APPENDIX I

SOLUTIONS TO PROBLEMS

Problems from Chapter 6

1/ (a) $\int z^5 + 2z + 3\, dz = \boxed{\frac{z^6}{6} + z^2 + 3z + c}$

(b) $\int z e^{2z} dz = \int z \, d\frac{e^{2z}}{2} =$ INTEGRATING BY PARTS

$$= z\frac{e^{2z}}{2} - \frac{1}{2}\int e^{2z} dz$$

$$= \boxed{\frac{z e^{2z}}{2} - \frac{e^{2z}}{4} + c}$$

(c) $\int \frac{dz}{1+z^2} = \boxed{\arctan z + c,}$

2/ (a) using the result of problem 1(a) we have

$$\left.\frac{z^6}{6} + z^2 + 3z \right|_0^{1+i} = \frac{(\sqrt{2}e^{i\frac{\pi}{4}})^6}{6} + (\sqrt{2}e^{i\frac{\pi}{4}})^2 + 3(1+i)$$

$$= \frac{8e^{i\frac{3\pi}{2}}}{6} + 2e^{i\frac{\pi}{2}} + 3 + 3i$$

$$= \frac{4}{3}(-i) + 2i + 3 + 3i = \boxed{3 + \frac{11}{3}i}$$

(b) using the result of problem 1(b) we have

$$\left.\frac{ze^{2z}}{2} - \frac{e^{2z}}{4}\right|_0^{1+i} = \frac{(1+i)e^{2+2i}}{2} - \frac{e^{2+2i}}{4} + \frac{1}{4}$$

$$= e^2(\cos 2 + i\sin 2)\left(\frac{1}{4} + \frac{i}{2}\right) + \frac{1}{4}$$

$$= \boxed{\frac{1}{4}e^2\cos 2 - \frac{1}{2}e^2\sin 2 + \frac{1}{4} + i\left[\frac{e^2}{2}\cos 2 + \frac{e^2}{4}\sin 2\right]}$$

3/ (a) $\displaystyle\int_{C_2} 2f(z) - 3g(z)\,dz = 2\int_{C_2} f(z) - 3\int_{C_2} g(z)\,dz$

$= 2(-i) - 3(2) = \boxed{-6-2i}$

(b) $\displaystyle\int_{1+i}^{1} 2f(z) - 3g(z)\,dz = -\int_{C_2} 2f(z) - 3g(z)\,dz$

$= -(-6-2i) = \boxed{6+2i}$

(c) $\displaystyle\int_{C} 2f(z) - 3g(z)\,dz = \int_{C_1} 2f(z) - 3g(z)\,dz + \int_{C_2} 2f(z) - 3g(z)\,dz$

$\displaystyle = 2\int_{C_1} f(z)\,dz - 3\int_{C_1} g(z)\,dz \quad -6-2i$

$= 2(3) - 3(2i) - 6 - 2i = \boxed{-8i}$

4/(a) Since $\cos z$ is analytic for all z, this integral is zero by Cauchy's Integral Theorem.

(b) The singularity is outside the contour, and thus the integral vanishes.

(c) The singularity is inside the contour and thus

$2\pi i \cos z\Big|_{z=0} = 2\pi i$ is the value of the integral.

(d) $\displaystyle 2\pi i \left.\frac{D^2 \cos z}{2!}\right|_{z=0} = \left.\frac{2\pi i(-\cos z)}{2}\right|_{z=0} = -\pi i$

4/(e) $$\oint_{|z-1|=1} \frac{(z+1)^{-1}\,dz}{z-1} = 2\pi i\,(z+1)^{-1}\Big|_{z=1} = \cancel{4}\pi i.$$

(f) $$2\pi i \left.\frac{D^N e^z}{N!}\right|_{z=0} = 2\pi i \left.\frac{e^z}{N!}\right|_{z=0} = \frac{2\pi i}{N!}$$

(g) $$2\pi i \left.\frac{D^N e^z}{N!}\right|_{z=2} = 2\pi i \left.\frac{e^z}{N!}\right|_{z=2} = \frac{2\pi i\, e^2}{N!}$$

(h) This integral has two simple poles inside the contour at $z=i$ and $z=-i$. The methods of this section do not apply to this problem. In the next section we will learn that

$$I = \oint_{|z|=2} \frac{z\,dz}{z^2+1} = \oint_{C_1} \frac{z(z+i)^{-1}}{z-i}\,dz + \oint_{C_2} \frac{z(z-i)^{-1}\,dz}{z+i}$$

where C_1 contains only the singularity at i,
and C_2 " " " " $-i$,

$$I = 2\pi i\, z(z+i)^{-1}\Big|_{z=i} + 2\pi i\, z(z-i)^{-1}\Big|_{z=-i}$$

$$= 2\pi i\left(\tfrac{1}{2}\right) + 2\pi i\left(\tfrac{1}{2}\right) = 2\pi i.$$

5/ $$\oint_{|z|=4} = \oint_{|z|=\frac{1}{2}} + \oint_{|z-1|=\frac{1}{2}}$$

5/ (continued)

$$= 2\pi i \left. \frac{\cos z}{z-1} \right|_{z=0} + 2\pi i \left. \frac{\cos z}{z} \right|_{z=1}$$

$$= -2\pi i + 2\pi i \cos 1 = 2\pi i \left(\cos(1) - 1\right),$$

6/

$$\oint_{|z|=2} = \oint_{|z|=\frac{1}{2}} + \oint_{|z-1|=\frac{1}{2}}$$

$$= 2\pi i \left. \frac{\frac{d}{dz}\left(\frac{1}{z-1}\right)}{1!} \right|_{z=0} + 2\pi i \left. \frac{1}{z^2} \right|_{z=1}$$

$$= 2\pi i \left. \left(\frac{-1}{(z-1)^2}\right) \right|_{z=0} + 2\pi i = 0$$

7/

$$\oint_{|z|=2} \frac{dz}{z(z+3)(z-1)} = \oint_{|z|=\frac{1}{2}} + \oint_{|z+3|=\frac{1}{2}} + \oint_{|z-1|=\frac{1}{2}}$$

$$= 2\pi i \left. \left(\frac{1}{(z+3)(z-1)}\right) \right|_{z=0} + 2\pi i \left. \frac{1}{z(z-1)} \right|_{z=-3} + 2\pi i \left. \frac{1}{z(z+3)} \right|_{z=1}$$

$$= 2\pi i \left(-\frac{1}{3}\right) + 2\pi i \frac{1}{12} + 2\pi i \frac{1}{4}$$

$$= 2\pi i \frac{0}{12} = 0$$

8/ $\dfrac{z}{z^2+1} = \dfrac{z}{(z+i)(z-i)}$

$$\mathrm{Res}\left(\frac{z}{z^2+1}, -i\right) = \left.\frac{z}{z-i}\right|_{z=-i} = \frac{1}{2}$$

$$\mathrm{Res}\left(\frac{z}{z^2+1}, i\right) = \left.\frac{z}{z+i}\right|_{z=i} = \frac{1}{2}$$

$$\oint_{|z|=2} \frac{z}{z^2+1}\, dz = 2\pi i \text{ (sum of residues)}$$

$$= 2\pi i \left(\frac{1}{2}+\frac{1}{2}\right) = 2\pi i$$

9/ $\mathrm{Res}(\csc z, \pi) = -1$ from Example 2,

$$\mathrm{Res}(\csc z, 0) = \lim_{z\to 0} \frac{z}{\sin z}$$

$$= \lim_{z\to 0} \frac{\frac{d}{dz}(z)}{\frac{d}{dz}\sin z} = \left.\frac{1}{\cos z}\right|_{z=0} = 1$$

$$\mathrm{Res}(\csc z, -\pi) = \lim_{z\to -\pi} \frac{z+\pi}{\sin z}$$

$$= \lim_{z\to -\pi} \frac{1}{\cos z} = \frac{1}{-1} = -1,$$

$$\oint_{|z|=4} \csc z\, dz = 2\pi i\left[\mathrm{Res}(-\pi) + \mathrm{Res}(0) + \mathrm{Res}(\pi)\right]$$

$$= 2\pi i\left[-1 + 1 - 1\right] = -2\pi i$$

10/ $$\frac{1}{z^3-1} = \frac{1}{(z-1)(z-e^{2\pi i/3})(z-e^{-2\pi i/3})}$$

$$\text{Res}\left(\frac{1}{z^3-1}, 1\right) = \lim_{z\to 1} \frac{z-1}{z^3-1}$$

$$= \lim_{z\to 1} \frac{1}{3z^2} = \boxed{\frac{1}{3}}$$

$$\text{Res}\left(\frac{1}{z^3-1}, e^{\frac{2\pi i}{3}}\right) = \lim_{z\to e^{\frac{2\pi i}{3}}} \frac{z-e^{\frac{2\pi i}{3}}}{z^3-1}$$

$$= \lim_{z\to e^{\frac{2\pi i}{3}}} \frac{1}{3z^2}$$

$$= \frac{1}{3e^{4\pi i/3}} = \boxed{\frac{e^{2\pi i/3}}{3}}$$

$$\text{Res}\left(\frac{1}{z^3-1}, e^{-\frac{2\pi i}{3}}\right) = \lim_{z\to e^{-\frac{2\pi i}{3}}} \frac{z-e^{-\frac{2\pi i}{3}}}{z^3-1}$$

$$= \lim_{z\to e^{-\frac{2\pi i}{3}}} \frac{1}{3z^2}$$

$$= \lim_{z\to e^{-\frac{2\pi i}{3}}} \frac{1}{3e^{-\frac{4\pi i}{3}}} = \boxed{\frac{e^{-\frac{2\pi i}{3}}}{3}}$$

$$\oint_{|z|=2} \frac{dz}{z^3-1} = 2\pi i\left[\frac{1}{3} + \frac{1}{3}e^{2\pi i/3} + \frac{1}{3}e^{-\frac{2\pi i}{3}}\right]$$

$$= 0$$

11/ $$\operatorname{Res}\left(\frac{e^z}{z^2(z-2)^3}, 0\right) = \left.\frac{\frac{d}{dz}\left(\frac{e^z}{(z-2)^3}\right)}{1!}\right|_{z=0}$$

$$= \left.\frac{(z-2)^3 e^z - e^z 3(z-2)^2}{(z-2)^6}\right|_{z=0} = \frac{e^0(-8-12)}{2^6}$$

$$\cancel{= -\frac{1}{2^5} = -\frac{1}{32}} = -\frac{5}{16}$$

Since only the singularity at $z=0$ *is inside* $|z|=1$ *we get*

$$\oint_{|z|=1} \frac{e^z}{z^2(z-2)^3}\, dz = 2\pi i \operatorname{Res}(0) = 2\pi i \left(\overset{-\frac{5}{16}}{\cancel{-\frac{1}{32}}}\right)$$

$$= \cancel{-\frac{\pi i}{16}} - \frac{5\pi i}{8}$$

12/ $$\operatorname{Res}\left(\frac{e^z}{z^2(z-2)^3}, 2\right) = \left.\frac{\frac{d^2}{dz^2}\left(\frac{e^z}{z^2}\right)}{2!}\right|_{z=2} =$$

$$= \frac{1}{2}\frac{d}{dz}\left(\frac{e^z(z-2)}{z^3}\right)\Big|_{z=2} = \frac{1}{2}\left[\frac{e^z(z^2-4z+6)}{z^4}\right]\Big|_{z=2}$$

$$= \frac{1}{2}\left[\frac{e^2(2)}{16}\right] = \frac{e^2}{16}$$

$$\oint \frac{e^z}{z^2(z-2)^3}\, dz = 2\pi i\left[\operatorname{Res}(0) + \operatorname{Res}(2)\right]$$

$$= 2\pi i\left[\overset{-\frac{5}{16}}{\cancel{-\frac{1}{32}}} + \frac{e^2}{16}\right]$$

$$= \cancel{\frac{\pi i}{16}\left[2e^2 - 1\right]} \qquad = \frac{\pi i}{8}\left[e^2 - 5\right]$$

13/ $\sin \frac{1}{z} = \frac{1}{z} - \frac{1}{6 z^3} + \cdots$

Thus $\text{Res}\left(\sin \frac{1}{z}, 0\right) = 1$,

$$\oint_{|z|=4} \sin \frac{1}{z} \, dz = 2\pi i \, \text{Res}(0) = 2\pi i$$

14/ Multiply the series for $\sin \frac{1}{z}$ by itself and get

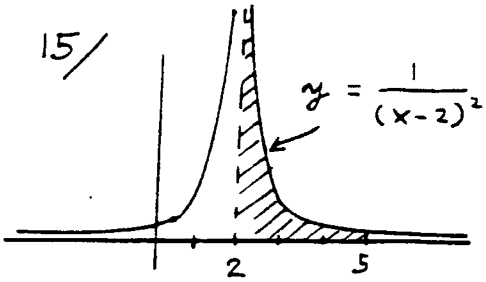
$$\begin{array}{rcl} \sin \frac{1}{z} & = & \frac{1}{z} - \frac{1}{6z^3} + \cdots \\ \sin \frac{1}{z} & = & \frac{1}{z} - \frac{1}{6z^3} + \cdots \\ \hline & & \frac{1}{z^2} - \frac{1}{6z^4} + \cdots \\ & & \quad - \frac{1}{6z^4} + \cdots \\ & & \qquad \cdots \\ \hline \sin^2 \frac{1}{z} & = & \frac{1}{z^2} - \frac{1}{3z^4} + \cdots \end{array}$$

Since there is no term involving $\frac{1}{z}$ in this last series, $\text{Res}\left(\sin^2 \frac{1}{z}, 0\right) = 0$.

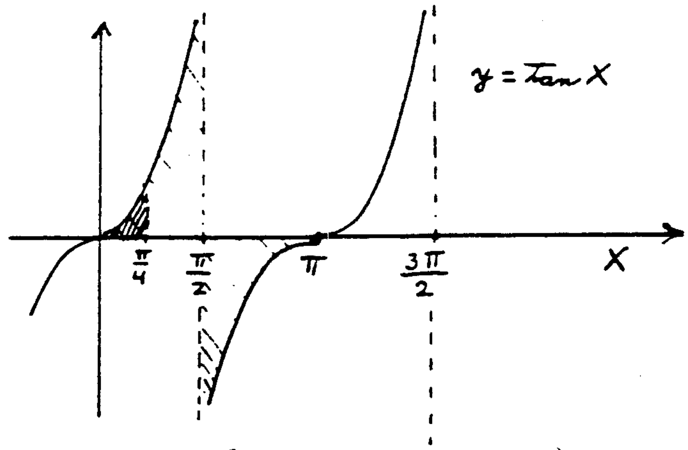
Thus

$$\oint_{|z|=4} \sin^2\left(\frac{1}{z}\right) dz = 0.$$

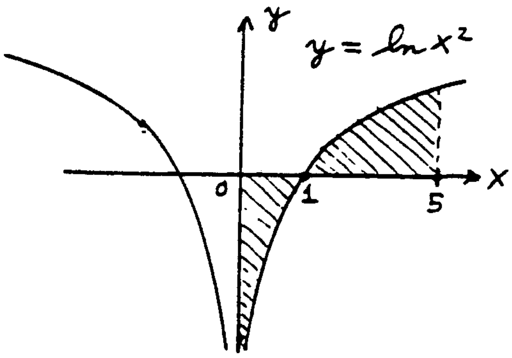
15/



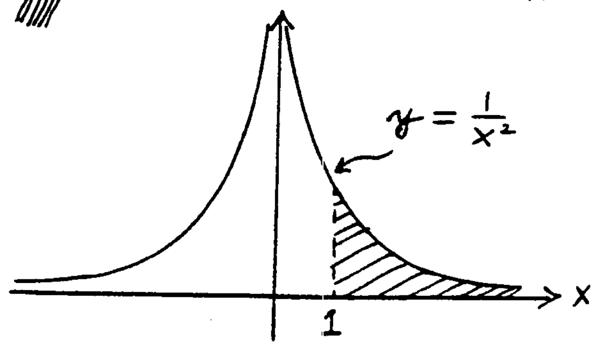
(a) IMPROPER



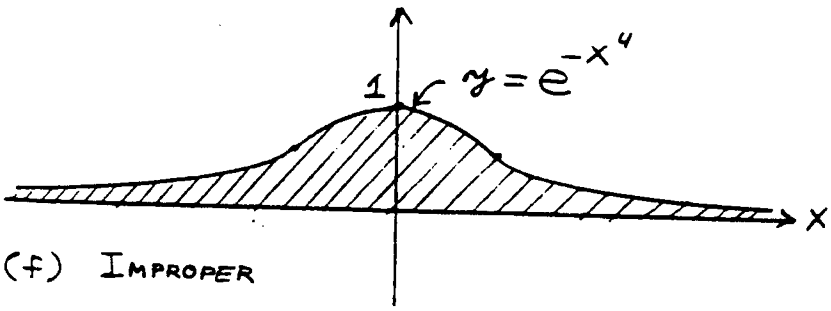
(b) PROPER, (c) IMPROPER



(d) IMPROPER



(e) IMPROPER



(f) IMPROPER

16/ (a) CONVERGES BECAUSE $P = -5 < -1$.

(b) DIVERGES BECAUSE $\frac{x}{5+x^2} \approx \frac{1}{x}$ FOR LARGE x.

(c) LET $u = x-2$, THEN WE GET $\int_0^3 u^{-\frac{1}{2}}\,du$ WHICH CONVERGES BECAUSE $P = -\frac{1}{2} > -1$.

(d) CONVERGES BECAUSE $\left|\frac{\sin x}{3+x^3}\right| < \frac{1}{x^3}$.

16/ (continued)

(e) LET $u = x-1$ AND GET $\int_{-1}^{0} \frac{du}{u}$ WHICH DIVERGES,

(f) NEAR $x=0$, $\sin x \approx x$. THUS $\frac{1}{\sin x} \approx \frac{1}{x}$ NEAR $x=0$,
THEREFORE THE INTEGRAL DIVERGES.

17/ (a) SINCE $\cos x$ VARIES BETWEEN -1 and $+1$, $e^{\cos x}$ VARIES BETWEEN e^{-1} AND e^{1}. THUS

$\frac{e^{\cos x}}{x} \geq \frac{e^{-1}}{x}$ AND SINCE $\int_{2}^{\infty} \frac{e^{-1}}{x}\,dx$ DIVERGES,

OUR INTEGRAL ALSO DIVERGES.

(b) e^{-2t} DOMINATES, OUR INTEGRAL CONVERGES SINCE $\int_0^\infty e^{-2t}\,dt$ CONVERGES,

(c) e^{-x} DOMINATES AND THE INTEGRAL CONVERGES.

(d) FOR SMALL x, $\frac{1}{\sqrt{x+x^3}} = \frac{1}{\sqrt{x}\sqrt{1+x^2}} \approx \frac{1}{\sqrt{x}}$

THUS NEAR $x=0$, OUR INTEGRAND BEHAVES LIKE $x^{-\frac{1}{2}}$

AND THE AREA IS FINITE HERE.

FOR LARGE x, $\frac{1}{\sqrt{x+x^3}} \approx \frac{1}{\sqrt{x^3}} = x^{-\frac{3}{2}}$

THUS THE AREA IS FINITE FOR LARGE x ALSO,

THUS THE INTEGRAL CONVERGES.

(e) SET $4-x=u$ AND GET $\int_0^4 u^{-\frac{1}{2}}\,du$ WHICH CONVERGES.

(f) $\frac{1}{e^x+6} \approx e^{-x}$ FOR LARGE x. THUS IT CONVERGES.

17/ (continued)

(g) x^x DOMINATES AND THUS $\dfrac{e^{2x}}{x^x+8} \approx x^{-x}$ FOR LARGE x. THE INTEGRAL <u>CONVERGES</u>.

(h) FOR LARGE t, $\dfrac{1}{(x^3+t^3)^{5/3}} \approx t^{-5}$

THUS IT <u>CONVERGES</u>.

(i) FOR LARGE x, $\ln x > e$, THUS $(\ln x)^x > e^x$.

THEREFORE $\dfrac{x^4}{(\ln x)^x} < \dfrac{x^4}{e^x}$. THE INTEGRAL <u>CONVERGES</u>.

(j) $\tan x = \dfrac{\sin x}{\cos x}$. NEAR $x = \frac{\pi}{2}$, $\sin\frac{\pi}{2} = 1$ AND $\cos x \approx \frac{\pi}{2} - x$. THUS NEAR $x = \frac{\pi}{2}$, $\tan x \approx \dfrac{1}{\frac{\pi}{2}-x}$.

THE INTEGRAL THEREFORE <u>DIVERGES</u>.

18/ The solution here is similar to the solution of Example 1, only now the integrand

$$\frac{1}{a^2+z^2} = \frac{1}{(z+ia)(z-ia)}$$

has a singularity at $z = ia$ inside the contour. Thus

$$\int_{-\infty}^{\infty} \frac{dx}{a^2+x^2} = \int \frac{dz}{a^2+z^2} = 2\pi i \operatorname{Res}\left(\frac{1}{a^2+z^2},\ ia\right)$$

$$= 2\pi i \frac{1}{ia+ia} = \frac{2\pi i}{2ia} = \boxed{\frac{\pi}{a}}.$$

19/ The solution is similar to the solution of Example 2, only now the integrand

$\dfrac{1}{a^4+z^4}$ has singularities at

$z = a e^{i\frac{\pi}{4}}$ and $z = a e^{\frac{3\pi i}{4}}$. Thus

$$\int_{-\infty}^{\infty} \frac{dx}{a^4+x^4} = 2\pi i \left[\operatorname{Res}\left(\frac{1}{a^4+z^4}, a e^{i\frac{\pi}{4}}\right) + \operatorname{Res}\left(a e^{\frac{i3\pi}{4}}\right)\right].$$

$$\operatorname{Res}\left(a e^{i\frac{\pi}{4}}\right) = \lim_{z \to a e^{i\frac{\pi}{4}}} \frac{z - a e^{i\frac{\pi}{4}}}{a^4+z^4}$$

$$= \left.\frac{1}{4z^3}\right|_{z = a e^{i\frac{\pi}{4}}} = \frac{e^{-i\frac{3\pi}{4}}}{4a^3}$$

$$\operatorname{Res}\left(a e^{\frac{3\pi i}{4}}\right) = \left.\frac{1}{4z^3}\right|_{z = a e^{\frac{3\pi i}{4}}} = \frac{e^{-\frac{i\pi}{4}}}{4a^3}$$

$$\int_{-\infty}^{\infty} \frac{dx}{a^4+x^4} = 2\pi i \left[\frac{e^{-i\frac{3\pi}{4}} + e^{-\frac{i\pi}{4}}}{4a^3}\right]$$

$$= \frac{2\pi i}{4a^3}\left[-\sqrt{2}\, i\right] = \boxed{\frac{\pi}{\sqrt{2}\, a^3}}.$$

20/ We use the same contour as before. The integrand $\dfrac{1}{(1+z^2)^2} = \dfrac{1}{(z-i)^2(z+i)^2}$ has a pole of order two at $z = i$. Thus

20/ (continued)

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = \oint \frac{dz}{(1+z^2)^2} = 2\pi i \, \mathrm{Res}\left(\frac{1}{(1+z^2)^2}, i\right)$$

$$= 2\pi i \left\{ \frac{d}{dz} (z+i)^{-2} \right\}\Bigg|_{z=i} = 2\pi i \left\{ -2(z+i)^{-3} \right\}\Bigg|_{z=i}$$

$$= \frac{2\pi i(-2)}{(2i)^3} = \frac{-4\pi i}{-8i} = \frac{\pi}{2},$$

Since $\int_0^{\infty} = \frac{1}{2}\int_{-\infty}^{\infty} = \frac{1}{2}\left(\frac{\pi}{2}\right) = \boxed{\frac{\pi}{4}}$.

21/ Use the contour as before. The integrand

$\dfrac{z^2}{(1+z^2)^2} = \dfrac{z^2}{(z-i)^2(z+i)^2}$ has a singularity of order two at $z=i$ inside. Thus

$$\int_{-\infty}^{\infty} \frac{x^2\,dx}{(1+x^2)^2} = 2\pi i \, \mathrm{Res}\left(\frac{z^2}{(1+z^2)^2}, i\right)$$

$$= 2\pi i \left\{ \frac{d}{dz}\left[\frac{z^2}{(z+i)^2}\right] \right\}\Bigg|_{z=i}$$

$$= 2\pi i \left\{ -2(z+i)^{-3} z^2 + 2z(z+i)^{-2} \right\}\Bigg|_{z=i}$$

$$= 2\pi i \left\{ \frac{2}{-8i} + \frac{2i}{-4} \right\} = 2\pi i \left\{ \frac{i}{4} - \frac{i}{2} \right\}$$

$= \frac{\pi}{2}$. Thus $\int_0^{\infty} = \frac{1}{2}\int_{-\infty}^{\infty} = \boxed{\frac{\pi}{4}}$.

22/ As in Example 3 we have

$$I = \int_{-\infty}^{\infty} \frac{\cos x \, dx}{1+x^4} = \lim_{R\to\infty} \left\{ \frac{1}{2} \int \frac{e^{iz}\,dz}{1+z^4} + \frac{1}{2} \int \frac{e^{-iz}\,dz}{1+z^4} \right\}$$

But $z^4+1 = (z-e^{i\frac{\pi}{4}})(z-e^{i\frac{3\pi}{4}})(z-e^{-i\frac{\pi}{4}})(z-e^{-\frac{3\pi i}{4}})$.

Thus

$$I = 2\pi i \left\{ \frac{1}{2} \operatorname{Res}\left(\frac{e^{iz}}{1+z^4}, e^{i\frac{\pi}{4}}\right) + \frac{1}{2} \operatorname{Res}\left(\frac{e^{iz}}{1+z^4}, e^{i\frac{3\pi}{4}}\right) \right\}$$

$$-2\pi i \left\{ \frac{1}{2} \operatorname{Res}\left(\frac{e^{-iz}}{1+z^4}, e^{-i\frac{\pi}{4}}\right) + \frac{1}{2} \operatorname{Res}\left(\frac{e^{-iz}}{1+z^4}, e^{-i\frac{3\pi}{4}}\right) \right\}$$

Note that the minus sign is used because is in the negative sense.

To find the residue of $\dfrac{e^{\pm iz}}{1+z^4}$ at z_0 we use

$$\lim_{z\to z_0} \frac{(z-z_0)e^{\pm iz}}{1+z^4} = \left. \frac{e^{\pm iz} \pm i(z-z_0)e^{\pm iz}}{4z^3} \right|_{z=z_0}$$

$$= \frac{e^{\pm i z_0}}{4z_0^3}$$

Thus we have

$$I = \frac{\pi i}{4} \left\{ e^{-\frac{3\pi i}{4}} e^{i e^{i\frac{\pi}{4}}} + e^{-\frac{\pi i}{4}} e^{i e^{i\frac{3\pi}{4}}} - e^{\frac{3\pi i}{4}} e^{-i e^{-i\frac{\pi}{4}}} - e^{\frac{\pi i}{4}} e^{-i e^{-i\frac{3\pi}{4}}} \right\}$$

22/ (continued)

This last expression is of the form

$$I = \frac{\pi i}{4}\left\{ a + b - \bar{a} - \bar{b} \right\}$$

where the bar denotes "complex conjugate".

Recall that $a - \bar{a} = 2i\,\mathrm{Im}(a)$. Thus

$$I = \frac{\pi i}{4}\left\{ 2i\,\mathrm{Im}\left(e^{-\frac{3\pi i}{4}} e^{ie^{i\frac{\pi}{4}}}\right) + 2i\,\mathrm{Im}\left(e^{-\frac{\pi i}{4}} e^{ie^{i\frac{3\pi}{4}}}\right)\right\}$$

$$I = \frac{\pi i}{4}\left\{ 2i\,\mathrm{Im}\left(\left(\frac{-1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) e^{i\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)}\right) + 2i\,\mathrm{Im}\left(\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) e^{i\left(-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)}\right)\right\}$$

$$I = \frac{-\pi}{2}\left\{ \mathrm{Im}\left(\left(-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) e^{-\frac{1}{\sqrt{2}}}\left(\cos\frac{1}{\sqrt{2}} + i\sin\frac{1}{\sqrt{2}}\right)\right) + \mathrm{Im}\left(\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) e^{-\frac{1}{\sqrt{2}}}\left(\cos\frac{1}{\sqrt{2}} - i\sin\frac{1}{\sqrt{2}}\right)\right)\right.$$

$$I = -\frac{\pi}{2}\,\mathrm{Im}\left(-\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\left(\cos\frac{1}{\sqrt{2}} + i\sin\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)\left(\cos\frac{1}{\sqrt{2}} - i\sin\frac{1}{\sqrt{2}}\right)\right) e^{-\frac{1}{\sqrt{2}}}$$

This last expression is of the form

$$I = -\frac{\pi}{2}\,\mathrm{Im}\left(-b + \bar{b}\right) e^{-\frac{1}{\sqrt{2}}}$$ which simplifies

to

22/ (continued)

$$I = +\frac{\pi}{2} e^{-\frac{1}{\sqrt{2}}} \operatorname{Im}\left(b - \bar{b}\right) = \frac{\pi}{2} e^{-\frac{1}{\sqrt{2}}} \operatorname{Im}\left(2i \operatorname{Im}(b)\right)$$

$$= \frac{\pi}{2} e^{-\frac{1}{\sqrt{2}}}\, 2 \operatorname{Im}(b)$$

$$= \pi e^{-\frac{1}{\sqrt{2}}} \operatorname{Im}\left(\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\left(\cos\frac{1}{\sqrt{2}} + i \sin\frac{1}{\sqrt{2}}\right)\right)$$

$$= \pi e^{-\frac{1}{\sqrt{2}}} \frac{1}{\sqrt{2}} \left(\cos\frac{1}{\sqrt{2}} + \sin\frac{1}{\sqrt{2}}\right) = \boxed{\pi e^{-\frac{1}{\sqrt{2}}} \sin\left(\frac{1}{\sqrt{2}} + \frac{\pi}{4}\right)}$$

23/ $$I = \int_0^\infty \frac{\cos x}{(1+x^2)^2}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos x}{(1+x^2)^2}\,dx$$

This problem is similar to Example 3, only now the poles at $z = \pm i$ are of order two.

$$I = \frac{1}{2}\left\{ \frac{1}{2}\int_{\circlearrowleft} \frac{e^{iz}\,dz}{(1+z^2)^2} + \frac{1}{2}\int_{\circlearrowright} \frac{e^{-iz}\,dz}{(1+z^2)^2} \right\}$$

Since $(1+z^2)^2 = (z-i)^2(z+i)^2$ we have

$$I = \frac{1}{4}\, 2\pi i \left\{ \operatorname{Res}\left(\frac{e^{iz}}{(1+z^2)^2}, i\right) - \operatorname{Res}\left(\frac{e^{-iz}}{(1+z^2)^2}, -i\right) \right\}$$

$$I = \frac{\pi i}{2}\left\{ \frac{d}{dz}\left[\frac{e^{iz}}{(z+i)^2}\right]\Bigg|_{z=i} - \frac{d}{dz}\left[\frac{e^{-iz}}{(z-i)^2}\right]\Bigg|_{z=-i} \right\}$$