6.7 The evaluation of real definite integrals

We will now apply our knowledge of contour integration to the evaluation of certain real improper integrals of the form

$$\int_{-\infty}^{\infty} f(x)\,dx \qquad \text{and} \qquad \int_{0}^{\infty} f(x)\,dx \quad .$$

The methods used are best illustrated through examples.

Example 1

Evaluate the integral $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1 + x^2}$.

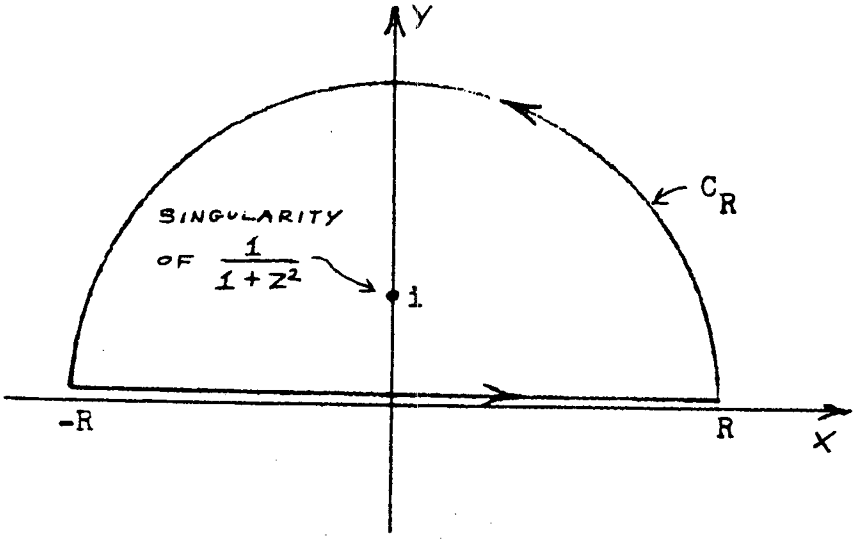
Solution

Before presenting a complex variable solution, we observe that this example is easily solved by the methods of the elementary calculus.

$$\int_{-\infty}^{\infty} \frac{dx}{1 + x^2} = \tan^{-1}x \Big|_{-\infty}^{\infty} = \frac{\pi}{2} - \left(- \frac{\pi}{2} \right) = \pi \quad .$$

We now replace x by the complex variable z and consider integrating over the closed contour $C_R = \longrightarrow + \frown$ as shown..

$$\oint_{C_R} \frac{dz}{1 + z^2}$$



Our integral can be written as the sum of two integrals, one over the portion of the real axis from $-R$ to R, and the other over the semi-circle of radius R . Thus

$$(1)\quad \oint_{C_R} \frac{dz}{1+z^2} = \int_{-R}^{R} \frac{dx}{1+x^2} + \int_{\frown} \frac{dz}{1+z^2} \; .$$

The first integral on the right hand side has x rather than z because the integral is over real variables. We can easily evaluate the left hand side of (1) by means of the Residue Theorem. If we let R tend to infinity, the first integral on the right hand side becomes the integral we wish to evaluate. As R tends to infinity, the second integral on the right will tend to zero because the term z^2 in the denominator grows large so fast. Let us examine this more closely:

$$\left| \int_{\frown} \frac{dz}{1+z^2} \right| \le \int_{\frown} \frac{|dz|}{|1+z^2|} \qquad \text{(Sec. 6.1 , (5)) ,}$$

On the semi-circle we have $z = R\,e^{i\theta}$, where $0 \le \theta \le \pi$, and therefore $dz = i\,R\,e^{i\theta}d\theta$. Substituting these into the right hand side of the last inequality we get

$$\left| \int_{\frown} \frac{dz}{1+z^2} \right| \le \int_{0}^{\pi} \frac{R\,d\theta}{\left|1 + R^2\,e^{i2\theta}\right|} \; .$$

Since $\left|1 + R^2 e^{i2\theta}\right| \geq R^2 - 1$ for all θ and $R > 1$ we have

$$\left|\int_{\frown} \frac{dz}{1 + z^2}\right| \leq \int_0^{\pi} \frac{R\,d\theta}{R^2 - 1} = \frac{R\pi}{R^2 - 1} .$$

Since this last expression approaches zero as R grows large, we see that the integral over the semi-circle tends to vanish as the radius grows larger and larger.

From (1) we now have

$$(2) \quad \oint_{C_R} \frac{dz}{1 + z^2} = \int_{-\infty}^{\infty} \frac{dx}{1 + x^2}$$

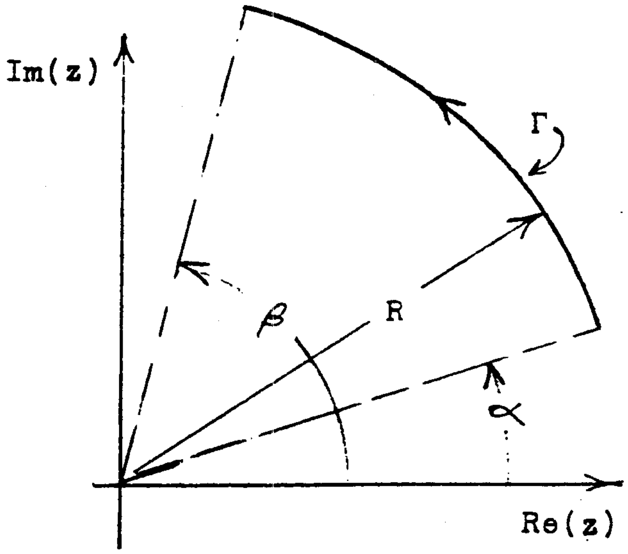
$$= 2\pi i \operatorname{Res}\left(\frac{1}{1 + z^2} , i \right)$$

$$= 2\pi i \left[\frac{1}{2i} \right]$$

$$= \pi .$$

It is interesting to note that the original integral $\int_{-\infty}^{\infty} \frac{dx}{1 + x^2}$ seems to have nothing to do with $i = \sqrt{-1}$. Yet the value of this integral was determined from the residue of $(1+z^2)^{-1}$ at $z = i$. How is it that an integral of a function spread over the real axis from negative to positive infinity can be determined by a residue at a single point ? Recall from page 4.40 that an analytic function is determined everywhere by its values near a single point. This result resembles Gauss's Theorem in electrostatics.

In the problems we shall consider, we will encounter integrals over the arc of a circle of radius R from $\theta = \alpha$ to $\theta = \beta$ as shown. by the arc Γ. The following theorem tells us when this integral vanishes as R approaches infinity.



Theorem 1

Let $f(z)$ satisfy the inequality

$$(3) \qquad \left| z^{p+1} f(z) \right| \leq C \quad , \quad p > 0 \, ,$$

where C is a constant (independent of R) on all arcs Γ of sufficiently large radius R. Then

$$\lim_{R \to \infty} \int_\Gamma f(z)\, dz = 0 \, .$$

Proof

Using the inequality for integrals and (3) we have

$$\left| \int_\Gamma f(z)\, dz \right| \leq \int_\Gamma |f(z)|\, |dz| = \int_\alpha^\beta |f(z)|\, R\, d\theta \leq \int_\alpha^\beta \frac{CR\, d\theta}{R^{p+1}} = \frac{C(\beta-\alpha)}{R^p}$$

Thus the limit follows.

Remark

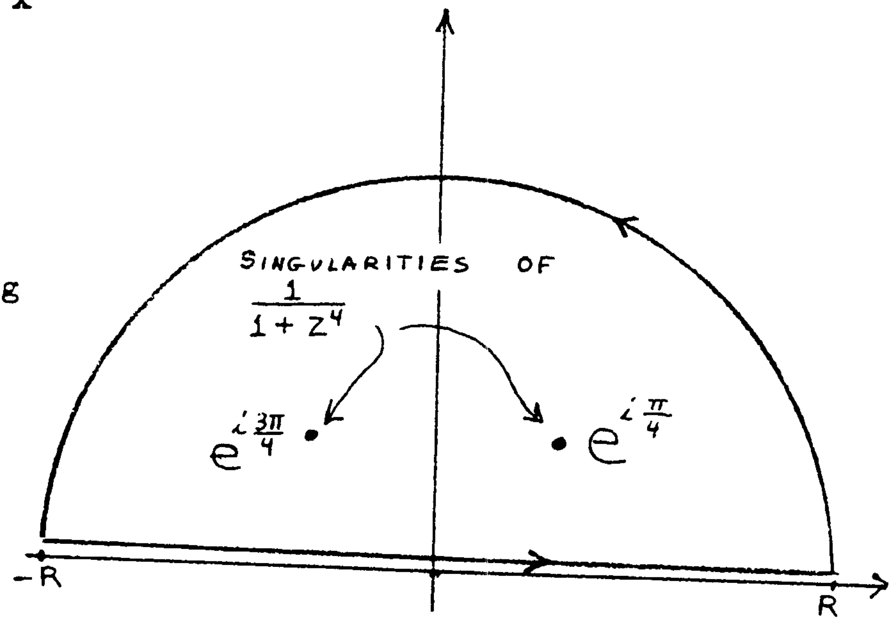
The inequality (3) of the previous theorem simply says that the function $f(z)$ approaches zero faster than $1/z$ as z approaches infinity on the circular arc. Examples of such functions are $(1+z^2)^{-1}$, $\frac{z}{z^3+4}$, $\frac{z^2+1}{z^5+z^3}$, etc. . An example of a function which does NOT satisfy (3) is $\frac{z}{z^2+4}$ since for large z this function behaves like $1/z$ which is just a bit too slow.

Example 2

Evaluate $\int_0^\infty \frac{dx}{1+x^4}$.

Solution

As in the previous example, we consider the closed contour consisting of a semicircular arc and a straight line from $-R$ to R along the real axis. It is clear from the Theorem 1 just considered that $\int_{\curvearrowleft} \frac{dz}{1+z^4}$ approaches zero as R approaches infinity. Thus from



$$\oint_{\circlearrowleft} \frac{dz}{1+z^4} = \int_{\rightarrow} + \int_{\curvearrowleft}$$

we get

$$\lim_{R\to\infty}\int_{\frown}\frac{dz}{1+z^4} = \int_{-\infty}^{\infty}\frac{dx}{1+x^4} + 0 .$$

The left hand side of this last expression is evaluated using the Residue Theorem.yielding

$$(4)\quad \int_{-\infty}^{\infty}\frac{dx}{1+x^4} = 2\pi i[\ \mathrm{Res}(\frac{1}{1+z^4},\ e^{i\pi/4}) + \mathrm{Res}(\frac{1}{1+z^4},\ e^{i3\pi/4})\ ]$$

The residue at $e^{i\pi/4}$ is

$$\lim_{z\to e^{i\pi/4}}\frac{z-e^{i\pi/4}}{1+z^4} ,$$

and using L'Hospital's rule we differentiate both the numerator and the denominator to get

$$(5)\quad \left.\frac{1}{4\ z^3}\right|_{z=e^{i\pi/4}} = \frac{e^{-i3\pi/4}}{4} = \mathrm{Res}(\ e^{i\pi/4}) .$$

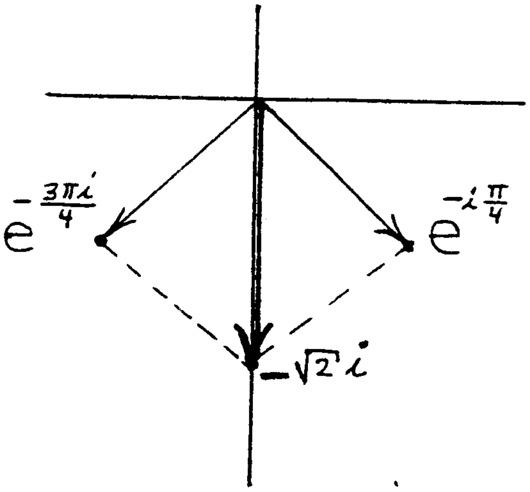
The residue at $e^{i3\pi/4}$ is obtained in the same way ;

$$(6)\quad \mathrm{Res}(\ e^{i3\pi/4}) = \left.\frac{1}{4\ z^3}\right|_{z=e^{i3\pi/4}} = \frac{e^{-i9\pi/4}}{4} = \frac{e^{-i\pi/4}}{4} .$$

Substituting (5) and (6) into (4) we have

$$(7)\quad \int_{-\infty}^{\infty}\frac{dx}{1+x^4} = 2\pi i\ [\ \frac{e^{-i3\pi/4}}{4} + \frac{e^{-i\pi/4}}{4}\ ] .$$

The sum of these last two exponentials is obtained at once from the vector diagram and we have



$$\int_{-\infty}^{\infty} \frac{dx}{1 + x^4} = 2\pi i \left[\frac{-\sqrt{2}\, i}{4} \right]$$

$$= \pi/\sqrt{2},$$

Since $(1 + x^4)^{-1}$ is an even function we have

$$\int_{-\infty}^{\infty} = 2\int_{0}^{\infty}$$

and thus $\displaystyle\int_0^{\infty} \frac{dx}{1 + x^4} = \frac{\pi}{2\sqrt{2}}$.

Problems

18. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{a^2 + x^2}$.

19. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{a^4 + x^4}$

20. Evaluate $\displaystyle\int_0^{\infty} \frac{dx}{(1+x^2)^2}$

21. Evaluate $\displaystyle\int_0^{\infty} \frac{x^2\,dx}{(1+x^2)^2}$

The next example will require two contours of integration, one in the upper half plane, and the other in the lower half plane.

Example 3

Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x \quad dx}{1 + x^2}$.

Solution

We cannot employ the contour  used previously with the integral

$$\oint \frac{\cos z \; dz}{1 + z^2}$$

because the function cos z tends to infinity with exponential speed as z approaches infinity in the direction of the imaginary axis. To see this, look back at Figure 2.8 on page 2.22 . Because cos z grows so rapidly along the semicircular part of the contour, the integral will not tend to zero along this circular arc and the method employed previously will fail.

However, if we write

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \quad ,$$

the integrand breaks up into the two integrands

$$(8) \qquad \frac{\cos z}{1 + z^2} = \frac{1}{2} \frac{e^{iz}}{1 + z^2} + \frac{1}{2} \frac{e^{-iz}}{1 + z^2} \quad .$$

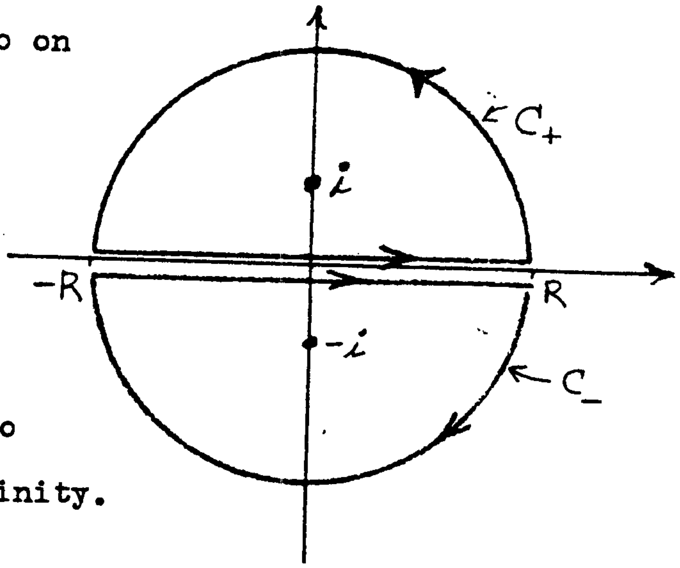
Let us investigate how e^{iz} and e^{-iz} grow when we replace z by x + iy and then let y grow large.

$$\left|e^{iz}\right| = \left|e^{i(x+iy)}\right| = \left|e^{-y+ix}\right| = e^{-y} \longrightarrow \begin{cases} 0 & \text{as} \quad y \to +\infty \\ \infty & \text{as} \quad y \to -\infty \end{cases}$$

and

$$\left|e^{-iz}\right| = \left|e^{-i(x+iy)}\right| = \left|e^{y-ix}\right| = e^{y} \longrightarrow \begin{cases} \infty & \text{as} \quad y \to +\infty \\ 0 & \text{as} \quad y \to -\infty \end{cases}$$

Since $e^{iz}(1+z^2)^{-1}$ approaches zero on $\curvearrowleft$ as R tends to infinity, we select C_+ as the contour for the first term on the right of (8). We select C_- as the contour for the second term because $e^{-iz}(1+z^2)^{-1}$ tends to zero on $\smile$ as R tends to infinity. We have



$$(9) \quad \lim_{R\to\infty} \frac{1}{2}\oint_{C_+} \frac{e^{iz}\,dz}{1+z^2} + \frac{1}{2}\oint_{C_-} \frac{e^{-iz}\,dz}{1+z^2}$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{ix}\,dx}{1+x^2} + \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{-ix}\,dx}{1+x^2}$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{ix}+e^{-ix}}{1+x^2}\,dx$$

$$= \int_{-\infty}^{\infty} \frac{\cos x\;dx}{1+x^2}$$

Using (9) and the Residue Theorem we have

$$(10)\quad \int_{-\infty}^{\infty} \frac{\cos x\,dx}{1+x^2} = 2\pi i\left[\tfrac{1}{2}\mathrm{Res}\left(\frac{e^{iz}}{1+z^2}, i\right) - \tfrac{1}{2}\mathrm{Res}\left(\frac{e^{-iz}}{1+z^2}, -i\right)\right]$$

Notice that a minus sign is used in front of the second residue because the contour C_- is in the clockwise of negative sense.

Now

$$\mathrm{Res}\left(\frac{e^{iz}}{1+z^2}, i\right) = \left.\frac{e^{iz}}{z+i}\right|_{z=i} = \frac{e^{-1}}{2i}$$

and

$$\mathrm{Res}\left(\frac{e^{-iz}}{1+z^2}, -i\right) = \left.\frac{e^{-iz}}{z-i}\right|_{z=-i} = \frac{e^{-1}}{-2i}.$$

Substituting these last two residues into (10) we have

$$\int_{-\infty}^{\infty} \frac{\cos x\ \ dx}{1+x^2} = \pi/e .$$

Problems

22\. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x\,dx}{1+x^4}$.

23\. Evaluate $\displaystyle\int_{0}^{\infty} \frac{\cos x\,dx}{(1+x^2)^2}$

24\. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{\sin^2 x\,dx}{1+x^2}$.

We next consider integrals of the type $\int_0^{2\pi} F(\sin\theta, \cos\theta)\,d\theta$, where F is a rational function. If we make the substitution

$$z = e^{i\theta}$$

we find that

$$(1) \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - z^{-1}}{2i},$$

$$(2) \qquad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + z^{-1}}{2}, \text{ and}$$

$dz = i\,e^{i\theta}\,d\theta$ which implies that

$$(3) \qquad d\theta = -i\,z^{-1}\,dz \quad .$$

As θ varies from 0 to 2π, $z = e^{i\theta}$ will travel once over the unit circle in the positive sense. Thus our original integral is converted into

$$\int_0^{2\pi} F(\sin\theta, \cos\theta)\,d\theta = \oint_{|z|=1} F\left(\frac{z - z^{-1}}{2i}, \frac{z + z^{-1}}{2}\right)\frac{dz}{zi},$$

and this last integral can be evaluated by means of the Residue Theorem.

Example 4

Evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{2 + \sin\theta}$.

Solution

Making the substitutions (1), (2) and (3) above we get

$$I = \int_0^{2\pi} \frac{d\theta}{2+\sin\theta} = \oint_{|z|=1} \frac{1}{2 + \frac{z - z^{-1}}{2i}} \frac{dz}{iz}$$

$$= \oint_{|z|=1} \frac{2\,dz}{z^2 + 4iz - 1}$$

To locate the poles of the integrand we use the quadratic formula

$$z = \frac{-4i \pm \sqrt{-16 + 4}}{2} = (-2 \pm \sqrt{3})i \;.$$

Thus we have

$$I = \oint_{|z|=1} \frac{2\,dz}{(z + (2 - \sqrt{3})i)(z + (2 + \sqrt{3})i)} \;.$$

Only the singularity at $z = -(2 - \sqrt{3})i$ is inside the unit circle and thus the Residue Theorem gives us

$$I = 2\pi i \; 2 \operatorname{Res}\left(\frac{1}{(z+(2-\sqrt{3})i)(z+(2+\sqrt{3})i)},\; -(2-\sqrt{3})i \right)$$

$$= 4\pi i \frac{1}{[-(2-\sqrt{3}) + (2+\sqrt{3})]i} = \frac{2\pi}{\sqrt{3}} \;.$$

Example 5

Evaluate $\displaystyle\int_0^{\pi} \frac{\cos 2\theta \, d\theta}{a^2 - 2a\cos\theta + 1} = I$ where $a^2 < 1$.

Solution

Since $\cos\theta$ is an even function,

$$\int_0^{\pi} = \frac{1}{2}\int_{-\pi}^{\pi}$$

With the substitution $z = e^{i\theta}$, $\cos 2\theta = [\,e^{i2\theta} + e^{-i2\theta}]/2 = [\,z^2 + z^{-2}]/2$. Thus we have for our integral I

$$I = \frac{1}{2}\oint_{|z|=1} \frac{[\,z^2 + z^{-2}\,]/2}{a^2 - 2a\,\frac{z + z^{-1}}{2} + 1}\,\frac{dz}{iz}\,.$$

After a little manipulation we have

$$I = \frac{1}{-4ai}\oint_{|z|=1} \frac{z^4 + 1 \qquad dz}{z^2\,(\,z^2 - (\,a + a^{-1})\,z + 1\,)}$$

The quadratic $z^2 - (\,a + a^{-1})\,z + 1$ is zero when $z = a$ and a^{-1}, therefore

$$I = \frac{1}{-4ai}\oint_{|z|=1} \frac{z^4 + 1 \qquad dz}{z^2\,(z - a)(z - a^{-1})}$$

We see that inside the unit circle there is a pole of order two at the origin and a simple pole at $z = a$ (recall that $a^2 < 1$). Using the Residue Theorem we have

$$I = \frac{2\pi i}{-4ai}\left[\ \text{Res}(\,a\,) + \text{Res}(\,0\,)\ \right]$$

$$= -\frac{\pi}{2a}\left[\ \frac{a^4+1}{a^2(a-a^{-1})} + \frac{d}{dz}\left\{\frac{z^4+1}{z^2-(a+a^{-1})z+1}\right\}_{z=0}\ \right]$$

$$= -\frac{\pi}{2a}\left[\ \frac{a^4+1}{a(a^2-1)} + \frac{a^2+1}{a}\ \right] = \frac{\pi a^2}{1-a^2}\ .$$

Problems

25. Evaluate $\displaystyle\int_0^{2\pi}\frac{d\theta}{a+b\sin\theta}$. $a^2 > b^2$

26. Evaluate $\displaystyle\int_0^{2\pi}\frac{d\theta}{a+b\cos\theta+c\sin\theta}$ $a^2 > b^2 + c^2$

27. Evaluate $\displaystyle\int_0^{\pi}\frac{d\theta}{(a+\cos\theta)^2}$. $a^2 > 1$

28. Evaluate $\displaystyle\int_0^{\pi}\sin^{2n}\theta\ d\theta$.

6.8 Contour integrals about branch cuts

We will now evaluate certain real definite integrals by complex contour integration which features multiple valued integrands. When the integrand is multiple valued, such as $z^{1/2}$, $\log z$, $z^{1/2}(1-z)^{1/2}$, we must select an appropriate single valued branch of the function with a corresponding branch cut. Because of this additional complexity, these techniques are more difficult than those considered in the previous section. As before, the methods are best learned by examining examples.

Example 1

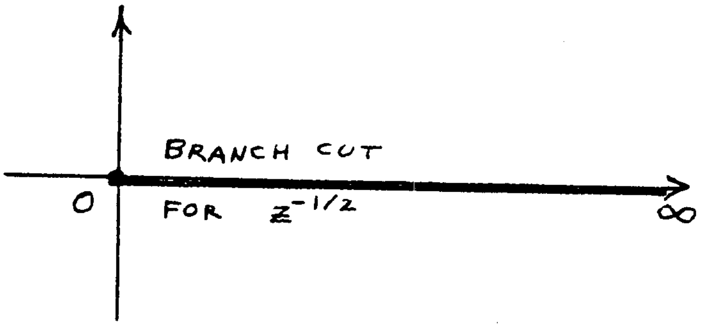
Evaluate $\displaystyle\int_0^\infty \frac{x^{-1/2}\,dx}{x+1} \doteq I$.

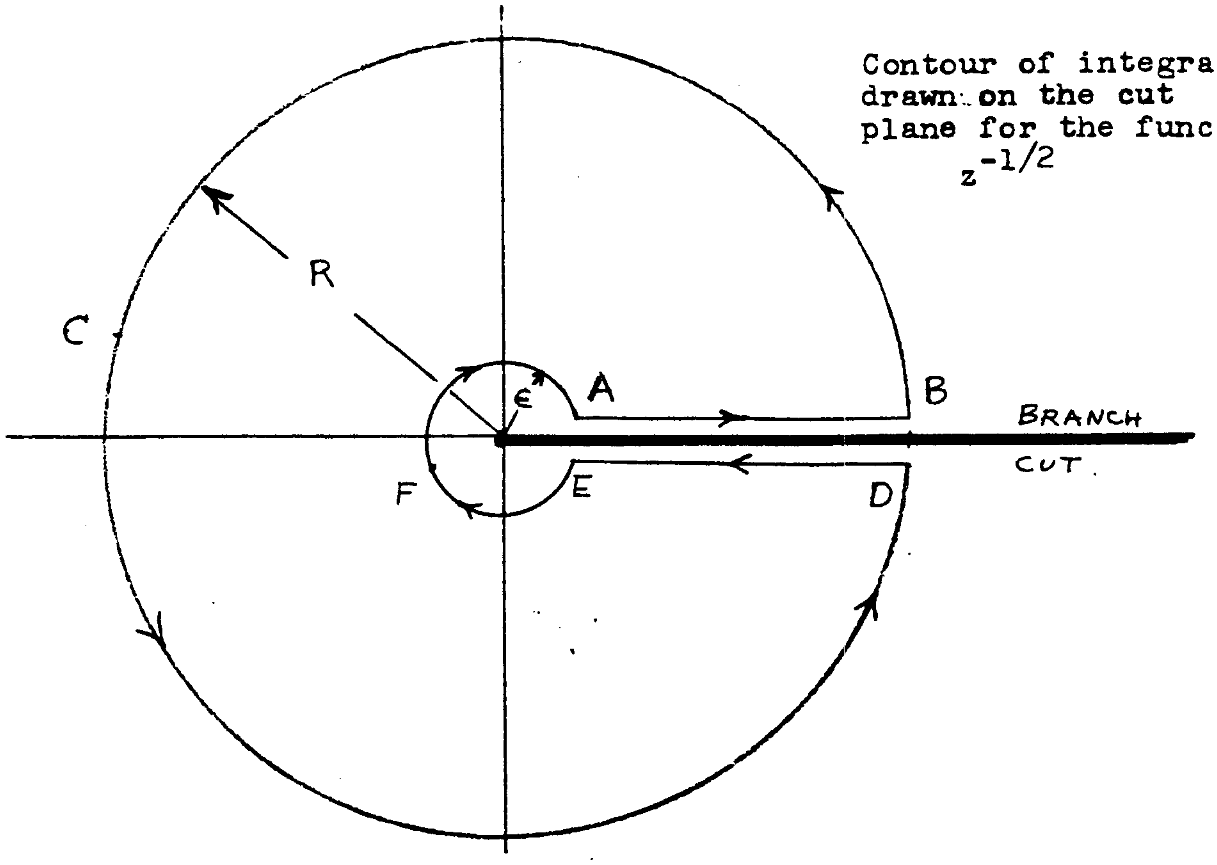
Solution

The function $z^{-1/2}$ is a multiple-valued function. If we set $z = r e^{i\theta}$ and select θ to cover the range $0 \le \theta < 2\pi$, we define the branch of the reciprocal square root such that

(1) $$z^{-1/2} = r^{-1/2} e^{-i\theta/2}, \quad \text{where } 0 \le \theta < 2\pi$$
and r is real and positive.

This branch of the function has its branch line along the positive real axis, and branch points at the origin and infinity. For example, if we wish to find $z^{-1/2}$ for $z = i$, we must select $r = 1$ and $\theta = \pi/2$ by the restrictions in (1) and we get $i^{-1/2} = e^{-i\pi/4}$.





Contour of integratio
drawn on the cut
plane for the function
$z^{-1/2}$

Examine now the contour of integration which is drawn on the branch cut just described. We will examine

$$(2)\quad \int_{\mathbb{C}} \frac{z^{-1/2}\,dz}{z+1} = \int_{A \to B} + \int_{C \circlearrowright^{B}_{D}} + \int_{E \leftarrow D} + \int_{F \circlearrowleft^{A}_{E}}$$

We think of ε, the radius of the small circle, as being very small and we will ultimately let ε approach zero. Likewise we think of R, the radius of the large circle, as being very large and we will finally let R approach infinity. Both of the integrals over the circles will tend to zero and will thus contribute nothing to the evaluation. (More on this later.) The

The integrals over AB and ED at first appear to cancel each other because they are over the same line segment in opposite directions. However, they do not cancel because the factor $z^{-1/2}$ in the integrand assumes different values on the top line AB than it does on the bottom line ED . (Both AB and ED coincide, but are on different ends of the branch cut.) The left side of (2) can be evaluated by means of the residue theorem since there is a simple pole at $z = -1$. Thus (2) will become an equation from which we will solve for the desired real integral I .

We will now examine each of the five integrals in (2) separately.

The entire contour integral

The left side of (2) has a simple pole at $z = -1$ and thus the Residue Theorem gives its value as $2\pi i$ times $z^{-1/2}$ with z equal to -1 . Now when $z = -1$, $r = 1$ and $\theta = \pi$ in (1) so that $z^{-1/2} = e^{-i\pi/2}$. Therefore

$$(3) \qquad \int_{\textcircled{c}} \frac{z^{-1/2}\,dz}{z+1} = 2\pi i\, e^{-i\pi/2} = 2\pi i/i = 2\pi .$$

The integral from A to B.

On this integral, z is the real variable $r\,e^{i0} = r$. Thus our integral is simply

$$\int_{\varepsilon}^{R} \frac{r^{-1/2}\,dr}{r+1}$$

and since we will ultimately let ε approach 0 and R approach infinity this will tend to the desired integral I .

$$(4) \qquad \int_{A \to B} \longrightarrow I$$