuous or differentiable for any x.

The Maclaurin series expansion of a function requires that we have every derivative of the function at the origin. We cannot hope to find a Maclaurin series expansion for any of the above three functions U, V, W since the derivatives of these functions do not exist at $x = 0$.

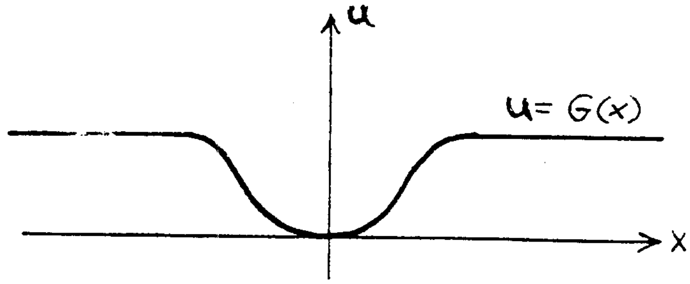
If a function $f(z)$ is known to be analytic at the point $z = z_0$, then $f(z)$ has derivatives of all orders at $z = z_0$ because the Taylor's series (1) requires them.

We might expect the converse of the above statement to be true. "If every derivative of $f(z)$ exists at the point $z = z_0$, then $f(z)$ is analytic at z_0." This might at first seem reasonable since the existence of every derivative is all that is required to write down the right hand side of the Taylor's series (1). However, just because we can write out the Taylor's series, there is <u>no</u> guarantee that this series will <u>converge to the original function</u> ! Indeed, the existence of every derivative of $f(z)$ at z_0 does not quarantee that $f(z)$ is analytic at z_0 . The following example illustrates this.

Example 1

Determine if the function

$$G(x) = \begin{cases} \exp(-1/z^2) & \text{for } z \neq 0 \\ 0 & \text{for } z = 0 \end{cases}$$



is analytic at $z = 0$.

Solution

To test G for analyticity at $z = 0$, we must perform two steps:

(i) Compute the Maclaurin expansion $\sum_{n=0}^{\infty} G^{(n)}(0)\, z^n/n!$.

(ii) Decide if the above Maclaurin expansion actually converges to G(x) for some range of the variable z , $|z| < r$.

We will not take the space here to actually compute every derivative of G(z) at z =0, but leave that computation to the interested student. The result is that $G^{(n)}(0) = 0$ for every $n = 0, 1, 2, 3, \ldots$. The Maclaurin series required above by (i) is thus

$$\sum_{n=0}^{\infty} G^{(n)}(0)\, z^n/n! = 0 + 0z + 0z^2 + 0z^3 + \ldots \quad .$$

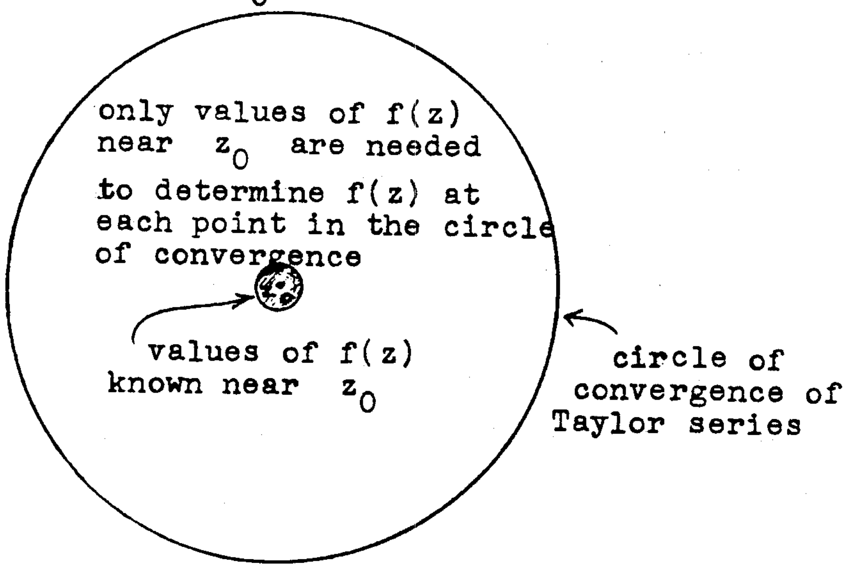
It is quite clear that this series converges to 0 for every value of z. Thus this series cannot converge to the function G(z) when z assumes every value inside some circle $|z| < r$ where $r \neq 0$. Thus G(z) is not analytic at $z = 0$.

Problem

38. Let $f(z) = 3\exp(-(z-1)^{-2}) + 2$ for $z \neq 1$, and let $f(1) = 2$. Determine if f(z) is analytic at $z = 1$.

B. Analytic continuation

Suppose $f(z)$ is analytic at $z = z_0$. To write down the Taylor's expansion of $f(z)$ about z_0 we need only know the values of the derivatives of $f(z)$ at the point z_0. To compute derivatives, one only needs to know how the function $f(z)$ behaves in the immediate vicinity of z_0. Points somewhat removed from z_0 do not effect the computation of $f^{(n)}(z_0)$. Yet the Taylor's series is valid for points away from z_0 ! This means that an analytic function is a very restricted type of function. Its values at points far removed from z_0 is determined only by knowing how $f(z)$ behaves very near z_0.



Example 2

Suppose $f(z)$ is known to be an analytic function for each z. Suppose however that only the values of $f(z)$ on the small segment of the real axis given by $0 \leq x < 0.1$ are known. On this small interval we know that $f(x) = 1 + 2x$. Find the values of $f(z)$ at all z.

Solution

Since $f(z)$ is analytic for all z, its Maclaurin series converges for all z. To write down this Maclaurin series, we need only the values of the derivatives of $f(z)$ at $z = 0$. Since on the interval $0 \leq x < 0.1$, $f(x) = 1 + 2x$, we see at once that $f(0) = 1$, $f'(0) = 2$, and all higher derivatives vanish at $z = 0$. Thus the Maclaurin series expansion of $f(z)$ is

$$f(z) = 1 + 2z$$

for all z.

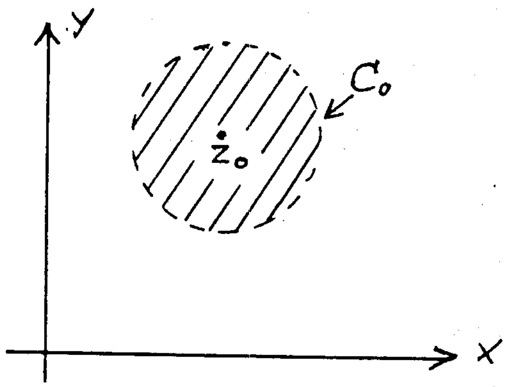
Problem

39. Does there exist an analytic function which assumes the values $f(x) = x^2$ for x on the interval $1 \leq x < 2$, and which assumes the value -1 at $z = 0$?

Consider now the Taylor's series

$$(2) \qquad f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

convergent for all z inside the circle C_0 given by $|z-z_0| < r_0$. How can we find the values of $f(z)$ at points z which are outside the circle C_0 ? In other words, we wish to find the natural extension of $f(z)$ to points outside the circle of convergence C_0 of the Taylor's series. There are some cases in which it is impossible to



extend the function beyond the original circular boundary. However, if it is possible to extend the domain of definition of $f(z)$, the following technique will achieve it.

Let z_1 be some point inside C_0. We can compute all the derivatives of $f(z)$ at z_1 by means of term by term differentiation of (2). This yields

$$f^{(0)}(z_1) = \sum_{n=0}^{\infty} a_n(z_1-z_0)^n$$

$$f^{(1)}(z_1) = \sum_{n=1}^{\infty} n\, a_n(z_1-z_0)^{n-1}$$

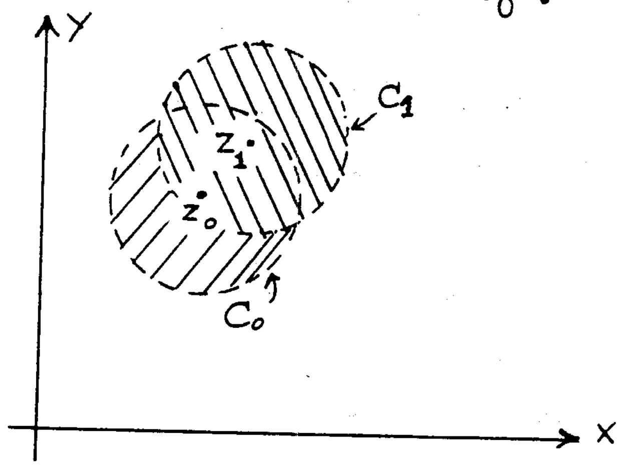
$$f^{(2)}(z_1) = \sum_{n=2}^{\infty} n(n-1)\, a_n(z_1-z_0)^{n-2}$$

...

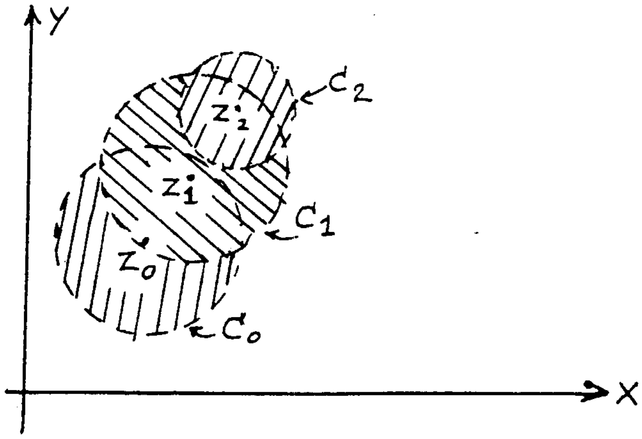
From these values we can now construct the Taylor's series about z_1 as

$$(3) \qquad f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_1)(z-z_1)^n}{n!} .$$

Suppose the series (3) converges in a circle C_1 given by $|z-z_1| < r_1$ which extends outside the original circle C_0. Using both (2) and (3) we have $f(z)$ defined in the open set $C_0 \cup C_1$. We say that we have <u>continued</u> $f(z)$ analytically from C_0 to the larger domain $C_0 \cup C_1$. Now we can select a point z_2 in C_1 and expand $f(z)$ about z_2 in yet another Taylor's series.



Suppose the circle of convergence C_2 of this new series extends outside $C_0 \cup C_1$ as shown. We have made now another analytic continuation of f(z). Doing this again and again, we can (in theory at least) extend the domain of definition of the original function from C_0 to every point in the z plane to which f(z) can be continued analytically. Since this largest possible domain of definition of f(z) is itself a union of open circles, it must be an open set.



Example 3

Suppose an analytic function is defined for all z in $0 \leq Re(z)$. Is it possible to extend f(z) to some values of z having negative real part ?

Solution.

The largest possible domain of definition of f(z) is always an open set. More precisely, we can continue f(z) analytically to some open set R which has the closed set $0 \leq Re(z)$ as a proper subset. To achieve this continuation, we can imagine f(z) being expanded in a convergent Taylor's series centered at the point $y_0 i$ which is on the imaginary axis. This Taylor's series must of necessity converge for some values of z which extend into the left half plane. Constructing one Taylor's series for each point on the y-axis we can achieve a desired continuation.

Problems

40. If an analytic function is defined on the region $|z| \leq 1$, must it be possible to continue $f(z)$ outside this region analytically ?

41. What is the analytic continuation of the function represented by $\sum_{n=1}^{\infty} nz^{n-1}$ outside the unit circle $|z| < 1$?

42. What is the analytic continuation of the function defined by $\sum_{n=2}^{\infty} n(n-1)z^{n-2}$ outside the unit circle ?

C. Liouville's Theorem

If an analytic function $f(z)$ has no singularities in the finite plane, we call the function "entire" . If $f(z)$ is entire, its Maclaurin series converges for all z . Consider now the three possible cases for an entire function:

(i) $f(z) = a_0$ for all z

(ii) $f(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_N z^N$ for all z

(iii) $f(z) = a_0 + a_1 z + a_2 z^2 + \ldots$ (infinitely many non-zero terms) for all z

In the case (i), $f(z)$ is identically a constant. In the case (ii), $f(z)$ has a pole at infinity and thus $|f(z)|$ approaches infinity as z approaches infinity. In case (iii), $f(z)$ has an essential singularity at infinity, and thus $f(z)$ assumes every value (except possibly one) for z in the exterior of any circle $r < |z|$.

Assume now that $f(z)$ is "bounded for all z". This means that there exists some positive constant M such that $|f(z)| < M$ for all z. Clearly this is not possible in cases (ii) and (iii). Only case (i) remains, and therefore $f(z)$ is identically a constant. This fact is summarized in the following theorem.

Liouville's Theorem

A bounded entire function is a constant.

Example 3

The function $f(z) = \sin z$ is bounded for all real z. In fact, we learned in a course in trigonometry that

$$-1 \leq \sin x \leq 1 \quad .$$

Can $\sin z$ be bounded for all complex z ?

Solution

The function sin z is an entire function, since its Maclaurin series converges for all z . By Liouvilles Theorem we know that a bounded entire function is a constant. Since sin z is surely not a constant function, it must be unbounded. (A glance at Figures 2.7 and 2.8 reveals that sin z grows quite rapidly in modulus as z approaches infinity on a line parallel to the imaginary axis) .

Review Problems for Chapter 4

1. An important Maclaurin series is

$$\frac{z}{e^z - 1} = 1 - \frac{z}{2} + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{B_n z^{2n}}{(2n)!} .$$

(a) Determine the the region of convergence of this series.

(b) By actually dividing the Maclaurin series for $e^z - 1$ into z, show that $B_1 = 1/6$, $B_2 = 1/30$. (The numbers B_n are important in mathematical analysis and are called the "Bernoulli numbers". Further Bernoulli numbers are $B_3 = 1/42$, $B_4 = 1/30$, $B_5 = 5/66$, $B_6 = 691/2730$.)

2. The hypergeometric series (studied in problem 27), is usually denoted by

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n z^n}{(c)_n \, n!} , \quad \text{for } |z| < 1 .$$

Show that ${}_2F_1(a,1;1;z) = (1-z)^{-a}$.

3. Expand $(1-z)^{-2}$ in a Maclaurin series and in a Laurent series in powers of z. In each case determine the region of convergence.

4. Locate and classify the singularities of $(\sin z)^3/z^5$ in the extended z - plane.

APPENDIX I

SOLUTIONS TO PROBLEMS

Problems from Chapter 4

1/ Replace z by z^2 in (1) and get the result at once.

2/ $$\frac{1}{a-z} = \frac{1}{a}\left(\frac{1}{1-\frac{z}{a}}\right) = \frac{1}{a}\sum_{n=0}^{\infty}\left(\frac{z}{a}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{z^n}{a^{n+1}}.$$

3/ $$\frac{1}{2-z} = \frac{1}{1+1-z} = \frac{1}{1-(z-1)} = \sum_{n=0}^{\infty}(z-1)^n.$$

4/ $$\frac{1}{4+z} = \frac{1}{3+(z+1)} = \frac{1}{3}\left(\frac{1}{1+\frac{z+1}{3}}\right) = \frac{1}{3}\left(\frac{1}{1-\left(-\frac{z+1}{3}\right)}\right)$$

$$= \frac{1}{3}\sum_{n=0}^{\infty}\left(-\frac{z+1}{3}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n(z+1)^n}{3^{n+1}}$$

5/ $$\frac{1}{a-z} = \frac{1}{a-z_0-z+z_0} = \frac{1}{(a-z_0)-(z-z_0)} =$$

$$\frac{1}{a-z_0}\left(\frac{1}{1-\frac{z-z_0}{a-z_0}}\right) = \frac{1}{a-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{a-z_0}\right)^n =$$

$$= \sum_{n=0}^{\infty}\frac{(z-z_0)^n}{(a-z_0)^{n+1}}$$

6/ Forming $\sin^2 z$ we have

$$z - \frac{z^3}{6} + \frac{z^5}{120} - \cdots$$
$$z - \frac{z^3}{6} + \frac{z^5}{120} - \cdots$$

$$z^2 - \frac{z^4}{6} + \frac{z^6}{120} - \cdots$$
$$- \frac{z^4}{6} + \frac{z^6}{36} - \cdots$$
$$+ \frac{z^6}{120} - \cdots$$

$$z^2 - \frac{z^4}{3} + \frac{2z^6}{45} - \cdots$$

Forming $\cos^2 z$ we have

$$1 - \frac{z^2}{2} + \frac{z^4}{24} - \frac{z^6}{720} + \cdots$$
$$1 - \frac{z^2}{2} + \frac{z^4}{24} - \frac{z^6}{720} + \cdots$$

$$1 - \frac{z^2}{2} + \frac{z^4}{24} - \frac{z^6}{720} + \cdots$$
$$- \frac{z^2}{2} + \frac{z^4}{4} - \frac{z^6}{48} + \cdots$$
$$+ \frac{z^4}{24} - \frac{z^6}{48} + \cdots$$
$$- \frac{z^6}{720} + \cdots$$

$$1 - z^2 + \frac{z^4}{3} - \frac{2z^6}{45} + \cdots$$

Adding these series for $\sin^2 z$ and $\cos^2 z$ we get

$$1 + 0 + 0 + 0 + \cdots = 1,$$

7/ $$e^{iz} = \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} = 1 + \frac{iz}{1} + \frac{(iz)^2}{2} + \frac{(iz)^3}{3!} + \frac{(iz)^4}{4!} + \cdots$$

$$e^{iz} = 1 + iz - \frac{z^2}{2} - i\frac{z^3}{3!} + \frac{z^4}{4!} + i\frac{z^5}{5!} + \cdots$$

$$e^{-iz} = 1 + (-iz) + \frac{(-iz)^2}{2} + \frac{(-iz)^3}{3!} + \frac{(-iz)^4}{4!} + \frac{(-iz)^5}{5!} + \cdots$$

$$e^{-iz} = 1 - iz - \frac{z^2}{2} + i\frac{z^3}{3!} + \frac{z^4}{4!} - i\frac{z^5}{5!} + \cdots$$

adding we get

$$e^{iz} + e^{-iz} = 2 - 2\frac{z^2}{2} + 2\frac{z^4}{4!} + \cdots$$

$$= 2\left(1 - \frac{z^2}{2} + \frac{z^4}{4!} + \cdots\right) = 2\cos z,$$

8/ Subtracting the series for e^{iz} and e^{-iz} obtained in problem 7 we get

$$e^{iz} - e^{-iz} = 2iz - 2i\frac{z^3}{3!} + 2i\frac{z^5}{5!} + \cdots$$

$$= 2i\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right)$$

$$= 2i \sin z ,$$

9/ $\cosh z = \frac{1}{2}\left(e^{z} + e^{-z}\right)$

$$= \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{z^n}{n!} + \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{n!}\right)$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{z^n}{n!} + \frac{(-1)^n z^n}{n!}\right)$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \left(1 + (-1)^n\right) \frac{z^n}{n!}$$

$$= \frac{1}{2}\left(2 + 2\frac{z^2}{2!} + 2\frac{z^4}{4!} + 2\frac{z^6}{6!} + \cdots\right)$$

$$= 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} .$$

10/ $\tan z = \dfrac{\sin z}{\cos z}$, Dividing the series for $\cos z$ into the series for $\sin z$ we get

$$
\begin{array}{r|l}
 & z + \frac{z^3}{3} + \frac{2z^5}{15} \\
\hline
1 - \frac{z^2}{2} + \frac{z^4}{24} - \cdots & z - \frac{z^3}{6} + \frac{z^5}{120} - \cdots \\
 & z - \frac{z^3}{2} + \frac{z^5}{24} - \cdots \\
\hline
 & \quad \frac{z^3}{3} - \frac{z^5}{30} + \cdots \\
 & \quad \frac{z^3}{3} - \frac{z^5}{6} + \cdots \\
\hline
 & \qquad \frac{2z^5}{15} + \cdots \\
 & \qquad \frac{2z^5}{15} + \cdots \\
\hline
 & \qquad \cdots
\end{array}
$$

Thus $\tan z = z + \dfrac{z^3}{3} + \dfrac{2z^5}{15} + \cdots$,

11/ $\cosh z = 1 + \dfrac{z^2}{2!} + \dfrac{z^4}{4!} + \dfrac{z^6}{6!} + \cdots$

$$\frac{d \cosh z}{dz} = \frac{2z}{2!} + \frac{4z^3}{4!} + \frac{6z^5}{6!} + \cdots$$

$$\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$$

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!},$$

12/
$$\frac{1}{1+t} = \frac{1}{1-(-t)} = \sum_{n=0}^{\infty} (-t)^n = \sum_{n=0}^{\infty} (-1)^n t^n$$

$$\int_0^z \frac{1}{1+t}\, dt = \sum_{n=0}^{\infty} (-1)^n \int_0^z t^n\, dt$$

$$\operatorname{Log}(z+1) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{n+1}}{n+1},$$

13/ Replacing z by $z-1$ in the previous result we get

$$\operatorname{Log}((z-1)+1) = \sum_{n=0}^{\infty} (-1)^n \frac{(z-1)^{n+1}}{n+1}$$

$$\operatorname{Log} z = \quad \text{"} ,$$

14/
$$\frac{d}{dz}(1-z)^{-1} = \sum_{n=0}^{\infty} \frac{d}{dz} z^n$$

$$(1-z)^{-2} = \sum_{n=1}^{\infty} n z^{n-1} = \sum_{n=0}^{\infty} (n+1) z^n$$

15/
$$\frac{d}{dz}(1-z)^{-2} = \sum_{n=0}^{\infty} (n+1) \frac{d}{dz} z^n$$

$$2(1-z)^{-3} = \sum_{n=1}^{\infty} (n+1) n z^{n-1} = \sum_{n=0}^{\infty} (n+2)(n+1) z^n$$

$$(1-z)^{-3} = \sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{2} z^n$$

16/ $D^n e^z\big|_{z=0} = e^z\big|_{z=0} = e^0 = 1.$

THUS $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$.

17/ FIRST; $e^z = e^{z_0} e^{z-z_0} = e^{z_0} \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{n!} = \sum_{n=0}^{\infty} \frac{e^{z_0}}{n!} (z-z_0)^n,$

SECOND; $D^n e^z\big|_{z=z_0} = e^{z_0},$ THUS $e^z = \sum_{n=0}^{\infty} \frac{e^{z_0}}{n!}(z-z_0)^n,$

18/ $D^0 (a-z)^{-1}\big|_{z_0} = (a-z)^{-1}\big|_{z_0} = (a-z_0)^{-1}$

$D^1 (a-z)^{-1}\big|_{z_0} = (a-z)^{-2}\big|_{z_0} = (a-z_0)^{-2}$

$D^2 (a-z)^{-1}\big|_{z_0} = 2(a-z)^{-3}\big|_{z_0} = 2(a-z_0)^{-3}$

$D^3 (a-z)^{-1}\big|_{z_0} = 2\cdot 3(a-z)^{-4}\big|_{z_0} = 3!(a-z_0)^{-4}$

$\vdots$

$D^n (a-z)^{-1}\big|_{z_0} = \qquad = n!(a-z_0)^{-n-1}$

THUS $(a-z)^{-1} = \sum_{n=0}^{\infty} (a-z_0)^{-n-1}(z-z_0)^n,$

19/ $D^0 \operatorname{Log} z\big|_1 = \operatorname{Log} z\big|_1 = 0$

$D^1 \operatorname{Log} z\big|_1 = z^{-1}\big|_1 = 1$

$D^2 \operatorname{Log} z\big|_1 = -z^{-2}\big|_1 = -1$

$D^3 \operatorname{Log} z\big|_1 = 2z^{-3}\big|_1 = 2$

$\vdots$

$D^n \operatorname{Log} z\big|_1 = (n-1)!(-1)^{n-1} z^{-n}\big|_1 = (-1)^{n-1}(n-1)!$

THUS $\operatorname{Log} z = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(z-1)^n$.

20/ $\sin i = i - \frac{(i)^3}{3!} + \frac{(i)^5}{5!} - \frac{(i)^7}{7!} + \frac{(i)^9}{9!} - \ldots$

$$\cong i + \frac{i}{3!} + \frac{i}{5!} + \frac{i}{7!} + \frac{i}{9!}$$

$$\cong i\left(1 + \frac{1}{3!} + \frac{1}{5!} + \frac{1}{7!} + \frac{1}{9!}\right)$$

Using the table in Example 1 we have

$\sin i \cong (1,1752012)i$, The series converges,

21/ $\text{Log}\, z = (z-1) - \frac{(z-1)^2}{2} + \frac{(z-1)^3}{3} - \frac{(z-1)^4}{4} + \frac{(z-1)^5}{5} - \ldots$

$$\text{Log}\,(1,1) = 0,1 - \frac{0,1^2}{2} + \frac{0,1^3}{3} - \frac{0,1^4}{4} + \frac{0,1^5}{5} - \ldots$$

adding these first five terms we have

$$\begin{array}{r} 0,100000 \\ -0,005000 \\ 0,000333 \\ -0,000025 \\ 0,000002 \\ \hline \end{array}$$

$\text{Log}\, 1,1 \approx 0,095300$ The series converges,

22/ $\text{Log}\,(-9) = (-10) - \frac{(-10)^2}{2} + \frac{(-10)^3}{3} - \ldots$

$$= -\left[10 + \frac{100}{2} + \frac{1000}{3} + \frac{10000}{4} + \ldots\right]$$

This series makes no sense at all, and diverges, 23/ $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \frac{z^{10}}{10!} + \cdots$

SET $z = \frac{1+i}{\sqrt{2}} = e^{i\frac{\pi}{4}}$, THEN $z^2 = e^{i\frac{2\pi}{4}} = i$,

$z^4 = (z^2)^2 = (i)^2 = -1$, $z^6 = -i$, $z^8 = 1$, $z^{10} = i$.

$$\cos\left(\frac{1+i}{\sqrt{2}}\right) = 1 - \frac{i}{2!} + \frac{-1}{4!} - \frac{-i}{6!} + \frac{1}{8!} - \frac{i}{10!} + \cdots$$

$$= \left(1 - \frac{1}{4!} + \frac{1}{8!}\right) + i\left(-\frac{1}{2!} + \frac{1}{6!} - \frac{1}{10!}\right) + \cdots$$

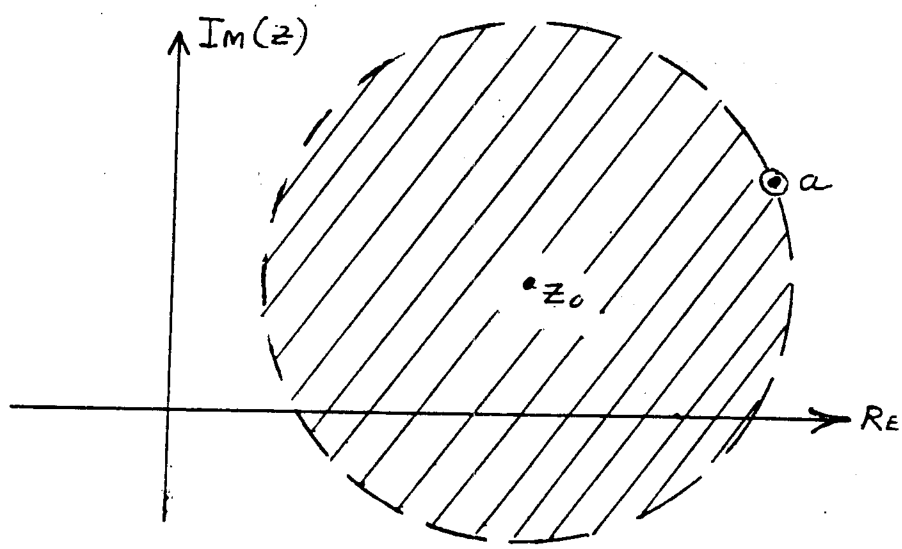
$$\cong (1.0000000 - 0.0416667 + 0.0000248)$$

$$+ i(-0.5000000 + 0.0013889 - 0.0000003)$$

$$\cong 0.9583581 - 0.4986114\, i$$

The series converges. The above result is accurate to at least the first five decimal places.

24/ WE REQUIRE THAT $\left|\frac{z - z_0}{a - z_0}\right| < 1$, WHICH MEANS THAT $|z - z_0| < |a - z_0|$.



25/ 1, $|z| < 1$

2, $|z| < |a|$

3, $|z-1| < 1$

4, $|z+1| < 3$

5, $|z - z_0| < |a - z_0|$

6, 7, and 8, The series for e^z, $\sin z$ and $\cos z$ converge for all z.

9, Converges for all z,

10, $|z| < \frac{\pi}{2}$

11, Converges for all z,

12, $|z| < 1$

13, $|z-1| < 1$

14, $|z| < 1$

15, $|z| < 1$

26/ $$\lim_{n \to \infty} \left| \frac{z^{n+1}}{z^n} \right| = |z|$$

Thus the series converges for $|z| < 1$ and diverges for $|z| > 1$,

27/ $$\lim_{n \to \infty} \left| \frac{(a)_{n+1} (b)_{n+1} \; z^{n+1}}{(c)_{n+1} \; (n+1)!} \; \frac{(c)_n \; n!}{(a)_n (b)_n \; z^n} \right|$$

$$= \lim_{n \to \infty} \left| \frac{(a+n)(b+n)\, z}{(c+n)(n+1)} \right| \quad \left(\text{since } \frac{(a)_{n+1}}{(a)_n} = (a+n) \right)$$

$$= \lim_{n\to\infty} \left| \frac{(1+\frac{a}{n})(1+\frac{b}{n})\ z}{(1+\frac{c}{n})(1+\frac{1}{n})} \right| = |z|$$

Thus the series converges for $|z|<1$ and diverges for $|z|>1$.

28/

$$\lim_{n\to\infty} \left| \frac{(a)_{n+1}\ z^{n+1}\quad (b)_n\ \ n!}{(b)_{n+1}\ (n+1)!\ \ (a)_n\ \ z^n} \right| =$$

$$\lim_{n\to\infty} \left| \frac{(a+n)\ z}{(b+n)(n+1)} \right| = \lim_{n\to\infty} \left| \frac{(1+\frac{a}{n})\ z}{(1+\frac{b}{n})\ n} \right| = 0.$$

Thus the series converges for all z,

29/ $$(a+z)^{\frac{1}{2}} = ((a+z_0)+(z-z_0))^{1/2} = (a+z_0)^{1/2}\left(1+\frac{z-z_0}{a+z_0}\right)^{1/2}$$

$$= (a+z_0)^{1/2} \sum_{n=0}^{\infty} \binom{1/2}{n} \left\{\frac{z-z_0}{a+z_0}\right\}^n$$

valid for $|z-z_0| < |a+z_0|$.

30/ $$(1-x^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \binom{-1/2}{n} (-x^2)^n = \sum_{n=0}^{\infty} \binom{-1/2}{n} (-1)^n x^{2n}$$

$$\int_0^x (1-x^2)^{-\frac{1}{2}} dx = \sum_{n=0}^{\infty} \binom{-1/2}{n} \frac{(-1)^n}{2n+1} x^{2n+1} = \arcsin x,$$

31/

$$D^0 (1+z)^p \Big|_{z=0} = (1+z)^p \Big|_{z=0} = 1$$

$$D^1 (1+z)^p \Big|_{z=0} = p(1+z)^{p-1} \Big|_{z=0} = p$$

$$D^2 (1+z)^p \Big|_{z=0} = p(p-1)(1+z)^{p-2} \Big|_{z=0} = p(p-1)$$

$$D^3 (1+z)^p \Big|_{z=0} = p(p-1)(p-2)(1+z)^{p-3} \Big|_{z=0} = p(p-1)(p-2)$$

$$\vdots$$

$$D^n (1+z)^p \Big|_{z=0} = \cdots = p(p-1)(p-2)\cdots(p-n+1)$$

THEREFORE

$$(1+z)^p = \sum_{n=0}^{\infty} \frac{D^n (1+z)^p \big|_{z=0}}{n!} z^n$$

$$= \sum_{n=0}^{\infty} \frac{p(p-1)(p-2)\cdots(p-n+1)}{n!} z^n$$

valid for $|z| < 1$.

32/ $$\frac{1}{a+z} = \frac{1}{(a+z_0)+(z-z_0)} = \frac{1}{z-z_0}\left[\frac{1}{1+\frac{a+z_0}{z-z_0}}\right]$$

$$= \frac{1}{(z-z_0)}\left[\frac{1}{1-\left\{-\frac{a+z_0}{z-z_0}\right\}}\right]$$

$$= \frac{1}{z-z_0}\sum_{n=0}^{\infty}\left\{-\frac{a+z_0}{z-z_0}\right\}^n$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n(a+z_0)^n}{(z-z_0)^{n+1}} \quad \text{for } \left|\frac{a+z_0}{z-z_0}\right| < 1$$

The region of convergence is $|a+z_0| < |z-z_0|$,

33/ Since $e^w = \sum_{n=0}^{\infty} \frac{w^n}{n!}$, we can replace

w by $\frac{1}{z}$ and get

$$z^5 e^{\frac{1}{z}} = z^5 \sum_{n=0}^{\infty} \frac{z^{-n}}{n!} = \sum_{n=0}^{\infty} \frac{z^{5-n}}{n!}$$

convergent for $0 < |z|$,

34/ $\text{Log}\,\frac{z-1}{z} = \text{Log}\left(1-\frac{1}{z}\right)$

Since $\text{Log}\, w = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{(w-1)^n}{n}$ for $|w$

we can set $w = 1-\frac{1}{z}$ and get $w-1 = -\frac{1}{z}$ so that

$$\text{Log}\,\frac{z-1}{z} = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{\left(-\frac{1}{z}\right)^n}{n} \quad \text{for} \quad \left|-\frac{1}{z}\right| < 1$$

$$= -\sum_{n=1}^{\infty} \frac{z^{-n}}{n} \quad \text{for} \quad 1 < |z|,$$

35/ $$\frac{1}{z^2+z-6} = \frac{1}{(z+3)(z-2)} = \frac{A}{z+3} + \frac{B}{z-2}$$

Multiply by $z+3$ and get

$$\frac{1}{z-2} = A + \frac{B(z+3)}{z-2}$$

Set $z = -3$ and get $A = -\frac{1}{5}$, Similarly, multiply by $z-2$ and get

$$\frac{1}{z+3} = \frac{A(z-2)}{z+3} + B,$$

Set $z = 2$ and get $B = \frac{1}{5}$. Therefore

$$\frac{1}{z^2+z+6} = -\frac{1}{5}\left(\frac{1}{z+3}\right) + \frac{1}{5}\left(\frac{1}{z-2}\right).$$

(A) $$\frac{1}{z+3} = \frac{1}{3\left(1-\left(-\frac{z}{3}\right)\right)} = \frac{1}{3}\sum_{n=0}^{\infty}\left(-\frac{z}{3}\right)^n \quad \text{for} \quad |z| < 3$$

(B) $$\text{"} = \frac{1}{z\left(1-\left(-\frac{3}{z}\right)\right)} = \frac{1}{z}\sum_{n=0}^{\infty}\left(-\frac{3}{z}\right)^n \quad \text{for} \quad 3 < |z|$$

(C) $$\frac{1}{z-2} = \frac{1}{-2\left(1-\frac{z}{2}\right)} = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n \quad \text{for} \quad |z| < 2$$

(D) $$\text{"} = \frac{1}{z\left(1-\frac{2}{z}\right)} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n \quad \text{for} \quad 2 < |z|,$$

Combining A *and* C *we get for* $|z| < 2$

$$\frac{1}{z^2+z-6} = -\frac{1}{15}\sum_{n=0}^{\infty}\frac{(-1)^n z^n}{3^n} - \frac{1}{10}\sum_{n=0}^{\infty}\frac{z^n}{2^n}$$

$$= -\frac{1}{5}\sum_{n=0}^{\infty}\left[\frac{(-1)^n}{3^{n+1}} + \frac{1}{2^{n+1}}\right] z^n ,$$

Combining A *and* D *we get for* $2 < |z| < 3$

$$\frac{1}{z^2+z-6} = -\frac{1}{15}\sum_{n=0}^{\infty}\frac{(-1)^n z^n}{3^n} + \frac{1}{5}\sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}} ,$$

Combining B *and* D *we get for* $3 < |z|$

$$\frac{1}{z^2+z-6} = -\frac{1}{5}\sum_{n=0}^{\infty}\frac{(-1)^n 3^n}{z^{n+1}} + \frac{1}{5}\sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}}$$

$$= \frac{1}{5}\sum_{n=0}^{\infty}\left[2^n - (-1)^n 3^n\right] z^{-n-1} .$$

36/ *Now we have*

(A) $\frac{1}{z+3} = \frac{1}{4+(z-1)} = \sum_{n=0}^{\infty}\frac{(-1)^n (z-1)^n}{4^{n+1}}$ *for* $|z-1| < 4$

(B) $\quad " \quad = \frac{1}{(z-1)\left(1+\frac{4}{z-1}\right)} = \sum_{n=0}^{\infty}\frac{(-1)^n 4^n}{(z-1)^{n+1}}$ *for* $4 < |z-1|$

(C) $\frac{1}{z-2} = \frac{-1}{1-(z-1)} = -\sum_{n=0}^{\infty}(z-1)^n$ *for* $|z-1| < 1$

(D) $\quad " \quad = \frac{1}{(z-1)\left(1-\frac{1}{z-1}\right)} = \sum_{n=0}^{\infty}(z-1)^{-n-1}$ *for* $|z-1| > 1$

36/ (Continued)

Combining A and C we get for $|z-1| < 1$

$$\frac{1}{z^2+z-6} = -\frac{1}{5}\sum_{n=0}^{\infty}\left[1+\frac{(-1)^n}{4^{n+1}}\right](z-1)^n .$$

Combining A and D we get for $1 < |z-1| < 4$

$$\frac{1}{z^2+z-6} = \frac{1}{5}\sum_{n=0}^{\infty}(z-1)^{-n-1} - \frac{1}{5}\sum_{n=0}^{\infty}\frac{(-1)^n(z-1)^n}{4^{n+1}} ,$$

Combining B and D we get for $4 < |z-1|$

$$\frac{1}{z^2+z-6} = \frac{1}{5}\sum_{n=0}^{\infty}\left[1-(-1)^n 4^n\right](z-1)^{-n-1} ,$$

37/

(a) $$\frac{\sin z}{z} = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$$

convergent for $|z| < \infty$, Thus there are no singularities in the finite z-plane, Set $z = \frac{1}{\zeta}$ in this last series and get $1 - \frac{1}{3!\,\zeta^2} + \frac{1}{5!\,\zeta^4} - \cdots$, Since this is an essential singularity at $\zeta = 0$, we say that $\frac{\sin z}{z}$ has an essential singularity at infinity,

(b) $$\sqrt{z}\,\sinh\sqrt{z} = \sqrt{z}\left(\sqrt{z} + \frac{z\sqrt{z}}{3!} + \frac{z^2\sqrt{z}}{5!} + \cdots\right)$$

$$= z + \frac{z^2}{3!} + \frac{z^3}{5!} + \cdots \quad \text{for } |z| < \infty,$$

Setting $z = \frac{1}{\zeta}$ we find that the only singularity is an essential singularity at infinity, 37/ (c) $$\frac{1-\cos z}{z^2} = \frac{1}{z^2}\left(1 - 1 + \frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} - \cdots\right)$$

$$= \frac{1}{2!} - \frac{z^2}{4!} + \frac{z^4}{6!} - \cdots \quad \text{for } |z| < \infty.$$

The only singularity is an essential singularity at infinity.

(d) $$\frac{e^z - 1}{z^2} = \frac{1}{z^2}\left(1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots \quad - 1\right)$$

$$= \frac{1}{z} + \frac{1}{2!} + \frac{z}{3!} + \frac{z^2}{4!} + \cdots \quad 0 < |z| < \infty$$

There is a simple pole at $z = 0$ and an essential singularity at infinity.

38/ We see that $f(z) = 3\,G(z-1) + 2$, where G is defined in the Example 1. Since $G^{(n)}(0) = 0$ for $n = 0, 1, 2, \ldots$, then $f(1) = 2$, $f^{(1)}(1) = 0$, $f^{(2)}(1) = 0, \ldots$. Therefore

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(z-1)^n = 2 + 0\,(z-1) + 0\,(z-1)^2 + \cdots$$

$$\equiv 2.$$

Since $f(z)$ is not identically the constant 2, the Taylor's series does not represent $f(z)$. Therefore $f(z)$ is not analytic at $z = 1$.

39/ Since $f(x) = x^2$ for $1 \le x < 2$, we know that $f(1) = 1$, $f^{(1)}(1) = 2$, $f^{(2)}(1) = 2$, $f^{(3)}(1) = 0$, $f^{(4)}(1) = 0, \dots$. Therefore, if $f(z)$ is analytic, it can be expanded in a Taylor's series about $z = 1$ and we get

$$f(z) = 1 + 2(z-1) + (z-1)^2 + 0$$
$$= z^2 \qquad \text{for all } z.$$

Thus $f(0)$ cannot equal -1.

40/ YES, BECAUSE THE LARGEST POSSIBLE DOMAIN TO WHICH AN ANALYTIC FUNCTION CAN BE EXTENDED IS ALWAYS AN OPEN SET. SINCE $|z| \le 1$ IS A CLOSED SET, THERE EXISTS SOME OPEN SET $\mathcal{R}$ CONTAINING $|z| \le 1$ ON WHICH $f(z)$ IS ANALYTIC.

41/ SINCE $(1-z)^{-1} = \sum_{n=0}^{\infty} z^n$ FOR $|z| < 1$, WE SEE THAT TERM BY TERM DIFFERENTIATION YIELDS $(1-z)^{-2} = \sum_{n=0}^{\infty} n z^{n-1}$. THUS THE DESIRED ANALYTIC CONTINUATION IS GIVEN SIMPLY BY $(1-z)^{-2}$ FOR ALL z EXCEPT $z = \phi$.

42/ DIFFERENTIATING THE SERIES FOR $(1-z)^{-2}$ IN THE PREVIOUS PROBLEM WE GET $2(1-z)^{-3} = \sum_{n=2}^{\infty} n(n-1) z^{n-2}$. THUS THE DESIRED CONTINUATION IS GIVEN BY $2(1-z)^{-3}$ FOR ALL z EXCEPT $z = 0$.

Solutions to Review Problems from Chapter 4

1/ (a) $e^z - 1 = 0$ when $z = 2\pi n i$, with $n = 0, \pm 1, \pm 2, \ldots$. Thus the zeros closest to the origin are $z = \pm 2\pi i$. Therefore the circle of convergence is $|z| < 2\pi$.

(b) $$e^z - 1 = \sum_{n=0}^{\infty} \frac{z^n}{n!} - 1 = \sum_{n=1}^{\infty} \frac{z^n}{n!}$$

$$\begin{array}{r|l}
 & 1 - \frac{z}{2} + \frac{z^2}{12} - \frac{z^4}{720} + \cdots \\
\hline
z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + \frac{z^5}{120} + \cdots & z \\
 & z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + \frac{z^5}{120} + \cdots \\
\hline
 & -\frac{z^2}{2} - \frac{z^3}{6} - \frac{z^4}{24} - \frac{z^5}{120} - \cdots \\
 & -\frac{z^2}{2} - \frac{z^3}{4} - \frac{z^4}{12} - \frac{z^5}{48} \\
\hline
 & \frac{z^3}{12} + \frac{z^4}{24} + \frac{z^5}{80} + \cdots \\
 & \frac{z^3}{12} + \frac{z^4}{24} + \frac{z^5}{72} + \cdots \\
\hline
 & -\frac{z^5}{720} + \cdots
\end{array}$$

Since
$$\frac{z}{e^z - 1} = 1 - \frac{z}{2} + \frac{B_1 z^2}{2} - \frac{B_2 z^4}{24} + \cdots,$$

$B_1 = \frac{1}{6}$ and $B_2 = \frac{1}{30}$.

2/ $${}_2F_1(a, 1; 1; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} z^n$$

$$= \sum_{n=0}^{\infty} \frac{a(a+1)(a+2)\cdots(a+n-1)}{1 \cdot 2 \cdot 3 \cdots n} z^n =$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{(-a)}{1}\frac{(-a-1)}{2}\frac{(-a-2)}{3}\cdots\frac{(-a-n+1)}{n} z^n$$

$$= \sum_{n=0}^{\infty} \frac{(-a)}{1}\cdot\frac{(-a-1)}{2}\cdot\frac{(-a-2)}{3}\cdots\frac{(-a-n+1)}{n}(-z)^n$$

$$= \sum_{n=0}^{\infty} \binom{-a}{n}(-z)^n = (1-z)^{-a},$$

3/ $$\left(\frac{1}{1-z}\right)^2 = \left(\sum_{n=0}^{\infty} z^n\right)^2 = \sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n} 1\right\} z^n$$

$$= \sum_{n=0}^{\infty} (n+1) z^n \qquad \text{for } |z| < 1.$$

This is the desired Maclaurin Series.

To get the Laurent series we write

$$\left(\frac{1}{1-z}\right)^2 = \left(-\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}\right)^2 = \left(\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}\right)^2.$$

$$\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \cdots$$

$$\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \cdots$$

$$\frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \frac{1}{z^5} + \cdots$$

$$\frac{1}{z^3} + \frac{1}{z^4} + \frac{1}{z^5} + \cdots$$

$$+ \frac{1}{z^4} + \frac{1}{z^5} + \cdots$$

$$\frac{1}{z^5} + \cdots$$

$$\vdots$$

$$\frac{1}{z^2} + \frac{2}{z^3} + \frac{3}{z^4} + \frac{4}{z^5} + \cdots = \sum_{n=1}^{\infty}\frac{n}{z^{n+1}} \qquad \text{for } |z| > 1,$$

4/ Since $\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$ for all z,

$(\sin z)^3 = z^3 + a_1 z^5 + a_2 z^7 + \cdots$ where $a_1, a_2, \ldots$ can be found by multiplying the series for $\sin z$ times itself three times. Of necessity, the series for $(\sin z)^3$ converges over the same region as does the series for $\sin z$: $|z| < \infty$. Dividing our series for $(\sin z)^3$ by z^5 we get

$$(1) \quad \frac{(\sin z)^3}{z^5} = \frac{1}{z^2} + a_1 + a_2 z^2 + \cdots ,$$

THUS WE HAVE A POLE OF ORDER TWO AT $z=0$. The only other possible singularity is at infinity. We set $z = \frac{1}{\zeta}$ in (1) and get

$$\zeta^5 \left(\sin \frac{1}{\zeta}\right)^3 = \zeta^2 + a_1 + \frac{a_2}{\zeta^2} + \cdots .$$

Since this last series has INFINITELY MANY TERMS WITH NEGATIVE POWERS OF ζ, we have an essential singularity at $\zeta = 0$. This means that at $z = \infty$ we have an essential singularity.

APPENDIX II

ANSWERS TO CONJECTURES

Chapter 4

4.1 The circle of convergence of a Taylor's series

If $f(z)$ is analytic at the point $z=z_0$, then $f(z)$ can be expanded in a Taylor's series $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ which

converges to $f(z)$ for all z inside the largest open circle with center at $z = z_0$ containing no singularities of $f(z)$.

Discussion:

Let z_1 be the singular point of $f(z)$ that is nearest to the point z_0. (There might be several points which have this distinction, and in this case, z_1 is any one of them.) Then if we call $|z_1 - z_0| = R$, our Taylor's series converges for $|z - z_0| < R$, diverges for $|z - z_0| > R$, and might converge at some points and diverge at others on the boundary $|z - z_0| = R$ of the circle itself.

4.2 Formal manipulations of addition subtraction and multiplication

Let

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n \quad \text{for } |z-z_0| < r, \text{ and}$$

$$g(z) = \sum_{n=0}^{\infty} b_n(z-z_0)^n \quad \text{for } |z-z_0| < R .$$

The formal addition, subtraction, and multiplication of these series is valid inside the smaller of the two circles of convergence.

4.3 The formal division of two Taylor's series

The formal division yielding

$$\frac{f(z)}{g(z)} = \frac{a_0 + a_1 z + a_2 z^2 + \cdots}{b_0 + b_1 z + b_2 z^2 + \cdots} = \frac{a_0}{b_0} + \frac{a_1 b_0 - b_1}{b_0} z + \cdots$$

is valid for all z[1] inside the largest circle centered at $z = 0$ containing no singularities of the function $f(z)/g(z)$. We must assume that $b_0 \neq 0$, since then $f(z)/g(z)$ would have a pole at $z = 0$ when a_0 is not zero. If both a_0 and b_0 are zero, then we can divide the series for f and for g by z and then begin again.

4.4 The composite function

If the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ converges for $|z| < R$, then we can replace z by $h(z)$ provided $|h(z)| < R$. Thus there is some region in the z-plane for which $|h(z)| < R$, and for these z we have $f(\,h(z)\,) = \sum_{n=0}^{\infty} a_n h(z)^n$.

4.5 Term by term differentiation and integration of series

We may differentiate a power series term by term at each point inside the circle of convergence of the original series. We may integrate a power series term by term provided that the path of integration is strictly inside the circle of convergence. Differentiating or integrating a power series does not alter the size of the circle of convergence.

4.6 Laurent series expansions

The boundaries of the regions of convergence are circles with centers at z_0 passing through the singular points z_1 , z_2 and z_3 .

In region I all the $b_n = 0$ for we have a Taylor's series expansion about the regular point z_0 .

In region II we have both positive and negative powers of $z-z_0$ in the Laurent series expansion. In fact, there must be infinitely many non-zero a_n , or otherwise the series in positive powers would be finite and thus would converge for all z, contradicting the finite circle of convergence which passes through the point z_2 . Similarly, there must be infinitely many non-zero b_n . If not, the series in negative powers of $z-z_0$ would be finite and thus would converge right up to the point z_0 itself, contradicting the boundary which passes through z_1 .

In region III no Laurent series is possible because of the discontinuity at the branch line.

In region IV it might happen that all the $a_n = 0$.