AN INTUITIVE INTRODUCTION TO COMPLEX ANALYSIS

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Preface

After having taught the traditional senior level undergraduate complex variables course many times, and after writing some dozen research papers incorporating the elements of this subject, the author became aware of the meed for a "down to earth" presentation of the important applicable features. The development here is intuitive and inductive as opposed to the usual rigorous and deductive presentations. Mathematical maturity is not required of the reader, as no use is made of epsilon delta arguments. The inductive exposition offered here requires that the reader first study in detail specific concrete examples, then he is called upon to "conjecture" general truths based on his experience with special cases. In this way the essential facts needed for a good working knowledge of complex analysis are made to stand out clearly, and the intricacies of the subject are mastered from first hand experience. The only background required of the reader is the usual three semester intuitive level calculus course given at most colleges and universities in the freshman and sophomore years. Even then, it is assumed that the reader has only a very vague appreciation for the more subtle aspects of the calculus such as infinite series and improper integrals.

While a rigorous formulation of the subject is absent from these pages, there is no attempt to "water down" the information needed in practical applications. Indeed, use is made of material and intuitive insights which the author has gleaned from his own research in complex variables which is not usually found in text books. As an example, unusual stress is placed upon actually visualizing specific functions through graphical representations. The exact definition of an analytic function is not presented until the fourth chapter, even though the concept is used in the second and third chapters. The student is made to see that he can deal with a concept even though it is not precisely formulated, and that definitions often evolve slowly as experience is gained with special cases. Thus the reader gradually developes a "feel" for this subject.

It is hoped that this intuitive presentation will be of value to a wide audience of readers. It can be used as a text book for the usual one semester undergraduate complex variables course given in the junior or senior year. Since this intuitive presentation proceeds at a considerably faster pace than most rigorous texts, advanced topics not usually given in a one semester course can be included. A glance at the Table of Contents shows this. Mathematical maturity is not required of the student, and even advanced sophomores should be able to profit from this course. If the professor also wishes to introduce a rigorous development of complex analysis, this text can serve as a tool for "anchoring the students feet to the ground" so that they will better appreciate the need for a deductive development. Engineers and physicists usually welcome intuitive developments of advanced mathematics, and this presentation might be of value in a one semester course for them. This book is also intended for self-study. There are many example problems, and every problem posed for the reader is solved in detail in an appendix. In addition, each chapter is followed by review problems which are also solved in full in the appendix. Students who have taken the traditional course in complex analysis might find that reading this book helps to add concreteness to the general theoretical development they have witnessed.

The format usually used in this book is as follows:

(1) Simple example problems are formulated and solved.

(2) Problems for the student to solve, similar to the above examples , are posed to reinforce the ideas in the students mind. All problems are fully solved in Appendix I.

(3) Brief observations are made on these examples and problems.

(4) The nature of mathematical principles, as yet unknown to the student, can be conjectured from the experience just obtained in problem solving. The student is asked to answer specific conjectures. Answers and remarks concerning the conjectures are provided in Appendix II.

Often intuitive presentations of mathematical subjects are authored by non-mathematicians. The professional mathematician usually objects to these treatments on the ground that they appear to lie to the serious student. Such presentations often present arguments which are called "proofs", when they are really only heuristic reasoning. It is hoped that this intuitive presentation will not be rejected by mathematicians on these grounds . The author is himself a professional mathematician, and he knows well their reaction to false claims of proof. Here the student is constantly reminded of the deficiencies in the arguments presented. When a concept is only partially formulated, the student is so warned, and not made to feel that it is a final definition. When real proofs and definitions are presented, the student is made aware.

CHAPTER 1

COMPLEX NUMBERS AND COMPLEX ARITHMETIC

The reader has certainly encountered the expression $i = \sqrt{-1}$, the " imaginary unit " before. Its occurence is probably best remembered from the solution of the quadratic equation

$$a x^2 + b x + c = 0$$

given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} .$$

When $4ac$ is larger than b^2, we can find no real number equal to x. We have encountered a new type of number which we call "complex". The number $3 + 4i$ is such a number. We will now present the arithmetic of complex numbers. We will not present anything like an axiomatic developement. That is not our purpose. Rather, we will assume that an arithmetic of complex numbers exists in which most of the rules of the real arithmetic already familiar to us carry over. We will proceed often with a " faith " in the symmetry of mathematics, and in the repetition of plausible patterns. After all, it is not unlikely that much that is already familiar to us in the "real domain" will carry over to the "complex domain".

1.1 Elementary arithmetic

We first consider simple powers of the imaginary unit $i = \sqrt{-1}$.

$$i^0 = 1$$
$$i^1 = i$$
$$i^2 = \sqrt{-1}\,\sqrt{-1} = -1$$
$$i^3 = i^2\, i = -i$$
$$i^4 = i^2\, i^2 = (-1)(-1) = 1$$
$$i^5 = i^4\, i = i$$
$$\vdots$$

When performing the operations of addition, subtraction, multiplication and division, we treat i as though it was an algebraic constant subject to the interpretation given by the above list of powers of i. For example,

$$(3 + 2i) + (2 - i) = 5 + i,$$

$$(3 + 2i) - (2 - i) = 1 + 3i,$$

$$(3+2i)(2-i) = 6-3i+4i-2i^2 = 6+i+2 = 8 + i .$$

Division requires remembering a simple trick. To compute

$$\frac{3+2i}{2-i}$$

we multiply numerator and denominator by the "complex conjugate" of the denominator. The complex conjugate of the number $a + bi$ is the number $a - bi$, and we denote complex conjugation by $\overline{a+bi} = a-bi$. (Here a and b are real numbers). Therefore

$$\frac{3+2i}{2-i} = \frac{(3+2i)(2+i)}{(2-i)(2+i)} = \frac{6+3i+4i-2}{4+2i-2i+1} = \frac{4+7i}{5} = \frac{4}{5} + \frac{7}{5}i .$$

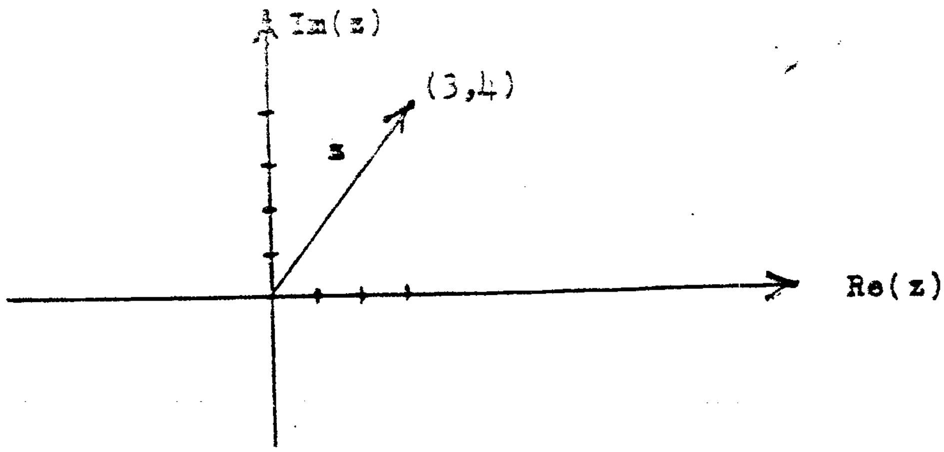
<u>Problem</u>

1. Let $z = 3-3i$, and $w = 1 + \sqrt{3}\,i$. Find (a) $z+w$, (b) $z-w$, (c) zw and (d) z/w .

2. Let $z = a + bi$ and $w = c + di$, where a, b, c, d, are real numbers. Show that (a) $\overline{z+w} = \overline{z} + \overline{w}$, (b) $\overline{z-w} = \overline{z} - \overline{w}$, (c) $\overline{zw} = \overline{z}\cdot\overline{w}$, and that (d) $\overline{(z/w)} = \overline{z} / \overline{w}$.

1.2 The Complex Plane

We are familiar with the "real line" as a geometric device for visualizing the real numbers. Since the complex number $z = x + yi$ is a two dimensional item (there is an x and a y), it is not surprising that mathematicians have found it useful to consider complex numbers as points or vectors on a plane. Thus, the complex number $z = 3+4i$ is represented by the following picture.

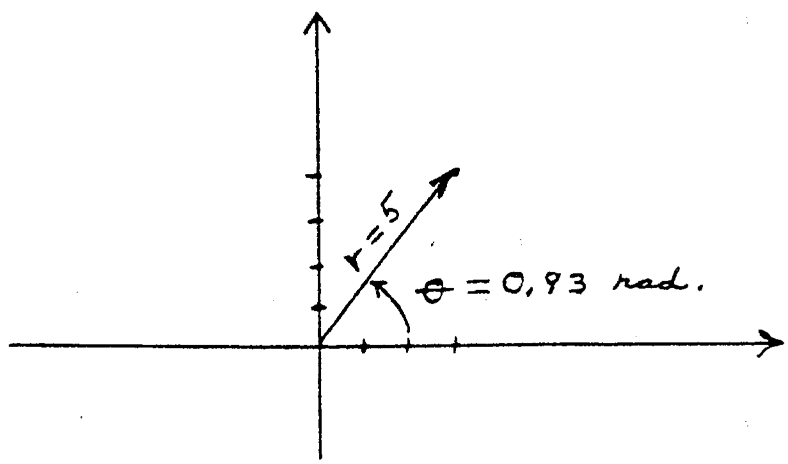


$z = 3+4i$ can be thought of as the point having Cartesian coordinates (3,4), or as the vector drawn from the origin to (3,4). We call the horizontal axis the "real axis" and the vertical axis the "imaginary axis". The entire plane is known as the "complex plane" or the "Argand diagram". In the case of the number $z = 3+4i$ we also write:

"real part of z" = $\mathrm{Re}(z) = 3$, and

"imaginary part of z" = $\mathrm{Im}(z) = 4$.

The complex number $z = 3+4i$ can also be defined by its polar coordinates r and θ.



Here $r = \sqrt{3^2 + 4^2} = 5$ and $\theta = \arctan(4/3) = 0.93$ radians. As in analytic geometry, the angle θ is not unique. We can add to θ any multiple of 2π radians. Thus

$$\theta = 0.93 + 2\pi n\ , \quad \text{where } n = 0, \pm 1, \pm 2, \ldots .$$

We often call r the "modulus of z", and denote it by use of absolute value symbol $|z|$. Thus $|3+4i| = 5$. The angle θ is often called the "argument of z", and is denoted by $\arg(z) = \theta$. Thus $\arg(3+4i) = 0.93 + 2\pi n$, where $n = 0, \pm 1, \pm 2, \ldots$.

Problem

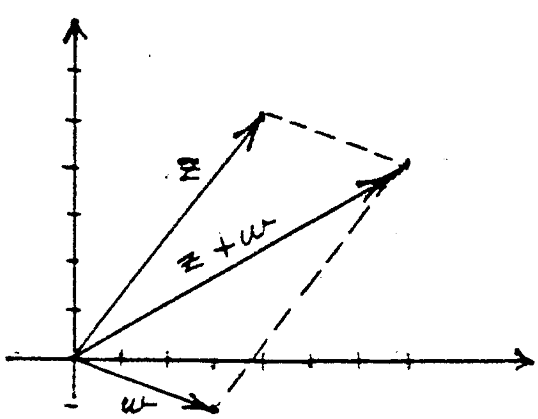
3. Let $z = -4 - 4\sqrt{3}\ i$. Find (a) $\mathrm{Re}(z)$, (b) $\mathrm{Im}(z)$, (c) $|z|$ and (d) $\arg(z)$.

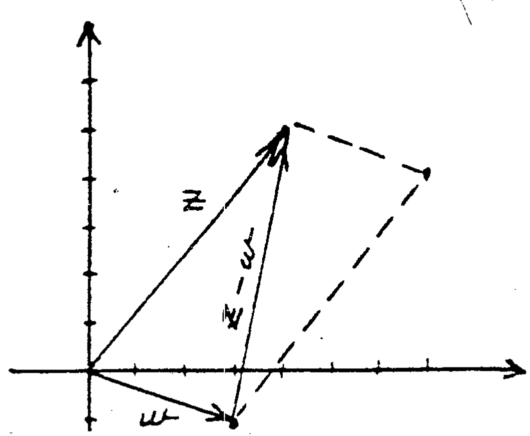
1.3 Geometric aspects of addition and subtraction

The reader is familiar with the addition and subtraction of vectors from earlier courses in physics and mathematics. Since complex numbers are represented by vectors on the Argand diagram, it comes as no surprise that the geometric description of the addition and subtraction of complex numbers is the same as that previously examined for ordinary vectors in other courses. We merely review the process by means of an example.

Example

Let $z = 4+5i$ and $w = 3-i$. Form $z+w$ and $z-w$.





From the above two diagrams we recall that the vectors $z+w$ and $z-w$ form the two diagonals of the parallelogram having as adjacent sides z and w. Note that the vector $z-w$ is drawn <u>from</u> the head of w <u>to</u> the head of z.

<u>Problem</u>

4. Let $a = 1+i$, $b = -1 + \sqrt{3}\,i$ and $c = -\sqrt{3} - i$. Find $a+b-c$ geometrically.

<u>1.4 Important regions in the complex plane</u>

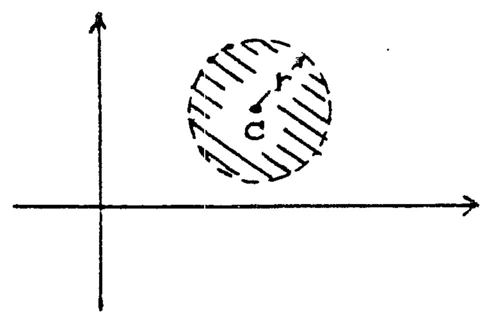
In the study of the calculus, we became familiar with certain important subsets of the real line:

open intervals such as $x-a < b$, $x < a$, and $x > b$;

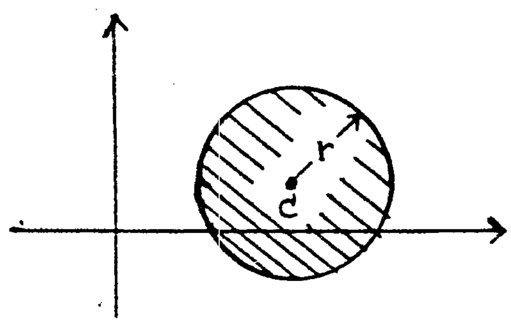
closed intervals such as $x-a \leq b$, $x \leq a$, and $x \geq b$.

In the complex plane we usually denote $z = x + iy$ as the "complex variable", where both x and y are real variables. The most important subsets of the complex plane are shown below.

(i) open disk $|z - c| < r$

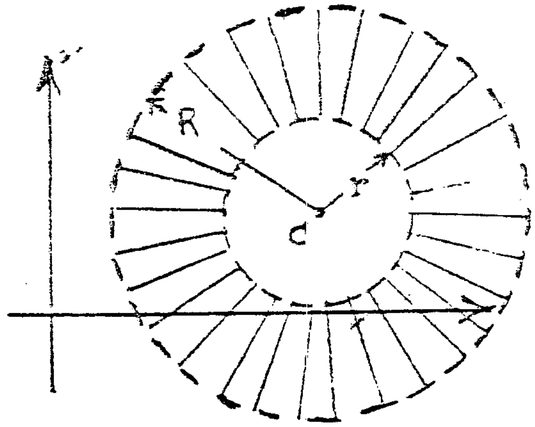


(ii) closed disk $|z - c| \leq r$



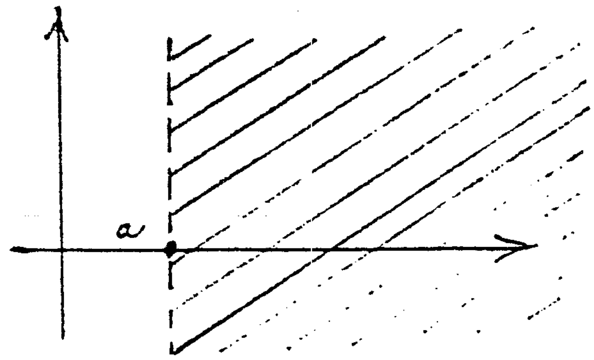
(iii) open annulus

$r < |z-c| < R$



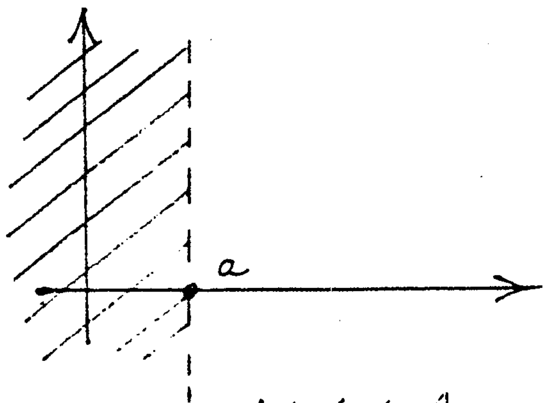
(iv) right half plane

$a < \mathrm{Re}(z)$



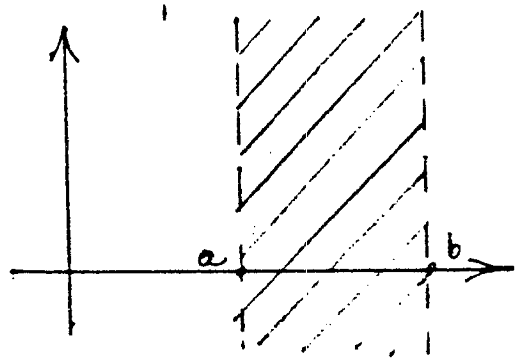
(v) left half plane

$\mathrm{Re}(z) < a$



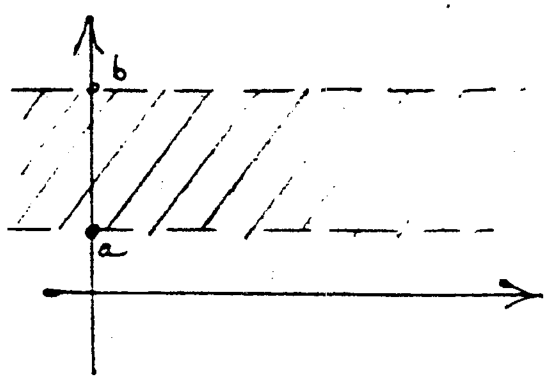
(vi) vertical strip

$a < \mathrm{Re}(z) < b$



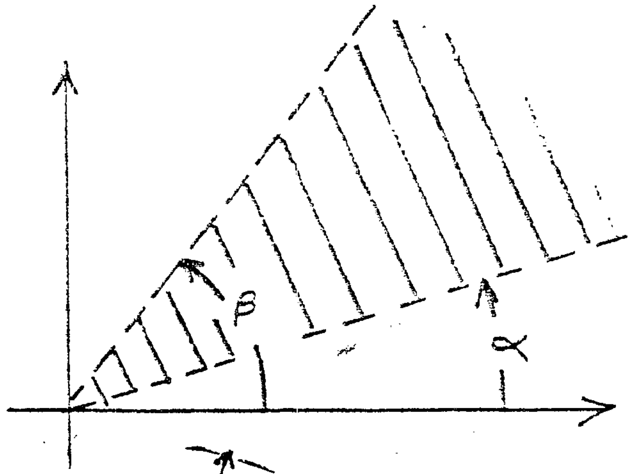
(vii) horizontal strip

$a < \mathrm{Im}(z) < b$



(viii) Infinite open sector

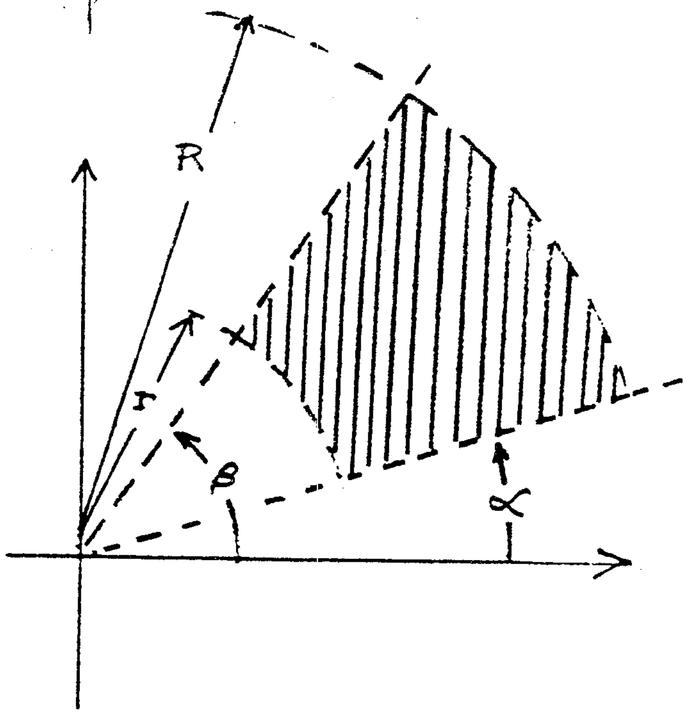
$\alpha < \arg(z) < \beta$



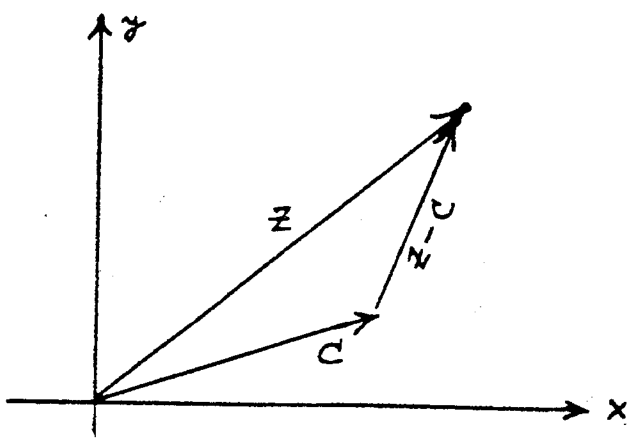
(ix) Finite open sector

$\alpha < \arg(z) < \beta$

$r < |z| < R$



It is easy to see how the analytical einequalities given in (iv) through (viii) above yield the corresponding geometric regions. The remaining regions are easily understood when we examine the geometric meaning of $|z-c|$. The diagram below shows the three vectors c, z, and $z-c$. The expression $|z-c|$ is the length of



the vector $z-c$. Now let us consider the inequality described in (i) . If c is held fixed, and the vector z is allowed to move under the restriction that the length of $z-c$ must stay less than some fixed given number r ($|z-c| < r$), we see at once that z may assume any location inside the circle centered at c of radius r as shown in (i) .

Often the notionsof open sets and closed sets of points in a plane are introduced in courses in Real Analysis, and in Topology. We will have need of this terminology for sets of points in the complex plane, but there is no need for us to give difficult general definitions. For our purposes an <u>open set</u> is a two dimentional region with the boundary points removed. The open set is made <u>closed</u> by adding to it the points on its boundary. As an example, the figure for region (i) is open because the circumference of the circle is excluded, while the figure for region (ii) is closed because the circumference is included.

<u>Problem</u>

5. Show the following regions in the complex plane and determine if the regions are (i) open, (ii) closed, or (iii) neither open nor closed. (a) $2 < |z|$, (b) $2 \leq \text{Re}(z)$, (c) $0 < \arg(z) \leq \pi/4$, (d) $0 < |z| < 1$, (e) $1 \leq |z-2| \leq 2$, (f) $\{z \mid 0 < \arg(z) < \pi/4,\ |z| > 1\}$.

We will sometimes have need to simplify an inequality before we attempt to visualize the corresponding region in the complex plane.

<u>Example</u>

Describe the region defined by $|3z - 6| \leq 9$.

<u>Solution</u>

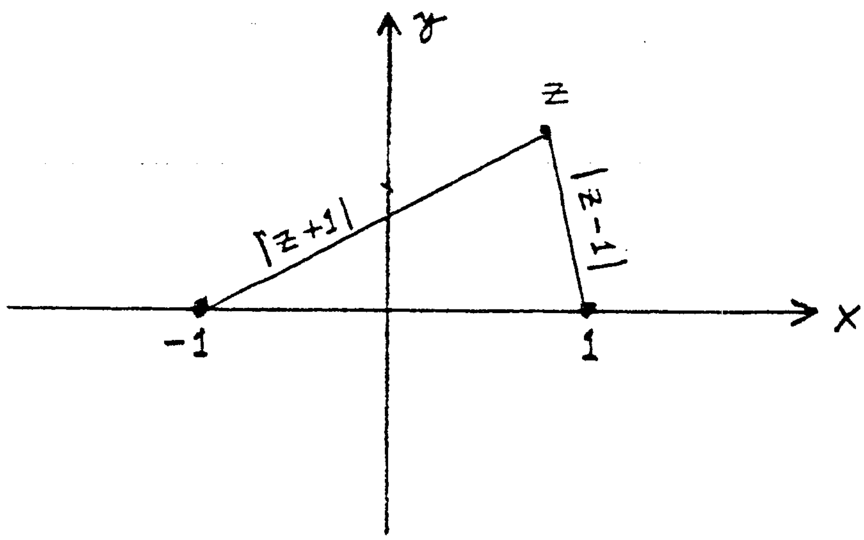
We can factor 3 from inside the absolute value sign to get $3\,|z-2| \leq 9$, or $|z-2| \leq 3$. This is the closed circle with center at $z = 2$ having radius 3.

Example

Describe the region in the z -plane defined by $\left|\frac{z-1}{1+z}\right| < 1$.

Solution

The inequality can be written as $|z-1| < |z+1|$. This inequality says " the distance of the point z from the point 1 is less than the distance of the point z from the point -1". We see



from the figure that this is true for points z in the right half plane described by $Re(z) > 0$. This is an open set.

Problem

6. Describe the region in the complex plane expressed by the inequality. (a) $|4 - 4z| > 8$, (b) $\left|\frac{z - z_0}{1 - z_0}\right| < 1$,

(c) $|z| \leq |1 - z|$.

1.5 The complex exponential

In a previous course we learned that the functions e^x , $\sin x$ and $\cos x$ could be expanded in power series as

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots ,$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots , \text{ and}$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots .$$

If we now set $x = i\theta$ into the above power series for e^x we get

$$e^{i\theta} = 1 + \frac{i\theta}{1!} + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \dots$$

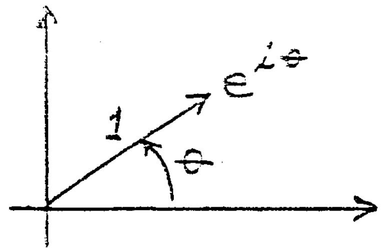
$$e^{i\theta} = 1 + \frac{i\theta}{1!} - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \dots$$

$$e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\right).$$

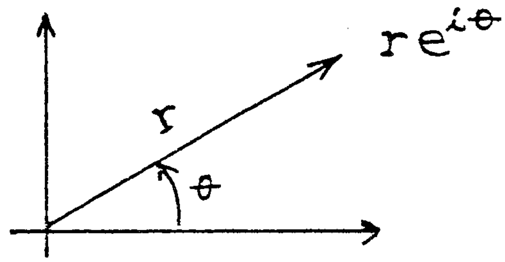
Comparing this last expression with the above power series for $\sin\theta$ and $\cos\theta$ we see that

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta} \quad (\text{ Euler's formula }).$$

Euler's formula defines $e^{i\theta}$ for real θ as the vector of unit length making angle θ with the real axis.



Note that multiplying $e^{i\theta}$ by the real number r gives $r\,e^{i\theta} = r\cos\theta + i\,r\sin\theta$, and since $|r\,e^{i\theta}| = r$, we see that $r\,e^{i\theta}$ is a complex number having modulus r and argument θ.



Thus the expression $z = r\,e^{i\theta}$ is convenient when we wish to display the "polar coordinates" of z, and the notation $z = x+iy$ is convenient when we wish to display the "Cartesian components" of z. For example,

$$1+i = \sqrt{2}\,e^{i\pi/4} = -\sqrt{2}\,e^{i5\pi/4} = -\sqrt{2}\,e^{-i3\pi/4}$$

are all equivalent ways of representing the same complex number. (Recall that the $x+iy$ form of the complex number is unique; while the $r\,e^{i\theta}$ form is not unique.)

Problems

7. Express the following numbers in the form $re^{i\theta}$, with $r > 0$, and $0 \le \theta < 2\pi$. (a) $5i$, (b) $-7+7i$, (c) $-4-4\sqrt{3}\,i$, (d) -1.

8. Express the following numbers in the form $x+iy$.
(a) $2e^{3\pi i}$, (b) $e^{13\pi i/4}$, (c) $7e^{-\pi i/6}$, (d) $e^{-\pi i/2}$.

1.6 Geometric aspects of multiplication and division.

Conjecture 1

Based on rules familiar from the study of exponents involving real numbers, conjecture the results of the folloing operations involving imaginary exponents:
$e^{ai}\, e^{bi}$, $e^{ai}/\, e^{bi}$, and $(e^{ai})^{b}$, where a and b are real numbers. (See Appendix II for the answers to Conjectures.)

Let us now examine multiplication. Let $z = r\, e^{i\theta}$ and $w = s\, e^{i\omega}$. Then

$$\begin{aligned} z\,w &= r\, e^{i\theta}\, s\, e^{i\omega} \\ &= rs\, e^{i\theta}\, e^{i\omega} \\ &= rs\, e^{i(\theta+\omega)}. \end{aligned}$$

This last expression tells us that $|zw| = |z|\cdot|w|$ and that $\arg(zw) = \arg(z) + \arg(w)$. In other words, when we multiply the vectors z and w, we form a new vector whose length is the product of the lengths of the original two vectors, and whose angle is the sum of the angles of the original two vectors.

Division is similar.

$$\frac{z}{w} = \frac{r\, e^{i\theta}}{s\, e^{i\omega}} = \frac{r}{s}\, e^{i(\theta-\omega)}$$

Thus $|z/w| = |z| / |w|$, and $\arg(z/w) = \arg (z) - \arg(w)$.

Example

Let $z = 6\, e^{i\pi/3}$ and $w = 3\, e^{i\pi/6}$. Describe zw and z/w .
zw has modulus 18 and argument $\pi/2$. Thus $zw = 18i$. z/w has

modulus 2 and argument $\pi/6$, which means that $z/w = \sqrt{3} + i$.

Problems

9. Returning to problems 1(c) and 1(d), form the required product and quotient using the $r\, e^{i\theta}$ form of the numbers, and compare the results with the computation performed previously using the $x+iy$ form of the numbers.

10. By squaring $e^{i\theta} = \cos\theta + i\sin\theta$, find formulas for $\cos 2\theta$ and $\sin 2\theta$ in terms of $\sin\theta$ and $\cos\theta$.

11. Find formulas for $\cos 3\theta$ and $\sin 3\theta$ in terms of $\sin\theta$ and $\cos\theta$.

1.7 Roots of complex numbers

Suppose we were asked for the cube root of $8i$. We would seek z such that $z^3 = 8i$. Writing $8i = 8\, e^{i(\frac{\pi}{2} + 2\pi n)}$, $n = 0, \pm 1, \pm 2, \ldots,$ we are led quite naturally to write

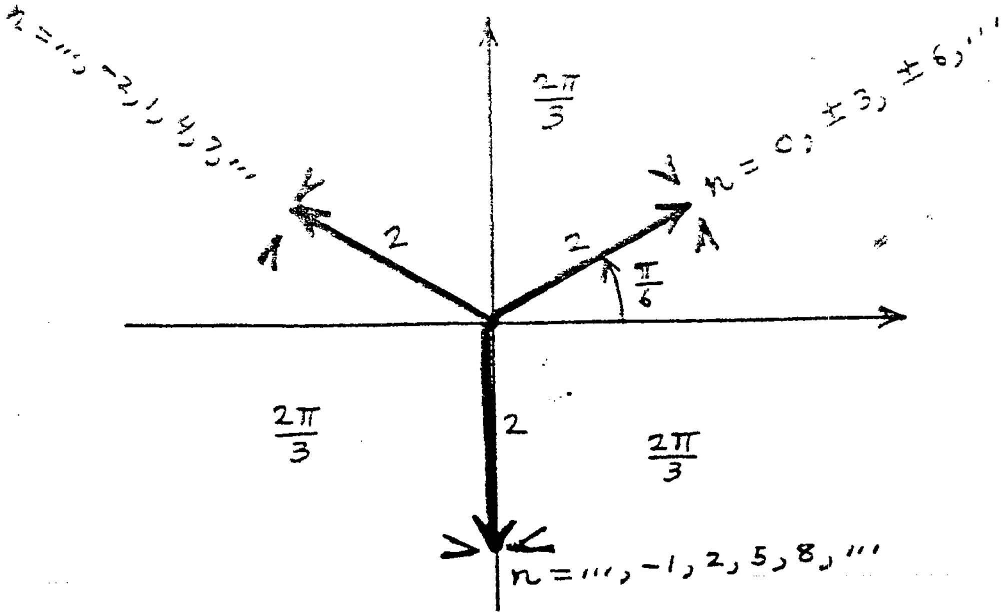
$$z^3 = 8\, e^{i(\frac{\pi}{2} + 2\pi n)}$$

$$z = [\,8\, e^{i(\frac{\pi}{2} + 2\pi n)}\,]^{\frac{1}{3}}$$

$$z = 8^{\frac{1}{3}}\, e^{i(\frac{\pi}{2} + 2\pi n)\frac{1}{3}}$$

$$z = 2\, e^{i(\frac{\pi}{6} + \frac{2\pi n}{3})}, \quad n = 0, \pm 1, \pm 2, \ldots .$$

At first glance, it looks as though there are infinitely many possible answers for z since n can be any integer. However, if we examine the answers on the Argand diagram we see that we have



These three values for the cube root of $8i$ can also be written as $\sqrt{3} + i$, $-\sqrt{3} + i$ and $-2i$. Thus there are three distinct answers. In a way, this is not surprising, since $z^3 = 8i$ is a cubic equation and we therefore expect three answers.

Problems

Solve the following equations and show the answers on the Argand diagram.

12. $z^2 + 4 = 0$, 13. $z^3 - 8 = 0$, 14. $z^4 = 81(1+i)/\sqrt{2}$,
15. $z^3 + i = 0$, 16. $z^6 = -64$.

Conjecture 2

You wish to find all z such that $z^N = a\, e^{ib}$, where N is a positive integer, a and b are real numbers, and $0 \leq b < 2\pi$.

(i) What is the number of distinct answers ? N

(ii) What is the modulus of each answer ? a

(iii) When the answers are displayed on the Argand diagram, what is the smallest positive angle, and what angle separates each of the vectors. $2\pi/n$

1.8 The Riemann sphere of numbers

We have seen that the complex plane is a convenient device for visualizing the operations of addition, subtraction, multiplication and division. For some purposes, it is convenient to use points on a sphere rather than points on the plane to visualize operations involving complex numbers. To see how this is achieved, look at Figure 1. Here a sphere having unit diameter is placed directly over the origin of coordinates of the complex z-plane. Each point P on the complex plane is made to correspond to a point P′ on the surface of the sphere in the following way:

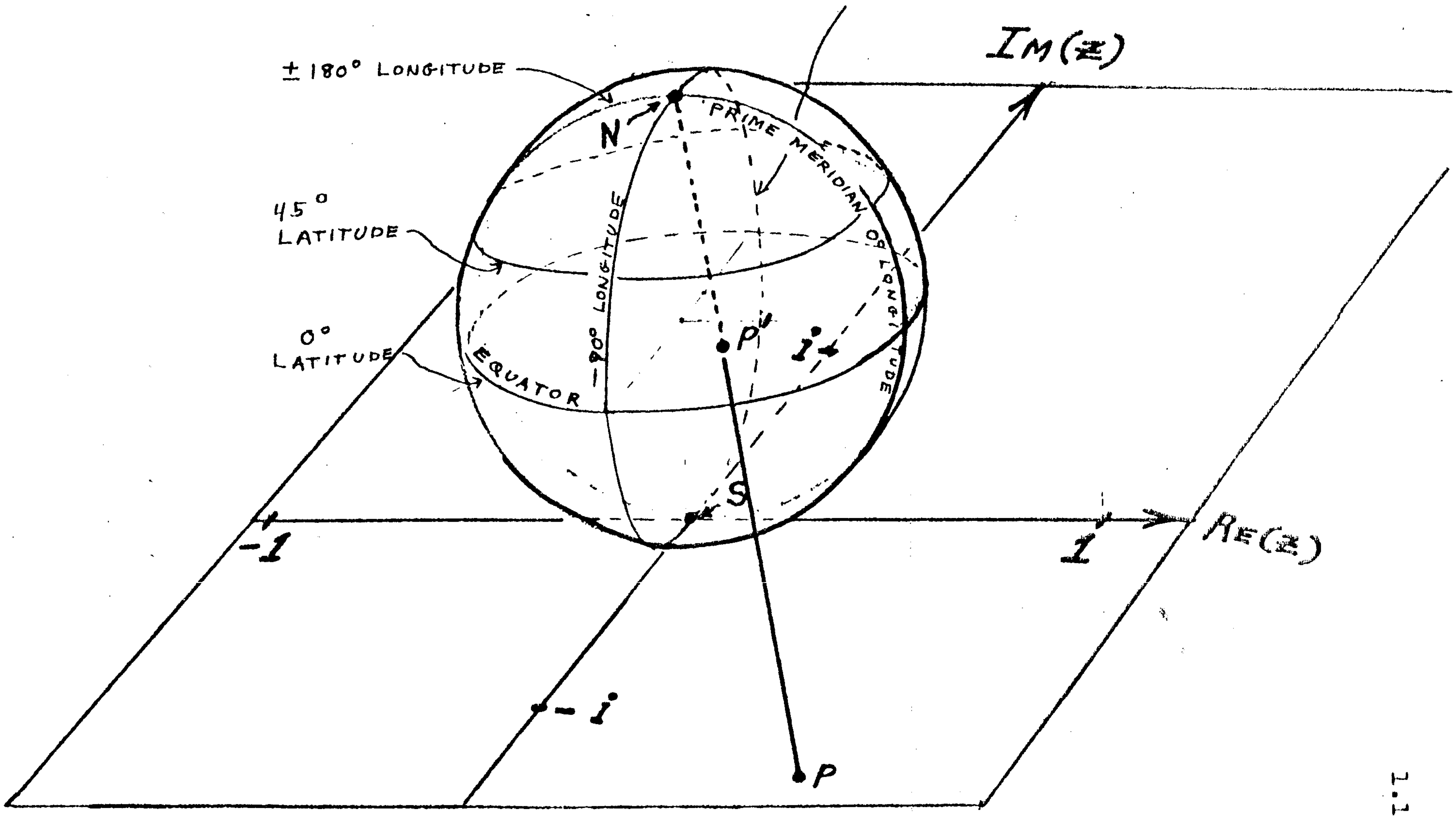
(1) A straight line is drawn from the north pole of the sphere "N" to the point P in the plane.

(2) The point P′ to which P is mapped is the point where the above straight line intersects the surface of the sphere.

The mapping of points P on the plane to points P′ on the sphere is called a stereographic projection. The sphere itself is called the Riemann sphere of the sphere of complex numbers.

To identify points on the Riemann sphere, we will use the terminology used to identify points on a globe of the earth. In Figure 1 we see the north pole "N" and the south pole "S". We also see the equator and circles of longitude and circles of latitude. An examination of Figure 1 reveals the following mapping of points from the plane to the sphere:

Figure 2 The Riemann sphere of numbers and the stereographic projection



Point on the Plane	Point on the Sphere
$z = 0$	South pole "S"
$z = 1$	Equator at 0° longitude
$z = i$	Equator at 90° longitude
$z = -1$	Equator at $\pm 180^\circ$ longitude
$z = -i$	Equator at -90° longitude

If we ignore the north pole "N", we see that the stereographic projection defines a one to one mapping of points on the plane with points on the sphere.

Notice that no point on the complex z-plane maps onto the north pole of the Riemann sphere. However, we do notice that points near N correspond to points on the z-plane very very far from the origin. If we introduce a new (improper) point $z = \infty$, it seems reasonable to identify the north pole with this point. We call the ordinary complex plane (without $z = \infty$) the <u>finite plane</u>. When we add the point $z = \infty$ to the complex plane we call it the <u>extended plane</u> or the <u>closed plane</u>. Thus we see that the concept of a point at infinity takes on a very tangible form on the Riemann sphere since it corresponds to a well defined point, the north pole. The Riemann sphere is very convenient for visualizing phenomena involving large z because these points occur near the north pole.

A curve on the complex z-plane now maps onto a curve on the surface of the Riemann sphere. It is easy to see that the circle $|z| = 1$ in the plane maps onto the equator of the sphere. The positive real axis maps onto the semi-circle of longitude 0°. The positive imaginary axis maps onto the semi-circle of longitude 90°. An examination of Figure 2 reveals that any straight line in the plane maps onto a circle through the north pole of the sphere. Suppose we call a straight line in the z-plane a circle of infinite radius. With this terminology, we have the following property of the stereographic projection:

Every circle on the plane maps onto a circle on the sphere

and

every circle on the sphere maps onto a circle on the plane.

This property of the mapping is summarized by saying that it is circular. We will not prove the circular property in this section. The student can prove this property by following the steps outlined in supplementary problem 1.8.

Problem

17. Look at Figure 1 and convince yourself that

    (a) A circle $|z| = R$ maps onto a circle of latitude on the sphere.

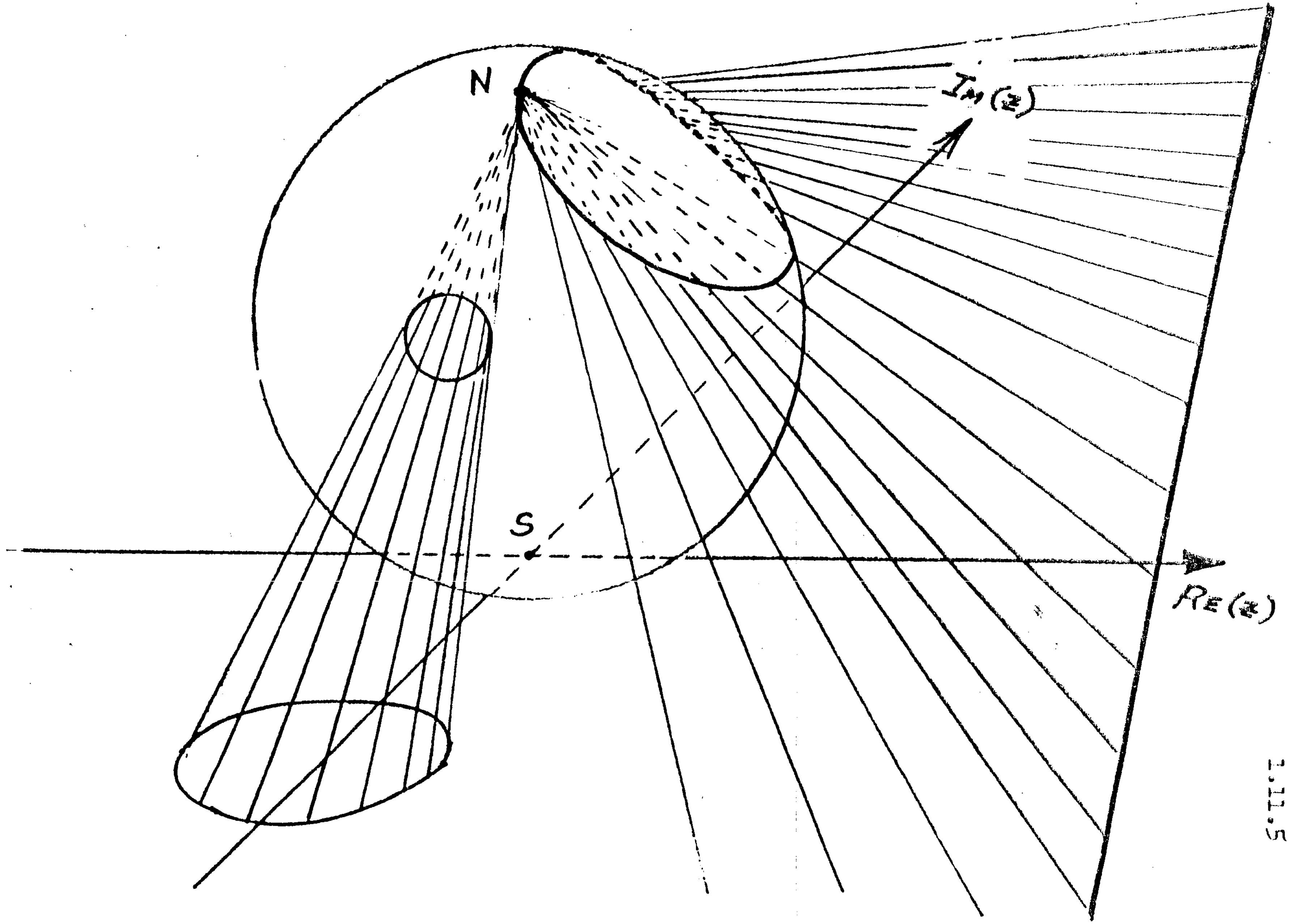
    (b) Two parallel lines on the plane map onto two circles on the sphere tangent at the north pole.

    (c) The region $|z| < 1$ maps onto the southern hemisphere.

    (d) The region $|z| > 1$ maps onto the northern hemisphere.

    (e) The half-plane $Im(z) > 0$ maps onto the hemisphere having longitude between 0° and 180°.

Figure 2 The circularity of the stereographic projection



Another important property of the stereographic projection is that it preserves angles. That is to say, angles on the plane map into equal angles on the sphere. We say that the mapping is <u>isogonal</u>. In Figure 3 we see the angle θ at the point P in the plane. This angle is defined by the straight lines AP and BP. The lines AP and BP map onto the arcs A'P' and B'P' on the surface of the sphere. The mapping is isogonal if the angle A'P'B' is also θ.

read this

It is easy to prove the isogonality of the mapping. First note that the sector APB in the plane (with the lines extended to infinity) maps onto a slice of the sphere which resembles a slice of the skin of an orange. Note that the angle at N must equal the angle at P'. But the angle at N is clearly θ since the tangents B''N and A''N are parallel to BP and AP. Thus the mapping is isogonal.

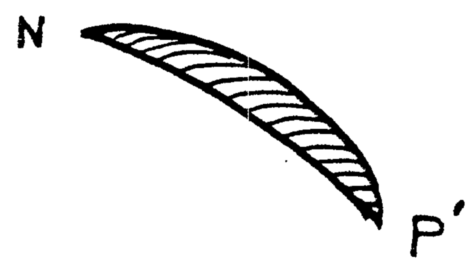
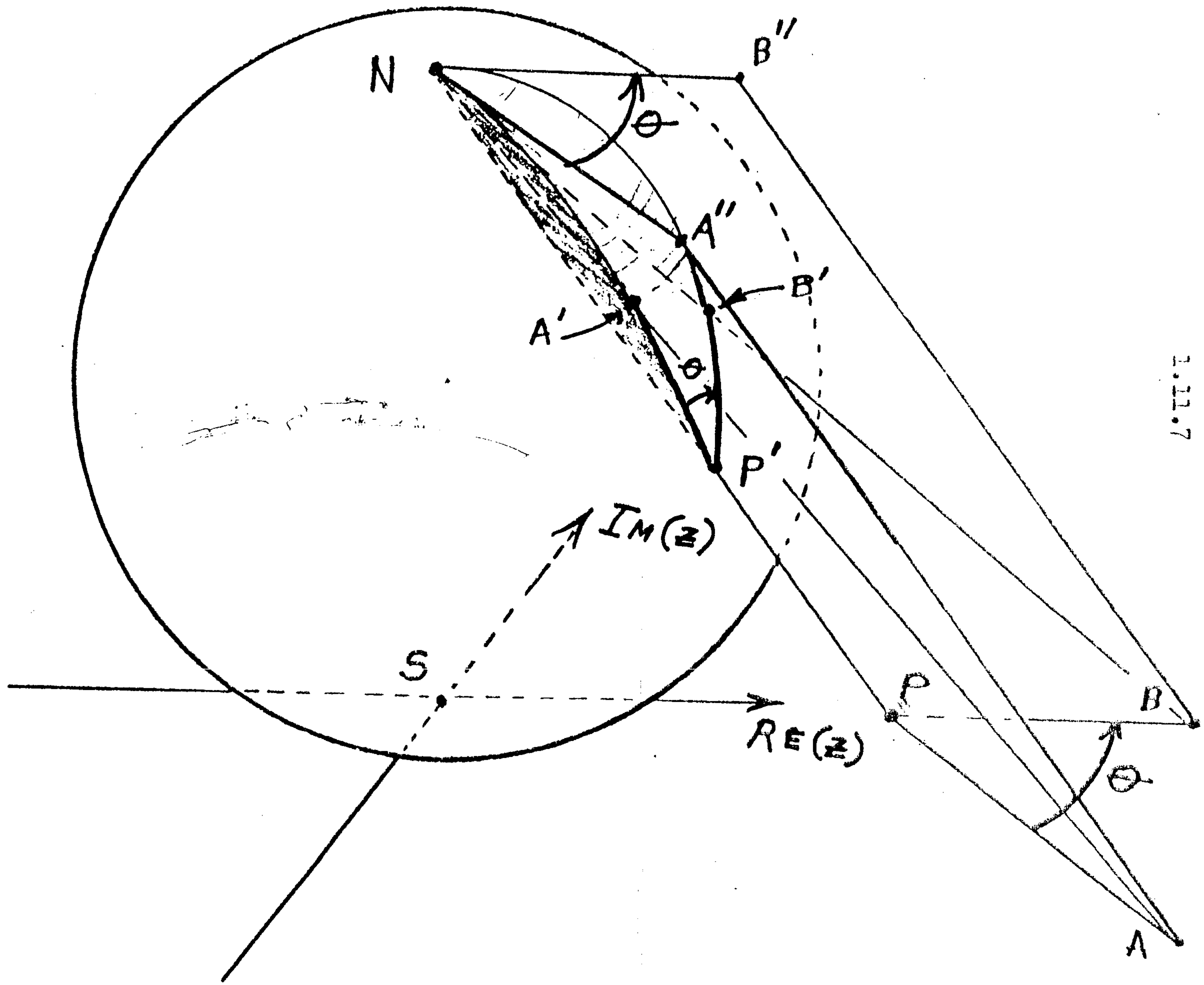


Figure 3 The isogonality of the stereographic projection.



Review problems for Chapter 1

1. Describe the regions in the z-plane and state if they are open or closed. (a) $2 \leq \mathrm{Re}(z)$, (b) $\{z \mid -\pi/4 < \arg(z) \leq \pi/4$, and $|z| < \pi\}$, (c) $2 / |z-1-i| < 1$; (d) $2 < |z-2| < 4$, (e) $|z-1-i| < |z|$.

2. Let $z = 1 + \sqrt{3}\,i$, $w = 3i$, $\zeta = 2\sqrt{3} - 2i$. (a) Express z, w, and ζ in polar form. (b) What is $|zw\zeta|$? (c) What is $\arg(zw\zeta)$?, (d) What is $\arg(\overline{zw\zeta})$?

3. Prove that $e^{i\theta} e^{i\omega} = e^{i(\theta+\omega)}$.

4. Find all values of z such that $z^4 = -81$. Express the results in Cartesian form.

SUPPLEMENTARY PROBLEMS

Note that all supplementary problems are identified by

CHAPTER - SECTION - PROBLEM NUMBER

Thus the item identified by 1.3.12 is the twelfth problem for section 3 of chapter one.

1.1.1 Find $z + w$, $z - w$, zw and z/w for each of the following pairs of complex numbers and express the result in the form $x + iy$. (a) $z = 4-4i$, $w = -8-8i$; (b) $z = 2+2\sqrt{3}\,i$, $w = 2-2\sqrt{3}\,i$; (c) $z = 3+4i$, $w = 4-3i$; (d) $z = 3-i$, $w = 2+3i$

1.1.2 Compute $(-1 + \sqrt{3}\,i)^3$. Ans. 8.

1.1.3 Compute $(-1 - \sqrt{3}\,i)^6$. Ans. 64.

1.1.4 Compute $(1 + i)^8$

1.1.5 Compute $(2 - 2i)^4$

1.2.1 Find $\mathrm{Re}(z)$, $\mathrm{Im}(z)$, $|z|$, $\arg(z)$ and $\bar{z}$ for each of the following complex numbers: (a) $z = 3$, (b) $z = 4i$, (c) $z = 3-3i$, (d) $z = \sqrt{3} - i$, (e) $z = 4 - 3i$.

1.2.2 (a) Show that $z + \bar{z} = 2\,\mathrm{Re}(z)$.

(b) Show that $z - \bar{z} = 2i\,\mathrm{Im}(z)$.

1.2.3 Show that $|z|^2 = z\bar{z}$

1.3.1 Add the numbers given in problem 1.1.1 vectorially.

1.3.2 Find the sum of the following numbers algebraically and vectorially: 3, $2-3i$, $4i$, $-6+i$, $2-2i$.

1.3.3 What conditions should be imposed on z_1 and z_2 so that $|z_1 + z_2| = |z_1| + |z_2|$?

1.3.4 What conditions should be imposed on z_1 and z_2 so that $|z_1 - z_2| = |z_1| - |z_2|$? the vectors share the same direction and $z_1 \ge z_2$

1.3.5 Give a geometric argument to demonstrate the identity

$$|z_1 - z_2| \ge \big||z_1| - |z_2|\big| .$$

1.4.1 Describe the following regions on the complex plane, and decide if they are (i) open, (ii) closed, (iii) neither open nor closed. (a) $\operatorname{Im}(z) < -1$; (b) $\operatorname{Re}(z) \ge 2$; (c) $1 \le \operatorname{Re}(z) < \pi$; (d) $\pi < \arg(z) \le 3\pi/2$; (e) $|z| < 5$; (f) $|z-2| \le 2$; (g) $3 \le |z| \le 4$; (h) $2 < |z-2| < 4$; (i) $\{z \mid 3 \le |z| \le 4, 0 \ \arg(z) < \pi/4\}$.

1.4.2 Describe the following regions on the complex plane: (a) $|3+3z| < 6$; (b) $|2z-4| \le 2$; (c) $8 < |4-4z| < 12$; (d) $|z| = |z-1|$; (e) $\left|\frac{z+i}{z-i}\right| < 1$; (f) $\left|\frac{z-2i}{z+2}\right| \ge 1$.

1.4.3 Show that the equation $|z+1| + |z-1| = 2\sqrt{2}$ defines the ellipse $x^2 + 2y^2 = 2$.

1.4.4 Show that $|z+i| - |z-i| = 2\sqrt{2}$ defines a hyperbola, and write its equation in terms of x and y .

1.4.5 Show that $|z| = 2|z-1|$ defines a circle and write its equation in terms of x and y .

1.5.1 Express the following numbers in the form $r\,e^{i\theta}$ with $r > 0$ and with $0 \leq \theta < 2\pi$: (a) $-3i$; (b) $1+i$; (c) $-1 - \sqrt{3}\,i$; (d) -3 .

1.5.2 Express the following numbers in the form $x + iy$: (a) $3\,e^{\pi i}$; (b) $2\,e^{\pi i/2}$; (c) $7\,e^{7\pi i}$; (d) $-2\,e^{-\pi i/4}$; (e) $4\,e^{\pi i/6}$; (f) $5\,e^{341\pi i}$.

1.6.1 Using the numbers given in problem 1.1.1, find zw and z/w vectorially .

1.6.2 Show that $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$. (De Moivre's Theorem)

1.6.3 Show that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$ and that $\sin 5\theta = \sin\theta\,[\,16\cos^4\theta - 12\cos^2\theta + 1\,]$.

1.7.1 Solve the following equations and express the results in the form $x+iy$: (a) $z^2 + 4i = 0$; (b) $z^2 + 1 = 0$; (c) $z^3 + 8 = 0$; (d) $z^4 + 16 = 0$; (e) $z^6 - 64 = 0$; (f) $z^4 + 1 - i = 0$. Answers (a) $\pm\sqrt{2}\,(-1+i)$; (b) $\pm i$; (d) $\pm\sqrt{2}\,(1+i)$, $\pm\sqrt{2}\,(-1+i)$.

1.7.2 Find the square roots of $5 + 12i$.

1.7.3 Find all the roots of the equation $z^4 + z^2 + 1 = 0$. Ans. $(1 \pm \sqrt{3})/2$, $(-1 \pm \sqrt{3})/2$.

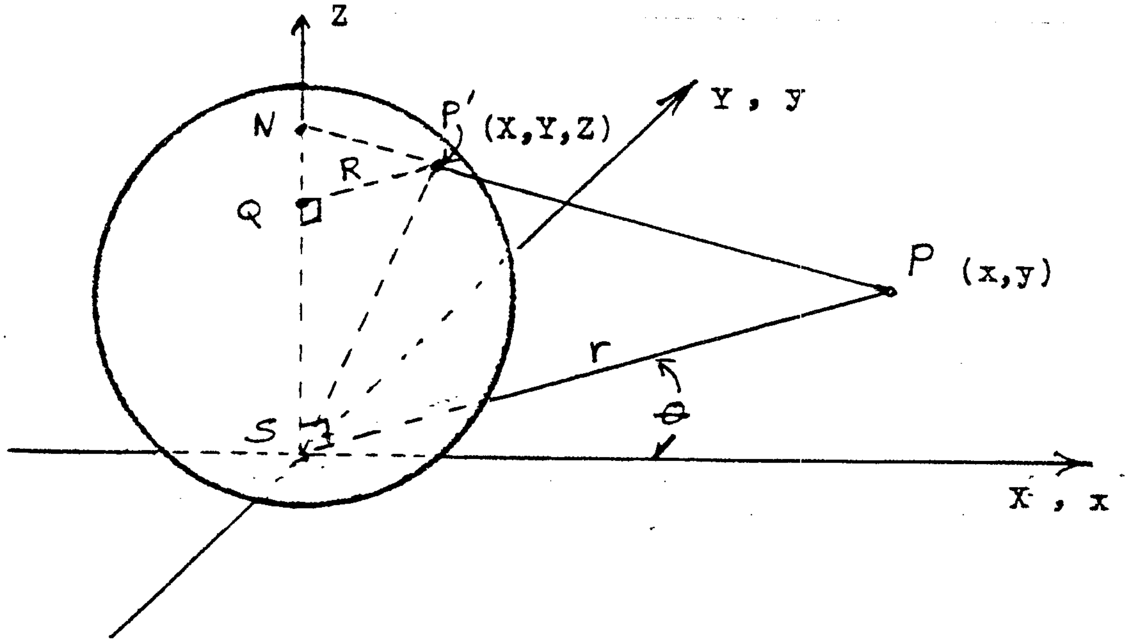
1.8.1 Argue that the straight line $x = 1$ maps onto a circle through the north pole and tangent to the equator at the point of 0° longitude.

1.8.2 Let C be a circle in the z-plane that intersects the unit circle $|z| = 1$ at diametrically opposite points. Argue that C maps onto a great circle on the Riemann sphere. (A great circle is the largest possible circle, i.e. a circle of diameter one.)

1.8.3 Describe the image of the circle of latitude 30°.

Ans. $|z| = \sqrt{3}$.

1.8.4 In this problem, the student will prove that the mapping effected by our stereographic projection from the plane to the sphere is circular, that is, it preserves circles.



In the figure, (x,y) refers to the point P in the plane, and (X,Y,Z) refers to its image P′ on the sphere. The length of the line segment QP′ is denoted by R, and the length of the line segment SP is denoted by r.

(a) Use similar triangles to show that $rR = Z$ and $r(1-Z) = R$.

(b) Use the results of (a) to show that

$$R = \frac{r}{1 + r^2} \quad \text{and} \quad Z = \frac{r^2}{1 + r^2}.$$

(c) Since $X = R\cos\theta$ and $Y = R\sin\theta$, and since $x = r\cos\theta$ and $y = r\sin\theta$, use (b) to show that

$$X = \frac{x}{1 + x^2 + y^2}, \quad Y = \frac{y}{1 + x^2 + y^2}, \quad Z = \frac{x^2 + y^2}{1 + x^2 + y^2}.$$

These equations give the coordinates of a point on the sphere when the coordinates (x,y) of the corresponding point on the plane are known.

(d) Use the results of (c) to show that

$$x^2 + y^2 = \frac{Z}{1 - Z}, \quad x = \frac{X}{1 - Z}, \quad y = \frac{Y}{1 - Z}.$$

These equations give the coordinates of a point on the plane when the coordinates (X,Y,Z) of the corresponding point on the sphere are known.

(e) The equation of a circle on the complex plane is

$A(x^2 + y^2) + Bx + Cy + D = 0$. Use the results of (d) to show that this circle maps to $AZ + BX + CY + D(1-Z) = 0$ which is the equation of a plane. This plane cuts the Riemann sphere to form a circle. Thus the stereographic projection preserves circles !

APPENDIX I

SOLUTIONS TO PROBLEMS

Problems from Chapter 1:

1/ (a) $z+w = 4+(\sqrt{3}-3)i$, (b) $z-w = 2-(\sqrt{3}+3)i$,

(c) $zw = 3+3\sqrt{3} + 3(\sqrt{3}-1)i$,

(d) $$\frac{3-3i}{1+\sqrt{3}i}\cdot\frac{1-\sqrt{3}i}{1-\sqrt{3}i} = \frac{3(1-\sqrt{3})-3(1+\sqrt{3})i}{1+3}$$

$$= \frac{3}{4}(1-\sqrt{3}) - \frac{3}{4}(1+\sqrt{3})i$$

2/ (a) $$\overline{z+w} = \overline{(a+c)+(b+d)i} = (a+c)-(b+d)i$$

$$\overline{z}+\overline{w} = a-bi + c-di = (a+c)-(b+d)i$$

THUS $\overline{z+w} = \overline{z}+\overline{w}$

(c) $$\overline{zw} = \overline{(a+bi)(c+di)} = \overline{(ac-bd)+(bc+ad)i}$$

$$= (ac-bd)-(ad+bc)i$$

$$\overline{z}\cdot\overline{w} = \overline{a+bi}\cdot\overline{c+di} = (a-bi)(c-di)$$

$$= (ac-bd)+(-ad-bc)i$$

THUS $\overline{zw} = \overline{z}\cdot\overline{w}$.

(d) $$\overline{\left(\frac{z}{w}\right)} = \overline{\left(\frac{a+bi}{c+di}\right)} = \overline{\frac{a+bi}{c+di}\cdot\frac{c-di}{c-di}} = \overline{\frac{(ac+bd)+(bc-ad)i}{c^2+d^2}}$$

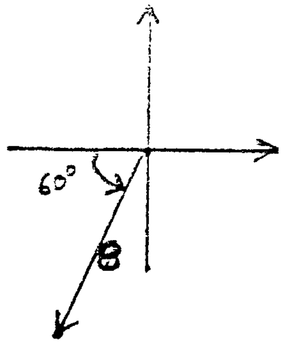
$$= \frac{ac+bd}{c^2+d^2} + \frac{ad-bc}{c^2+d^2}i, \quad \text{AND}$$

$$\frac{\overline{z}}{\overline{w}} = \frac{a-bi}{c-di} = \frac{a-bi}{c-di}\cdot\frac{c+di}{c+di} = \frac{(ac+bd)+(ad-bc)_{\cdot}}{c^2+d^2}$$

THUS $\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}$

3/ (a) -4, (b) $-4\sqrt{3}$, (c) $\sqrt{(-4)^2+(-4\sqrt{3})^2} = 8$.

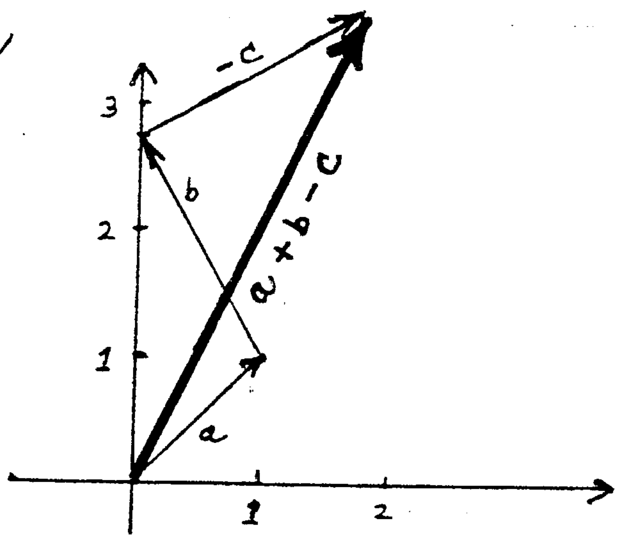
(d)



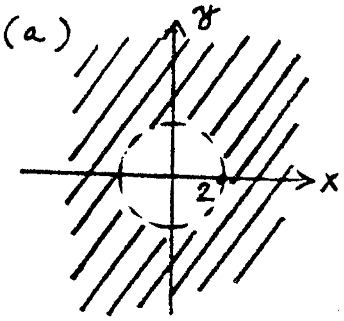
$\arg(-4-4\sqrt{3}i) = \frac{4\pi}{3} + 2\pi n,$

$n = 0, \pm 1, \pm 2, \cdots$.

4/

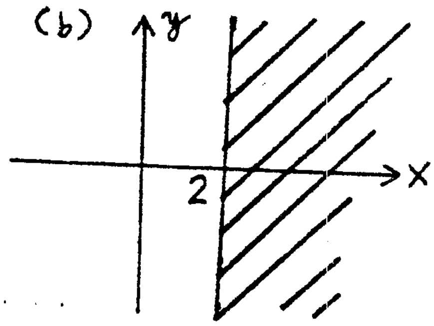


5/ (a)



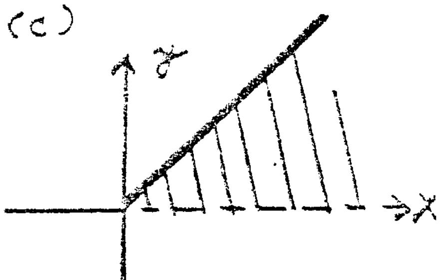
OPEN SET

(b)



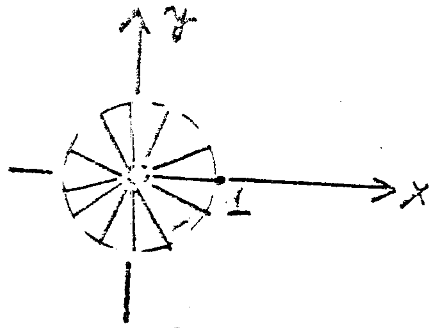
CLOSED SET

5/ (c)



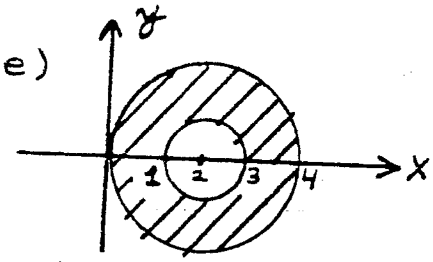
NOT OPEN OR CLOSED

(d)

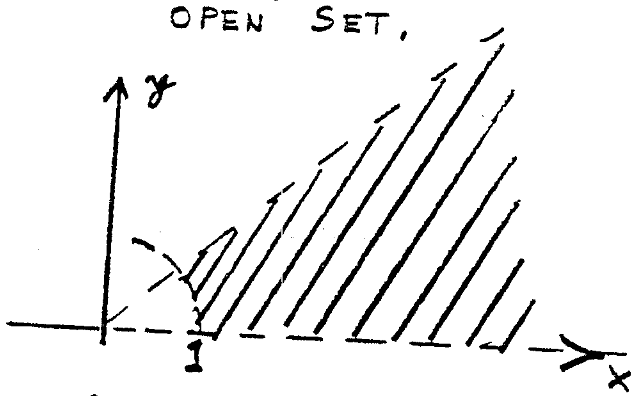


OPEN SET.

(e)



CLOSED SET

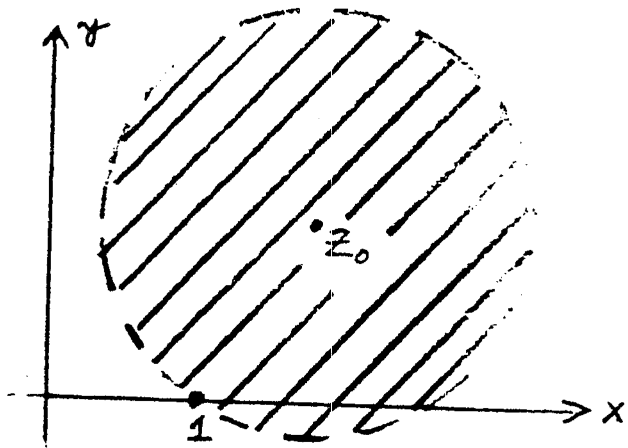


OPEN SET

6/ (a) $|z-1|>2$, open set, exterior of the circle centered at $z=1$ having radius 2.

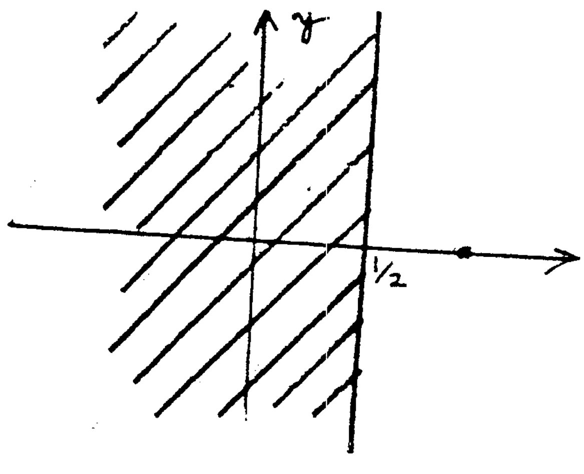
(b) $|z-z_0| < |1-z_0|$

Distance from z to z_0 is less than the distance from 1 to z_0



(c) $|z| \le |z-1|$

The distance from z to 0 is less than or equal to the distance from z to 1.



7/ (a) $5e^{i\frac{\pi}{2}}$, (b) $7\sqrt{2}\, e^{i\frac{3\pi}{4}}$, (c) $8e^{i\frac{4\pi}{3}}$, (d) $e^{i\pi}$

8/ (a) -2, (b) $-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i$, (c) $\frac{7\sqrt{3}}{2} - \frac{7}{2} i$,

(d) $-i$

9/ $z = 3\sqrt{2}\, e^{-i\frac{\pi}{4}}$ AND $w = 2e^{i\frac{\pi}{3}}$, THUS

$zw = 6\sqrt{2}\, e^{i\frac{\pi}{12}}$ AND $\frac{z}{w} = \frac{3}{\sqrt{2}} e^{-i\frac{7\pi}{12}}$.

10/ $(e^{i\theta})^2 = (\cos\theta + i\sin\theta)^2$

$$e^{i2\theta} = (\cos^2\theta - \sin^2\theta) + 2\sin\theta\cos\theta\, i.$$

SINCE $e^{i2\theta} = \cos 2\theta + i\sin 2\theta$ (EULER'S FORMULA), ON COMPARING THE LAST TWO EXPRESSIONS AND EQUATING REAL AND IMAGINARY PARTS WE GET

$$\cos 2\theta = \cos^2\theta - \sin^2\theta \quad \text{AND} \quad \sin 2\theta = 2\sin\theta\cos\theta$$

11/ $(e^{i\theta})^3 = (\cos\theta + i\sin\theta)^3$

$$e^{i3\theta} = \cos^3\theta + 3\cos^2\theta\, i\sin\theta + 3\cos\theta\, i^2\sin^2\theta + i^3\sin^3\theta$$

(1) $$e^{i3\theta} = (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta).$$

BUT

(2) $e^{i3\theta} = \cos 3\theta + i \sin 3\theta$.

COMPARING (1) AND (2) WE SEE THAT

$\cos 3\theta = \cos^3\theta - 3\cos\theta \sin^2\theta$ AND

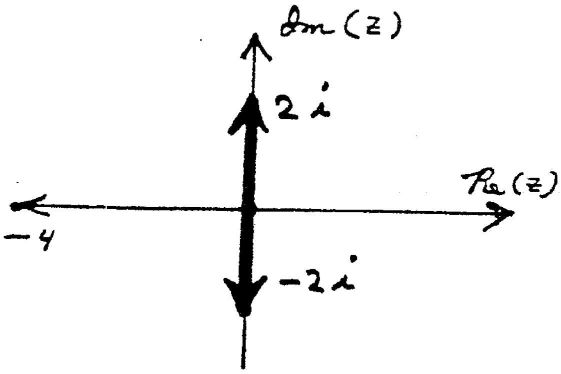
$\sin 3\theta = 3\cos^2\theta \sin\theta - \sin^3\theta$,

12/ $z^2 = -4$

$z^2 = 4e^{i(\pi + 2\pi n)}$, $n = 0, \pm 1, \ldots$

$z = 2e^{i(\frac{\pi}{2} + \pi n)}$

$z = 2i$ AND $-2i$

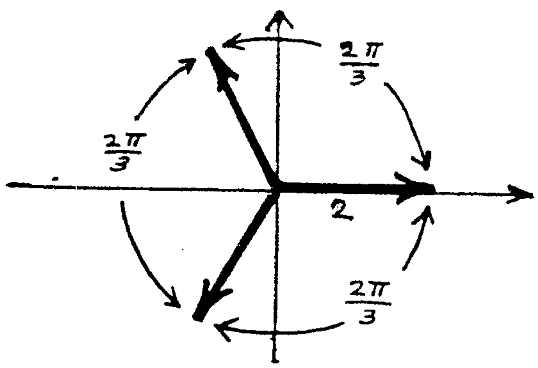


13/ $z^3 = 8$

$z^3 = 8e^{i2\pi n}$, $n = 0, \pm 1, \pm 2, \ldots$

$z = 2e^{i\frac{2\pi}{3}n}$

$z = 2,\ -1 + \sqrt{3}i$, AND $-1 - \sqrt{3}i$



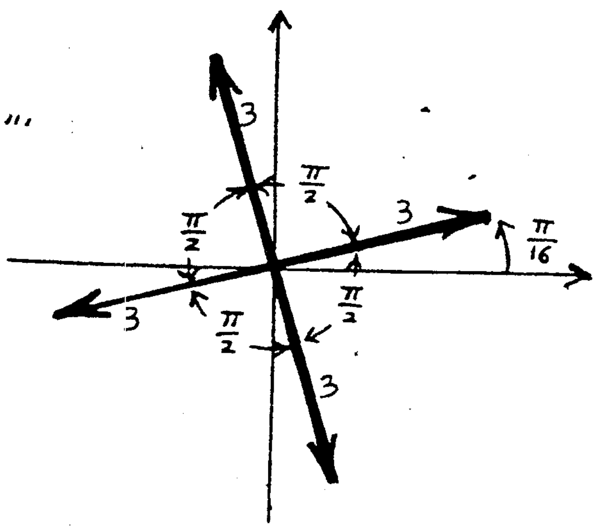
14/ $z^4 = \frac{81}{\sqrt{2}}(1 + i)$

$z^4 = 81e^{i(\frac{\pi}{4} + 2\pi n)}$ $n = 0, \pm 1, \ldots$

$z = 3e^{i(\frac{\pi}{16} + \frac{\pi n}{2})}$

$z = 3e^{i\frac{\pi}{16}},\ 3e^{i\frac{17\pi}{16}}$

$3e^{i\frac{9\pi}{16}},\ 3e^{i\frac{25\pi}{16}}$

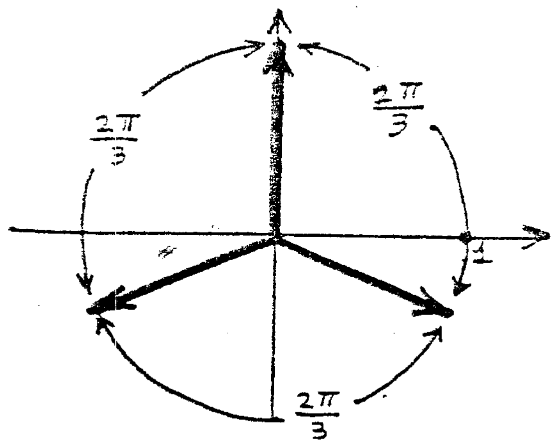


15/ $z^3 = -i$

$$z^3 = e^{\left(\frac{3\pi}{2} + 2\pi n\right) i}, \quad n = 0, \pm 1, \ldots$$

$$z = e^{\left(\frac{\pi}{2} + \frac{2\pi n}{3}\right) i}$$

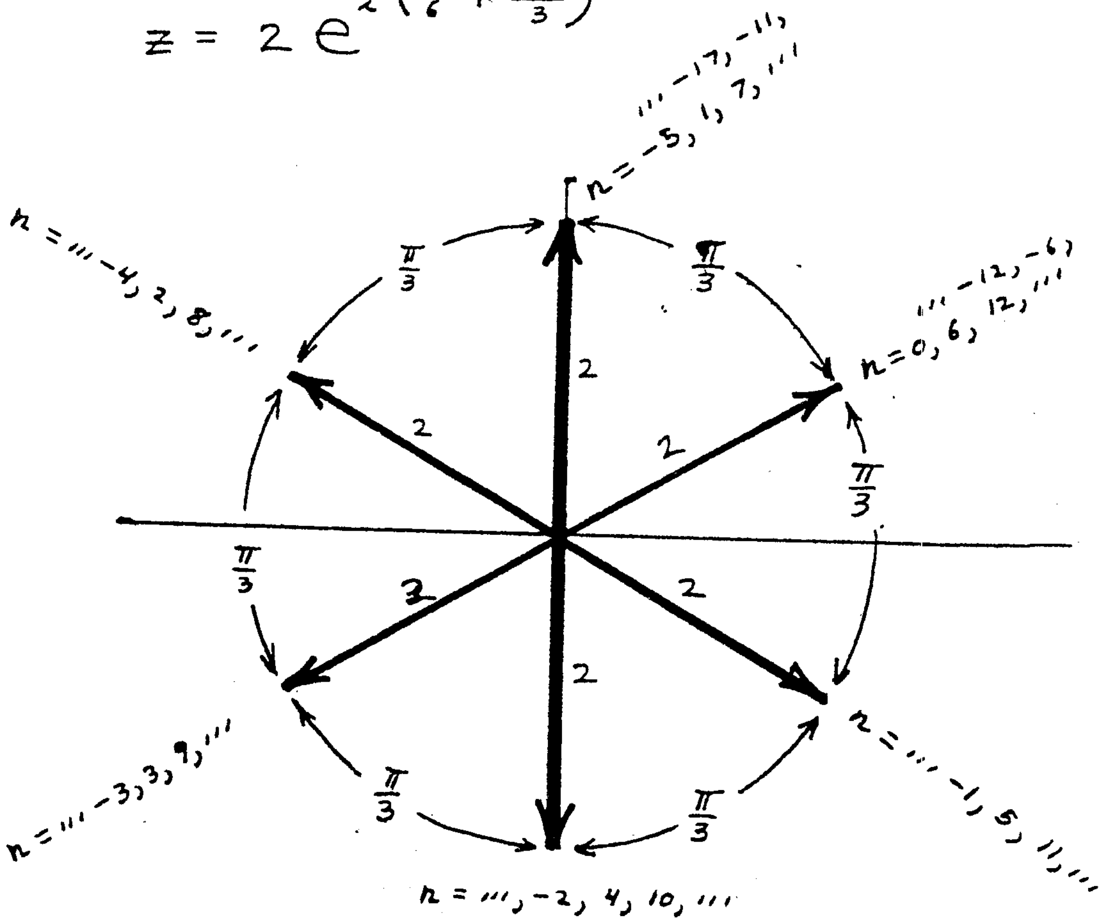
$$z = i,\ -\frac{\sqrt{3}}{2} - \frac{1}{2} i,\ \frac{\sqrt{3}}{2} - \frac{1}{2} i$$



16/ $z^6 = -64$

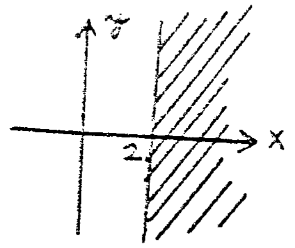
$$z^6 = 64\, e^{i(\pi + 2\pi n)}, \quad n = 0, \pm 1, \ldots$$

$$z = 2\, e^{i\left(\frac{\pi}{6} + \frac{\pi n}{3}\right)}$$



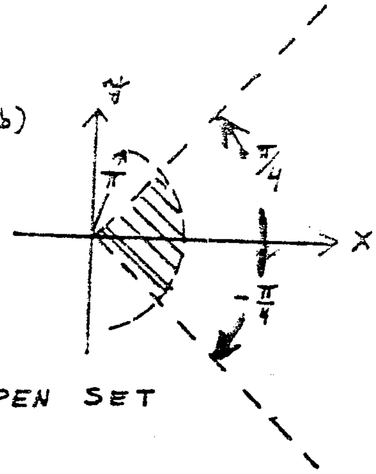
Solutions to Review Problems from Chapter 1

1 (a)



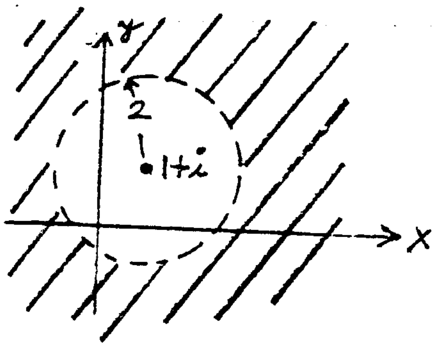
CLOSED SET

(b)



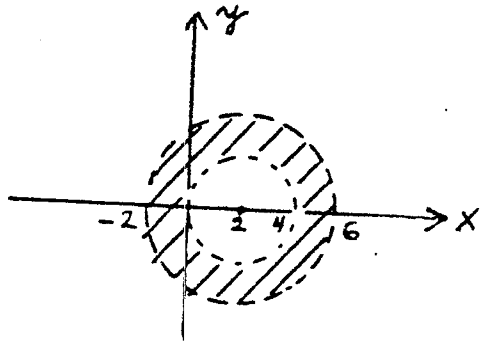
OPEN SET

(c) $2 < |z-(1+i)|$



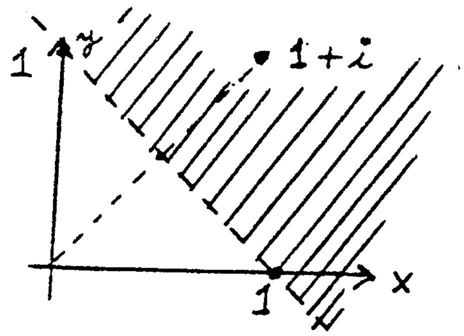
OPEN SET

(d)



OPEN SET

(e)



OPEN SET

2/ (a) $z = 2e^{i\frac{\pi}{3}}$, $w = 3e^{i\frac{\pi}{2}}$, $\zeta = 4e^{-i\frac{\pi}{6}}$.

WE CAN ALSO, IF WE WISH, ADD $2\pi n i$ TO EACH EXPONENT, WHERE $n = 0, \pm 1, \pm 2, \ldots$.

2/ (b) $|zw\zeta| = |z|\cdot|w|\cdot|\zeta| = 2\cdot 3\cdot 4 = 24$

(c) $\arg(zw\zeta) = \arg(z) + \arg(w) + \arg(\zeta)$

$$= \frac{\pi}{3} + \frac{\pi}{2} - \frac{\pi}{6} = \frac{2\pi}{3},$$

(d) $\arg(\overline{zw\zeta}) = -\arg(zw\zeta) = -\frac{2\pi}{3},$

3/ $e^{i\theta} = \cos\theta + i\sin\theta,\ e^{iw} = \cos w + i\sin w$

$$e^{i\theta}\cdot e^{iw} = (\cos\theta + i\sin\theta)(\cos w + i\sin w)$$

$$= (\cos\theta\cos w - \sin\theta\sin w) + i(\sin\theta\cos w + \cos\theta\sin w)$$

$$= \cos(\theta + w) + i\sin(\theta + w)$$

$$= e^{i(\theta + w)}$$

4/ $-81 = 81e^{i(\pi + 2\pi n)}$, WHERE $n = 0, \pm 1, \pm 2, \ldots$

$$z = (81e^{i(\pi+2\pi n)})^{\frac{1}{4}} = 81^{\frac{1}{4}} e^{i(\frac{\pi}{4} + \frac{\pi n}{2})}$$

$$= \begin{cases} 3e^{i\frac{\pi}{4}} \text{ (FOR } n=0) = 3\left(\frac{1+i}{\sqrt{2}}\right) = \frac{3}{\sqrt{2}} + \frac{3}{\sqrt{2}}i \\ 3e^{i(\frac{\pi}{4}+\frac{\pi}{2})} \text{ (FOR } n=1) = 3e^{i\frac{\pi}{4}}i = -\frac{3}{\sqrt{2}} + \frac{3}{\sqrt{2}}i \\ 3e^{i(\frac{\pi}{4}+\pi)} \text{ (FOR } n=2) = 3e^{i\frac{\pi}{4}}(-1) = -\frac{3}{\sqrt{2}} - \frac{3}{\sqrt{2}}i \\ 3e^{i(\frac{\pi}{4}+\frac{3\pi}{2})} \text{ (FOR } n=3) = 3e^{i\frac{\pi}{4}}(-i) = \frac{3}{\sqrt{2}} - \frac{3}{\sqrt{2}}i \end{cases}$$