23/ (continued)

$$I = \frac{\pi i}{2}\left\{\left[i e^{iz}(z+i)^{-2} - 2e^{iz}(z+i)^{-3}\right]_{z=i} - \left[-i e^{-iz}(z-i)^{-2} - 2e^{-iz}(z-i)^{-3}\right]_{z=-i}\right\}$$

$$I = \frac{\pi i}{2} e^{-1}\left\{\left[\frac{i}{-4} - \frac{2}{-8i}\right] - \left[\frac{-i}{-4} - \frac{2}{8i}\right]\right\}$$

$$I = \frac{\pi i}{2e}\left\{\frac{1}{i}\right\} = \boxed{\frac{\pi}{2e}}$$

24/ $I = \int_{-\infty}^{\infty} \frac{\sin^2 x \, dx}{1+x^2}$

$$\sin^2 z = \left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2 = -\frac{1}{4}\left(e^{2iz} - 2 + e^{-2iz}\right)$$

Thus

$$I = -\frac{1}{4}\int \frac{e^{2iz}\,dz}{1+z^2} + \frac{1}{2}\int \frac{dz}{1+z^2} - \frac{1}{4}\int \frac{e^{-2iz}\,dz}{1+z^2}$$

or

$$I = 2\pi i\left\{\left(-\frac{1}{4}\right)\text{Res}\left(\frac{e^{2iz}}{1+z^2}, i\right) + \frac{1}{2}\text{Res}\left(\frac{1}{1+z^2}, i\right) \underset{\uparrow}{-} \left(-\frac{1}{4}\right)\text{Res}\left(\frac{e^{-2iz}}{1+z^2}, -i\right)\right\}$$

minus because  is in the negative sense.

24/ (continued)

$$I = 2\pi i \left\{ -\frac{1}{4}\left(\frac{e^{-2}}{2i}\right) + \frac{1}{2}\left(\frac{1}{2i}\right) + \frac{1}{4}\left(\frac{e^{-2}}{-2i}\right) \right\}$$

$$I = \boxed{\frac{\pi}{2}\left(1 - e^{-2}\right)}$$

25/ $$I = \int_0^{2\pi} \frac{d\theta}{a + b\sin\theta} = \oint_{|z|=1} \frac{1}{a + b\,\frac{z - z^{-1}}{2i}} \quad \frac{dz}{iz}$$

$$= \frac{2}{b} \oint_{|z|=1} \frac{dz}{z^2 + \frac{2ia}{b} z - 1}$$

The singularities of the integrand occur at $z_\pm = \frac{-a \pm \sqrt{a^2 - b^2}}{b}$, but only z_+ is inside the unit circle.

$$I = \frac{2}{b} \oint \frac{dz}{(z - z_+)(z - z_-)} = (2\pi i)\left(\frac{2}{b}\right) \frac{1}{z_+ - z_-}$$

$$= \boxed{\frac{2\pi}{\sqrt{a^2 - b^2}}}$$

26/ $$I = \int_0^{2\pi} \frac{d\theta}{a + b\cos\theta + c\sin\theta} = \oint_{|z|=1} \frac{1}{\left[a + b\,\frac{z + z^{-1}}{2} + c\,\frac{z - z^{-1}}{2i}\right]} \frac{dz}{iz}$$

$$I = \frac{-2i}{b - ic} \oint_{|z|=1} \frac{dz}{z^2 + \frac{2a}{b - ic} z + \frac{b + ci}{b - ci}}$$

26/ (continued)

The singularities of the integrand occur at

$$z_{\pm} = \frac{-a \pm \sqrt{a^2-b^2-c^2}}{b-ci}, \text{ but only } z_+$$

is inside $|z|=1$. Therefore

$$I = \frac{-2i}{b-ic} \oint_{|z|=1} \frac{dz}{(z-z_+)(z-z_-)}$$

$$= \frac{-2i}{b-ic}(2\pi i)\,\mathrm{Res}\,(z_+) = \frac{-2i}{b-ic}(2\pi i)\frac{1}{z_+-z_-}$$

$$= \boxed{\frac{2\pi}{\sqrt{a^2-b^2-c^2}}}$$

27/ $I = \int_0^\pi \frac{d\theta}{(a+\cos\theta)^2} = \frac{1}{2}\int_{-\pi}^{\pi}(\text{same})$

$$= \frac{1}{2}\oint_{|z|=1} \frac{1}{\left(a+\frac{z+z^{-1}}{2}\right)^2}\cdot\frac{dz}{iz}$$

$$= \frac{2}{i}\oint \frac{z\,dz}{(z^2+2az+1)^2}$$

The singularities of the integrand occur at

$z_{\pm} = -a \pm \sqrt{a^2-1}$, and thus

27/ (continued)

$$I = \frac{2}{i}\oint \frac{z\,dz}{(z-z_+)^2(z-z_-)^2}$$

$$= \frac{2}{i}(2\pi i)\left\{\frac{d}{dz}\left[z(z-z_-)^{-2}\right]\right\}\Bigg|_{z=z_+}$$

$$= 4\pi\left\{(z-z_-)^{-2} - 2(z-z_-)^{-3}z\right\}\Bigg|_{z=z_+}$$

$$= \frac{-4\pi(z_+ + z_-)}{(z_+ - z_-)^3} = \boxed{\frac{\pi a}{(a^3-1)^{\frac{3}{2}}}}$$

28/ $I = \int_0^{\pi} \sin^{2n}\theta\,d\theta = \frac{1}{2}\int_0^{2\pi} \sin^{2n}\theta\,d\theta$

$$I = \frac{1}{2}\oint_{|z|=1}\left(\frac{z-z^{-1}}{2i}\right)^{2n}\frac{dz}{iz} = \frac{1}{2^{2n+1}(-1)^n i}\oint\frac{(z^2-1)^{2n}\,dz}{z^{2n+1}}$$

The integrand has a pole of order $2n+1$ at $z=0$. If we expand $(1-z^2)^{2n}$ by means of the binomial theorem, there will appear a term of the form $b\,z^{2n}$, from which we see that b is the desired residue at $z=0$.

$$(1-z^2)^{2n} = 1 + \cdots + \binom{2n}{n}(-z^2)^n + \cdots + (-z^2)^{2n},$$

28/ (continued)

We see now that the residue is $\binom{2n}{n}(-1)^n$.

Therefore

$$I = \frac{1}{2^{2n+1}(-1)^n i}(2\pi i)\binom{2n}{n}(-1)^n$$

$$= \boxed{\frac{\pi}{2^{2n}} \quad \frac{(2n)!}{n! \cdot n!}}$$

29/ ANSWER IS $\frac{2\pi}{\sqrt{3}}$. THIS INTEGRAL IS A SPECIAL CASE OF PROBLEM 30,

30/ THIS INTEGRAL IS A GENERALIZATION OF EXAMPLE 1. WE WILL USE THE SAME CONTOUR AND BRANCH CUT, ONLY NOW WE HAVE

$$z^{-k} = r^{-k} e^{-ik\theta} \qquad 0 \le \theta < 2\pi,$$

$$(1) \quad \int_{\ominus} \frac{z^{-k}\,dz}{z+1} = \int_{A\to B} + \int_{C\circlearrowleft{}^{B}_{D}} + \int_{E\leftarrow D} + \int_{F\circlearrowleft{}^{A}_{E}}$$

THE RESIDUE THEOREM GIVES FOR THE LEFT INTEGRAL IN (1)

$$2\pi i \operatorname{Res}\left(\frac{z^{-k}}{z+1}, -1\right) = 2\pi i\, z^{-k}\Big|_{z=-1=e^{i\pi}} = e^{-ik\pi}\, 2\pi i.$$

30/ (CONTINUED)

ON THE LARGE CIRCLE, $\left|\frac{z^{-k}}{z+1}\right| \approx R^{-k-1}$ AND SINCE $0<k<1$ THIS TENDS TO ZERO FASTER THAN $\frac{1}{R}$, THUS FROM REMARK 2 THIS INTEGRAL VANISHES AS $R\to\infty$,

ON THE SMALL CIRCLE $\left|\frac{z^{-k}}{z+1}\right| \approx |z^{-k}| \approx \epsilon^{-k}$

AND BY REMARK 3 THIS INTEGRAL ALSO VANISHES,

$\int\limits_{A\to B} \longrightarrow I$ BECAUSE $\theta=0$ AND $\frac{z^{-k}\,dz}{z+1} = \frac{r^{-k}\,dr}{r+1}$

ON $E \longleftarrow D$ HOWEVER, $\theta = 2\pi$ AND

$$\int_R^{\epsilon} \frac{z^{-k}\,dz}{z+1} = \int_R^{\epsilon} \frac{r^{-k}e^{-i2\pi k} \quad e^{i2\pi}\,dr}{re^{2\pi i}+1}$$

$$= \int_R^{\epsilon} \frac{r^{-k}\,dr}{r+1} \quad e^{-i2\pi k}$$

$$\underset{\substack{\text{as}\\ \epsilon\to 0\\ R\to\infty}}{\Longrightarrow} -e^{-i2\pi k}\,I$$

THUS (1) BECOMES

$$2\pi i\, e^{-ik\pi} = I + 0 - e^{-i2\pi k}\,I + 0$$

$$I = \frac{2\pi i\, e^{-ik\pi}}{1-e^{-i2\pi k}} = \frac{2\pi i}{e^{ik\pi}-e^{-ik\pi}} = \boxed{\frac{\pi}{\sin \pi k}}$$

31/ $I = \int_0^\infty \frac{x^{-1/2}\,dx}{1+x^2}$ WE USE THE SAME CONTOUR AND BRANCH CUT AS IN EXAMPLE 1.

$$(1) \quad \int \frac{z^{-1/2}\,dz}{1+z^2} = \int_{A \to B} + \int_{\substack{\text{LARGE}\\ \text{CIRCLE}}} + \int_{E \leftarrow D} + \int_{\substack{\text{SMALL}\\ \text{CIRCLE}}}$$

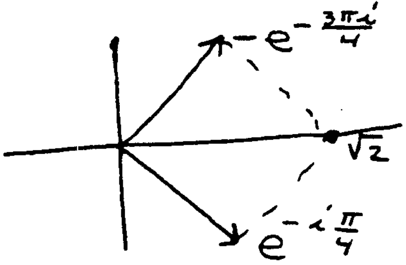
THE INTEGRAL ON THE LEFT OF (1) IS

$$2\pi i \left[\operatorname{Res}\left(\frac{z^{-\frac{1}{2}}}{1+z^2}, i \right) + \operatorname{Res}\left(\frac{z^{-\frac{1}{2}}}{1+z^2}, -i \right) \right] =$$

$$2\pi i \left[\left. \frac{z^{-\frac{1}{2}}}{z+i} \right|_{z=i=e^{i\frac{\pi}{2}}} + \left. \frac{z^{-\frac{1}{2}}}{z-i} \right|_{z=-i=e^{\frac{3\pi i}{2}}} \right] =$$

$$2\pi i \left[\frac{e^{-\frac{i\pi}{4}}}{2i} + \frac{e^{-\frac{3\pi i}{4}}}{-2i} \right] = \pi \left[e^{-\frac{i\pi}{4}} - e^{-\frac{3\pi i}{4}} \right]$$

$$= \pi\sqrt{2}$$



$\int_{A \to B} \longrightarrow I$ SINCE $\theta = 0$ HERE,

$$\int_{E \leftarrow D} \longrightarrow \int_\infty^0 \frac{r^{-\frac{1}{2}} e^{-\pi i}}{r^2+1}\,dr = I$$

THE INTEGRALS OVER THE TWO CIRCLES VANISH SINCE FOR LARGE R THE INTEGRAND BEHAVES LIKE $\frac{1}{R^{3/2}}$,

31/ (continued)

AND FOR SMALL ϵ IT BEHAVES LIKE $\frac{1}{\sqrt{\epsilon}}$. (See Remarks 2 and 3).

USING (1) WE NOW HAVE

$$\sqrt{2}\,\pi = I + 0 + I + 0 \quad , \quad I = \boxed{\frac{\pi}{\sqrt{2}}} .$$

32/ $I = \int_0^\infty \frac{x^{\frac{1}{2}}\,dx}{(1+x^2)^2}$. WE USE THE SAME CONTOUR AS IN EXAMPLE (1),

$$(1) \int_{C} \frac{z^{1/2}\,dz}{(1+z^2)^2} = \int_{A\to B} + \int_{\substack{\text{LARGE}\\ \text{CIRCLE}}} + \int_{E\leftarrow D} + \int_{\substack{\text{SMALL}\\ \text{CIRCLE}}}$$

$$\int_{C} \frac{z^{1/2}\,dz}{(1+z^2)^2} = 2\pi i\left[\operatorname{Res}(i) + \operatorname{Res}(-i)\right]$$

$$= 2\pi i\left[\frac{d}{dz}\left\{z^{1/2}(z+i)^{-2}\right\}\Big|_{z=i=e^{i\frac{\pi}{2}}} + \frac{d}{dz}\left\{z^{1/2}(z-i)^{-2}\right\}\Big|_{z=-i=e^{\frac{3\pi i}{2}}}\right]$$

$$= 2\pi i\left[\left\{\tfrac{1}{2}z^{-\frac{1}{2}}(z+i)^{-2} - 2z^{1/2}(z+i)^{-3}\right\}\Big|_{z=i=e^{i\frac{\pi}{2}}} + \left\{\tfrac{1}{2}z^{-\frac{1}{2}}(z-i)^{-2} - 2z^{1/2}(z-i)^{-3}\right\}\Big|_{z=-i=e^{i\frac{3\pi}{2}}}\right]$$

32/ (continued)

$$= 2\pi i\left[-\frac{\sqrt{2}\,i}{8}\right] = \frac{\sqrt{2}\,\pi}{4},$$

$$\int_{A\to B} \longrightarrow I$$ since $\theta = 0$. here,

$$\int_{E\leftarrow D} \longrightarrow \int_{\infty}^{0} \frac{r^{\frac{1}{2}}\, e^{i\pi}\, dr}{(1+r^2)^2} = I$$

The integrals over the two circles again tend to zero as $R \to \infty$ and $\epsilon \to 0$ since for large R the integrand resembles $R^{-7/2}$ and for small ϵ the integrand behaves like $\sqrt{\epsilon}$.

Combining all these values in (1) we get

$$\frac{\sqrt{2}\,\pi}{4} = I + 0 + I + 0,$$ Thus $$I = \boxed{\frac{\pi}{4\sqrt{2}}}.$$

33/ $I = \int_0^\infty \frac{\log x\, dx}{(x^2+a^2)^2}$ Here we use the same contour as in Example 2.

$$(1)\quad \int \frac{\log z\, dz}{(a^2+z^2)^2} = \int_{\epsilon}^{R} + \int_{\text{LARGE ARC}} + \int_{-R}^{-\epsilon} + \int_{\text{SMALL ARC.}}$$

33/ (CONTINUED)

$$\int_{\curvearrowleft} = 2\pi i \ \mathrm{Res}\left(\frac{\log z}{(a^2+z^2)^2}, \ i a\right)$$

$$= 2\pi i\left[\ \frac{d}{dz}\left\{\frac{\log z}{(ai+z)^2}\right\}\Bigg|_{z=ia}\right]$$

$$= 2\pi i\left[\ z^{-1}(ai+z)^{-2} - 2\log z \ (ai+z)^{-3}\right]_{z=ia}$$

$$= 2\pi i\left[\frac{-1}{4a^3 i} + \frac{\log a + i\frac{\pi}{2}}{4a^3 i}\right]$$

$$= \frac{\pi}{2a^3}\left[-1 + \log a + i\frac{\pi}{2}\right]$$

Now $\int_{\epsilon}^{R} \longrightarrow I$ and

$$\int_{-R}^{-\epsilon} \longrightarrow \int_{\infty}^{0} \frac{\log r + i\pi}{(a^2+r^2)^2}(-dr) = \int_0^{\infty}\frac{\log r}{(a^2+r^2)^2}dr + i\pi\int_0^{\infty}\frac{dr}{(a^2+r^2)^2}$$

$$= I + i\pi\int_0^{\infty}\frac{dr}{(a^2+r^2)^2}$$

To evaluate this last integral set $r = au$ and get

$$\int_0^{\infty}\frac{dr}{(a^2+r^2)^2} = \int_0^{\infty}\frac{a\,du}{a^4(1+u^2)^2} = \frac{1}{a^3}\int_0^{\infty}\frac{du}{(1+u^2)^2} = \frac{\pi}{4a^3}$$

where we have used the result of problem 20. Thus

$$\int_{-R}^{-\epsilon} \longrightarrow I + i\,\frac{\pi^2}{4a^3}$$

33/ (continued)

The integral over the large semicircle behaves like

$$\frac{\log R}{R^4}\,\pi R = \frac{\log R}{R^3} \longrightarrow 0 \quad \text{as} \quad R \to \infty,$$

The integral over the small semicircle behaves like

$$\left(\frac{\log \epsilon + i\theta}{a^4}\right)\pi\epsilon \longrightarrow 0 \quad \text{as} \quad \epsilon \to 0,$$

Combining these values in (1) we get

$$-\frac{\pi}{2a^3} + \frac{\pi \log a}{2a^3} + i\,\frac{\pi^2}{4a^3} = I + 0 + I + i\,\frac{\pi^2}{4a^3} + 0$$

which yields

$$I = \boxed{\frac{\pi}{4a^3}\left[-1 + \log a\right]}$$

34/ $I = \int_0^\infty \frac{\log x \, dx}{x^4+1}$, We use the same contour as in Example 2,

(1) $$\int_{\frown} \frac{\log z \, dz}{z^4+1} = \int_\epsilon^R + \int_{\substack{\text{large}\\ \text{semicircle}}} + \int_{-R}^{-\epsilon} + \int_{\substack{\text{small}\\ \text{arc}}}$$

Now

(2) $$\int_{\frown} = 2\pi i\left[\operatorname{Res}\left(\frac{\log z}{z^4+1}, e^{i\pi/4}\right) + \operatorname{Res}\left(e^{i\frac{3\pi}{4}}\right)\right]$$

34/ (continued)

$$\operatorname{Res}\left(e^{i\pi/4}\right) = \lim_{z\to e^{i\pi/4}} \frac{(z-e^{i\pi/4})\log z}{z^4+1} = \begin{pmatrix}\text{using}\\ \text{L'Hospital's}\\ \text{Rule}\end{pmatrix}$$

$$= \left.\frac{\log z}{4z^3}\right|_{z=e^{i\pi/4}} = \frac{i\,\pi/4}{4e^{\frac{3\pi i}{4}}}$$

$$\operatorname{Res}\left(e^{\frac{i3\pi}{4}}\right) = \left.\frac{\log z}{4z^3}\right|_{z=e^{\frac{3i\pi}{4}}} = \frac{i\,\frac{3\pi}{4}}{4\,e^{i\frac{9\pi}{4}}}$$

Thus from (2) we have

$$(3)\ \int_{\frown} = 2\pi i\left[\frac{i\pi}{16}e^{-\frac{3\pi}{4}i} + \frac{3\pi i}{16}e^{-\frac{i\pi}{4}}\right]$$

$$= \frac{-\pi^2}{8}\left[e^{-\frac{3\pi}{4}i} + 3e^{-i\frac{\pi}{4}}\right]$$

Now

$$\int_{\epsilon}^{R} \longrightarrow I \quad , \text{ and}$$

$$\int_{-R}^{-\epsilon} \longrightarrow \int_{\infty}^{0} \frac{\log r + i\pi}{r^4+1}(-dr) = \int_0^\infty \frac{\log r\,dr}{r^4+1} + i\pi\int_0^\infty \frac{dr}{1+r^4}$$

$$(4)\ \int_{-R}^{-\epsilon} \longrightarrow I + \frac{i\pi^2}{2\sqrt{2}} \quad \left(\begin{matrix}\text{where we use the}\\ \text{solution to problem 19}\end{matrix}\right)$$

As before, the integrals over the circular arcs tend to zero. Substituting (3) and (4) into (1) we get

$$-\frac{\pi^2}{8}\left[-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} + \frac{3}{\sqrt{2}} - \frac{3i}{\sqrt{2}}\right] = I + 0 + I + \frac{i\pi^2}{2\sqrt{2}}$$

$$-\frac{\pi^2\sqrt{2}}{8} = 2I \ , \quad \text{Thus} \quad I = \boxed{-\frac{\pi^2\sqrt{2}}{18}}$$

35/ $I = \int_0^1 x^{-\frac{2}{3}}(1-x)^{-\frac{1}{3}}\,dx$

As in Example 3, we define

$$z^{-\frac{2}{3}} = r^{-\frac{2}{3}} e^{-i\frac{2\theta}{3}} \quad \text{where} \quad 0 \le \theta < 2\pi$$

and

$$(1-z)^{-\frac{1}{3}} = s^{-\frac{1}{3}} e^{-i\frac{\omega}{3}} \quad \text{where} \quad 0 \le \omega < 2\pi$$

so that

$$z^{-\frac{2}{3}}(1-z)^{-\frac{1}{3}} = r^{-\frac{2}{3}} s^{-\frac{1}{3}} e^{-i\left(\frac{2}{3}\theta + \frac{1}{3}\omega\right)} \tag{1}$$

We must first locate the branch line for the function (1). Since θ is discontinuous on the positive real axis, it comes as no surprise that (1) is discontinuous on the line segment from $z=0$ to $z=1$. However, it is not discontinuous on the real axis beyond the point $z=1$. To see this, note that as we approach this segment from above, both θ and ω approach zero and thus (1) is a real number. As we approach from below, both θ and ω are 2π and the exponential term in (1) becomes

$$e^{-i\left(\frac{2}{3}(2\pi) + \frac{1}{3}(2\pi)\right)} = e^{-i2\pi} = 1,$$

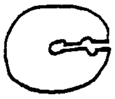
which means that (1) is again real.

Thus (1) has the very same branch cut as was found in Example 3.

Using the same contour as in Example 3 we have

35/ (continued)

$$(2)\quad \int z^{-\frac{2}{3}}(1-z)^{-\frac{1}{3}}dz = \int_{\substack{\text{large}\\ \text{circle}}} + \int_{\substack{\epsilon\\ \text{above}\\ \text{cut}}}^{1-\epsilon} + \int_{\substack{1-\epsilon\\ \text{below}\\ \text{cut}}}^{\epsilon} + \underbrace{\int + \int}_{\substack{\text{two small}\\ \text{circles}}}$$



The left side of (2) is zero by Cauchy's integral theorem since there are no singularities inside the contour.

On the large circle, $\theta \approx \omega$ and $r = R$ and $s \approx R$. Thus (1) becomes

$$(z)^{-\frac{2}{3}}(1-z)^{-\frac{1}{3}} \approx R^{-1}e^{-i\theta},$$ since $dz = iRe^{i\theta}d\theta$

we see that

$$(3)\quad \int_{\substack{\text{large}\\ \text{circle}}} \longrightarrow \int_0^{2\pi}\left(R^{-1}e^{-i\theta}\right)\left(iRe^{i\theta}d\theta\right) = i\int_0^{2\pi}d\theta = 2\pi i.$$

On the line segment above the cut, $\theta = 0$ and $\omega = \pi$, Thus (1) becomes

$$z^{-\frac{2}{3}}(1-z)^{-\frac{1}{3}} = r^{-\frac{2}{3}}s^{-\frac{1}{3}}e^{-i\frac{\pi}{3}} = x^{-\frac{2}{3}}(1-x)^{-\frac{1}{3}}e^{-i\frac{\pi}{3}},$$

Since $dz = dx$ we have as $\epsilon \to 0$

$$(4)\quad \int_{\substack{\epsilon\\ \text{above}\\ \text{cut}}}^{1-\epsilon} \xrightarrow[\epsilon\to 0]{} e^{-i\frac{\pi}{3}}\int_0^1 x^{-\frac{2}{3}}(1-x)^{-\frac{1}{3}}dx = e^{-i\frac{\pi}{3}} I$$

On the line segment below the cut, $\theta = 2\pi$, but $\omega = \pi$ and thus (1) becomes,

$$z^{-\frac{2}{3}}(1-z)^{-\frac{1}{3}} = r^{-\frac{2}{3}} s^{-\frac{1}{3}} e^{-i\left(\frac{2}{3}(2\pi)+\frac{1}{3}(\pi)\right)}$$

$$= r^{-\frac{2}{3}} s^{-\frac{1}{3}} e^{-\frac{5\pi}{3}i}$$

$$= r^{-\frac{2}{3}} s^{-\frac{1}{3}} e^{\frac{\pi}{3}i}$$

$$= x^{-\frac{2}{3}}(1-x)^{-\frac{1}{3}} e^{\frac{\pi}{3}i}$$

Since $dz = dx$ we have as $\epsilon \to 0$

$$(5) \quad \int_{\substack{1-\epsilon \\ \text{below} \\ \text{cut}}}^{\epsilon} \xrightarrow[\epsilon\to 0]{} \int_{1}^{0} e^{\frac{\pi}{3}i} x^{-\frac{2}{3}}(1-x)^{-\frac{1}{3}}\,dx$$

$$= -e^{\frac{\pi}{3}i}\, I$$

The integrals over the two small circles tend to zero as $\epsilon \to 0$ as in Example 3,

Combining (3), (4) and (5) in (2) we get

$$0 = 2\pi i + e^{-i\frac{\pi}{3}} I - e^{\frac{\pi}{3}i} I + 0 + 0$$

$$I = \frac{2\pi i}{e^{\frac{\pi}{3}i} - e^{-\frac{\pi}{3}i}} = \frac{\pi}{\sin\frac{\pi}{3}} = \boxed{\frac{2\pi}{\sqrt{3}}}.$$

Solutions to Review Problems from Chapter 6

1/ $$\oint_{|z|=2\pi} \frac{\sin z \, dz}{(z-\pi)^5} = \frac{2\pi i}{4!} \left. \frac{d^4 \sin z}{dz^4} \right|_{z=\pi}$$

$$= \frac{2\pi i}{24} \left. \sin z \right|_{z=\pi} = \boxed{0}$$

2/ (a) $$\frac{1}{(a^2+z^2)^{+4}} = \frac{1}{(z-ia)^4 (z+ia)^4}$$

$$\text{Res}\,(ia) = \frac{1}{3!} \left. \frac{d^3 (z+ia)^{-4}}{dz^3} \right|_{z=ia}$$

$$= \frac{1}{3!} (-4)(-5)(-6)(z+ia)^{-7} \Big|_{z=ia}$$

$$= -\frac{20}{(2ia)^7} = \frac{-20}{-2^7 a^7 i} = \boxed{\frac{-5i}{32a^7}}$$

(b) z^4+6z^2+1 has roots at
$$z^2 = -3 \pm \sqrt{9-1}$$
$$= -3 \pm 2\sqrt{2}$$

$$z^4+6z^2+1 = (z^2+3+2\sqrt{2})(z - \sqrt{3-2\sqrt{2}}\, i)(z+\sqrt{3-2\sqrt{2}}\, i)$$

WE SEE THAT THE ROOTS ARE SIMPLE,

$$\text{Res}\,(\sqrt{3-2\sqrt{2}}\, i) = \left. \frac{z}{(z^2+3+2\sqrt{2})(z+\sqrt{3-2\sqrt{2}}\, i)} \right|_{z=\sqrt{3-2\sqrt{2}}\, i}$$

$$= \frac{\sqrt{3-2\sqrt{2}}\, i}{(4\sqrt{2})\; 2\sqrt{3-2\sqrt{2}}\, i} = \boxed{\frac{1}{8\sqrt{2}}}$$

$$\text{Res}\,(-\sqrt{3-2\sqrt{2}}\, i) = \left. \frac{z}{(z^2+3+2\sqrt{2})(z-\sqrt{3-2\sqrt{2}}\, i)} \right|_{z=-\sqrt{3-2\sqrt{2}}\, i}$$

$$= \frac{-\sqrt{3-2\sqrt{2}}\, i}{(4\sqrt{2})(-2\sqrt{3-2\sqrt{2}}\, i)} = \boxed{\frac{1}{8\sqrt{2}}}$$

2/ (continued)

(c) The pole is simple, therefore

$$\text{Res}\,(-1) = z^{-k} \log z \Big|_{z=-1=e^{i\pi}}$$

$$= e^{-ik\pi}\left[\log 1 + i\pi\right]$$

$$= \boxed{i\pi\, e^{-ik\pi}}$$

3/ $I = \int_0^\infty \frac{dx}{(a^2+x^4)^4} = \frac{1}{2}\int_{-\infty}^{\infty}$

Using the contour of Section 6.7, Example 1 we have

$$\int_{\frown} \frac{dz}{(a^2+z^2)^4} = \int_{-\infty}^{\infty} + \int_{\frown}$$

$$2\pi i\ \text{Res}\left(\frac{1}{(a^2+z^2)^4}, ia\right) = 2I + 0$$

Use review problem 2 (a)

$$2\pi i\left(\frac{-5i}{32a^7}\right) = 2I$$

$$I = \boxed{\frac{5\pi}{32a^7}}$$

4/ $I = \int_0^\infty \frac{x^{-k}\log x\, dx}{1+x}$

Use the contour of Example 1, Section 6.8.

$$(1)\quad \int_{\ominus} \frac{z^{-k}\log z\, dz}{1+z} = \underset{\text{above cut}}{\int_0^\infty} + \underset{\text{below cut}}{\int_\infty^0} + \underset{\text{small circle}}{\int_{C}} + \underset{\text{big circle}}{\int_{\bigcirc}}$$

4/ (continued)

The left side of (1) is evaluated with the help of the Residue Theorem and problem 2 (c) above

$$\underset{\ominus}{\int} = 2\pi i\left[i\pi e^{-ik\pi}\right] = -2\pi^2 e^{-ik\pi}$$

Since $\theta = 0$ above the cut, $\underset{\substack{\text{above}\\ \text{cut}}}{\int_0^\infty} = I$

Below the cut, $\theta = 2\pi$ and

$$\frac{z^{-k}\log z}{1+z}dz = \frac{r^{-k}e^{-i2\pi k}\left[\log r + 2\pi i\right]}{1+r}(+dr)$$

Thus

$$\underset{\substack{\text{below}\\ \text{cut}}}{\int_\infty^0} = e^{-i2\pi k}\int_\infty^0 \frac{r^{-k}\log r}{r+1}dr + 2\pi i\, e^{-i2\pi k}\int_\infty^0 \frac{r^{-k}dr}{1+r}$$

$$= -e^{-i2\pi k}I - 2\pi i\, e^{-i2\pi k}\underbrace{\int_0^\infty \frac{r^{-k}dr}{1+r}}_{\frac{\pi}{\sin \pi k}\quad \text{problem 30 Chapter 6}}$$

$$= -e^{-i2\pi k}I - \frac{2\pi^2 i e^{-i2\pi k}}{\sin \pi k}$$

Now (1) becomes

$$-2\pi^2 e^{-ik\pi} = I - e^{-i2\pi k}I - \frac{2\pi^2 i e^{-i2\pi k}}{\sin \pi k} + 0 + 0$$

$$I = \left[\frac{2\pi^2 i e^{-i2\pi k}}{\sin \pi k} - 2\pi^2 e^{-ik\pi}\right]\frac{1}{1-e^{-i2\pi k}}$$

$$= 2\pi^2\left[\frac{2e^{-i2\pi k}}{e^{i\pi k}-e^{-i\pi k}} - e^{-ik\pi}\right]\frac{1}{1-e^{-i2\pi k}}$$

$$= 2\pi^2\left[\frac{1+e^{-i2\pi k}}{e^{i\pi k}-e^{-i\pi k}}\right]\frac{1}{1-e^{-i2\pi k}}$$

$$= \frac{2\pi^2(e^{i\pi k}+e^{-i\pi k})}{(e^{i\pi k}-e^{-i\pi k})^2} = \boxed{\frac{-\pi^2\cos \pi k}{\sin^2 \pi k}}$$

5/ $$I = \int_0^{2\pi} \frac{d\theta}{1+\cos^2\theta} = \oint_{|z|=1} \frac{1}{1+\left(\frac{z+z^{-1}}{2}\right)^2} \frac{dz}{iz}$$

$$= \frac{1}{i} \oint_{|z|=1} \frac{4z\,dz}{4z^2+z^4+2z^2+1}$$

$$= \frac{4}{i} \oint_{|z|=1} \frac{z\,dz}{z^4+6z^2+1}$$

$$= \frac{4}{i}\, 2\pi i \left[\text{sum of the residues of } \frac{z}{z^4+6z^2+1} \text{ inside } |z|=1 \right]$$

$$= 8\pi \left[\begin{array}{l} \text{sum of the two residues} \\ \text{computed above in problem } 2(b) \end{array} \right]$$

$$= 8\pi \left[\frac{1}{8\sqrt{2}} + \frac{1}{8\sqrt{2}} \right] = \frac{2\pi}{\sqrt{2}} = \boxed{\sqrt{2}\,\pi}$$

APPENDIX II

ANSWERS TO CONJECTURES

Chapter 6

6.1 The indefinite integral

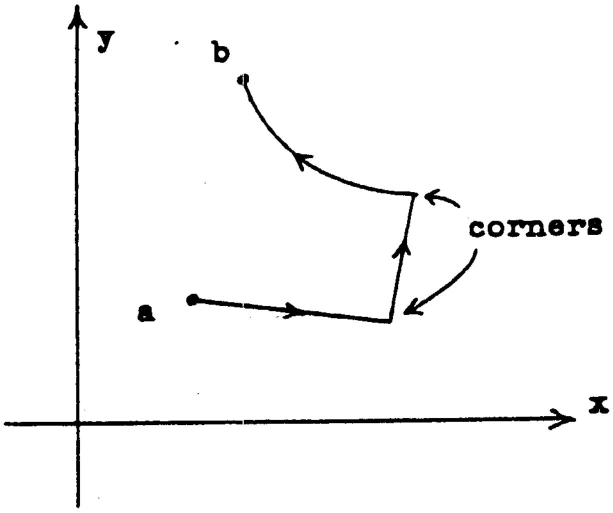
Let $f(z)$ be an analytic function defined on the open set R. Then there exists an analytic function $F(z)$ such that for each z on R, $F'(z) = f(z)$. We call $F(z)$ an **indefinite integral** of $f(z)$ and denote the family of all functions which when differentiated yield $f(z)$ by $\int f(z)\,dz$. It can be shown that this family is

$$\int f(z)\,dz = F(z) + c\,,$$

where c is an arbitrary (complex) constant.

6.2 The Riemann integral

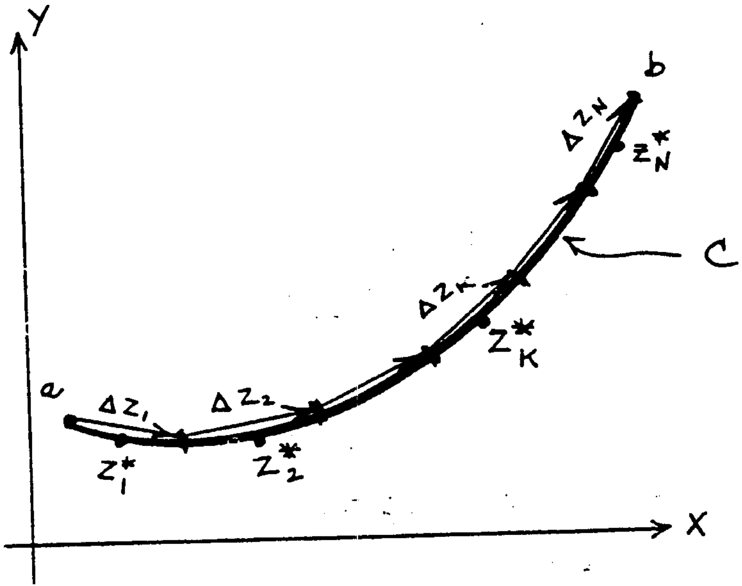
In real analysis we integrated over a segment of the x-axis from $x=a$ (the starting point) to $x=b$, (the final point) to form the definite integral $\int_a^b f(x)dx$. In the extension to the complex plane, this directed segment of the x-axis is replaced by a directed curve starting at $z=a$ and ending at $z=b$.



A directed curve (like the one in the figure) is called "nice" if it is smooth at most points. We do allow a few corners as shown in the figure where the curve fails to be smooth. (An exact definition of a nice curve would be one that is described by parametric equations $x = x(t)$ and $y = y(t)$, $t_0 \leq t \leq t_1$, where $x(t)$ and $y(t)$ have piecewise continuous derivatives.) We will not, however, have occasion to worry about the precise nature of a "nice" curve in this book, as the curves we usually encounter are composed of straight line segments and circular arcs.

Now let $f(z)$ be a complex valued function defined for all z on the nice directed curve C which starts at $z=a$ and ends at $z=b$.

(i) Subdivide the curve C into N small consecutive arcs as shown in the figure.



(ii) On each small arc select a point z_k^* and form $f(z_k^*)\Delta z_k$.

(iii) Form $\sum_{k=1}^{N} f(z_k^*)\Delta z_k$.

(iv) Continue to subdivide the curve C into more and more arcs $(N \to \infty)$ and simultaneously let the length of each small arc approach zero $(\Delta z_k \to 0)$. If the limit

$$\lim_{\substack{\Delta z_k \to 0 \\ N \to \infty}} \sum_{k=1}^{N} f(z_k^*)\Delta z_k$$

exists and gives but one value regardless of the manner in which the subdivision of the curve C is made, then we call this limit the Riemann integral of f(z) over the directed curve (or contour) from z=a to z=b and denote it by

$$\int_C f(z)\,dz = \int_{C\,a}^{b} f(z)\,dz = \int_a^b f(z)dz \quad .$$

Remarks

1. Each of the three expressions above for the definite integral over the curve C from a to b require that some description be given of the contour C which joins a to b . Sometimes other notations are also used which are often self-explanatory. For example, the notation

$$\oint_{|z| = 1} f(z)\,dz$$

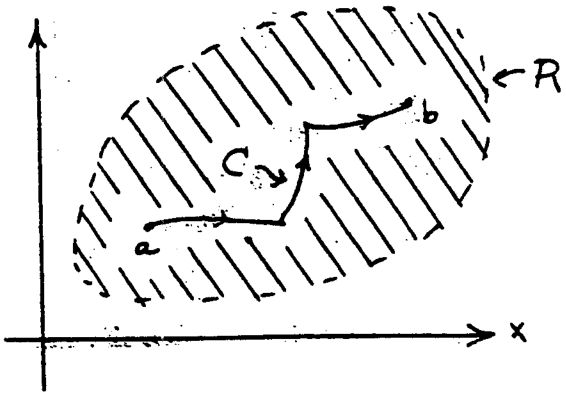
means that f(z) is integrated over the closed curve which is the circle of radius one centered at the origin in the counter-clockwise sense.

2. Note that there is no need to require that f(z) be analytic on the contour of integration C in the above definition of the Riemann integral. If f(z) is simply continuous on C, then the Riemann integral always exists provided the length of C is finite.

6.3 The Fundamental Theorem of the Integral Calculus

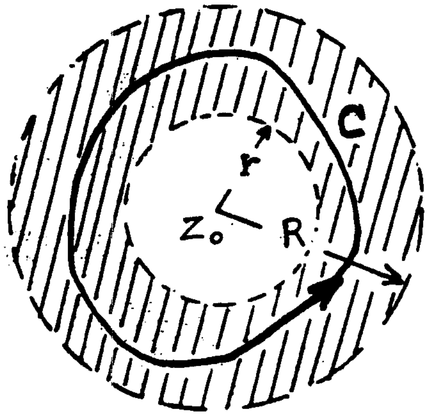
Let $f(z)$ be an analytic function for each z on an open set R and let C denote a directed contour from $z=a$ to $z=b$ in the set R. Then there exists an analytic function $F(z)$ on R such that $F'(z) = f(z)$ and

$$\int_{a}^{b} f(z)\, dz = F(b) - F(a) \, .$$
$$C$$



6.4 Laurent's Theorem

Let $f(z)$ be analytic for all z in the annulus $r < |z-z_0| < R$. Then $f(z)$ can be expanded in a convergent series at each point of this annulus given by



$$f(z) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi i} \oint_{C} \frac{f(t)\, dt}{(t-z)^{n+1}} \quad (z-z_0)^n \, ,$$

where the simple closed curve C circles the annulus as shown and does not extend outside the annulus.