AN INTUITIVE INTRODUCTION TO COMPLEX ANALYSIS

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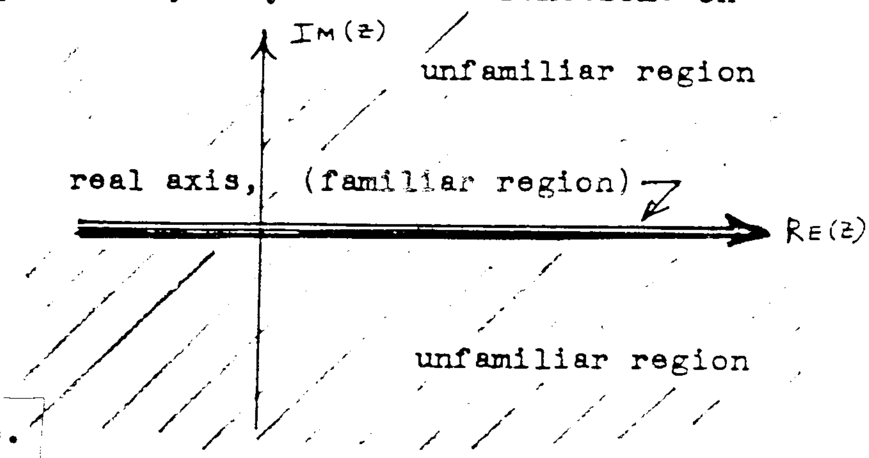
CHAPTER 2

ELEMENTARY FUNCTIONS

The reader has become familiar with many functions. They include x^2, $1/x$, e^x, $\sin x$, $\log x$, $\sinh x$, x^p, and many many more. In previous studies, the independent variable x was always real. Now we replace x by $z = x + iy$ and ask ourselves how the old familiar functions behave now that iy has been added to x. In other words, we previously only saw these functions on that narrow subset of the complex plane called the real axis. We now wish to enlarge our view of these functions by examining their behavior for all points in the complex plane.



There are, of course, many ways in which we can arbitrarily define a given function on the unfamiliar region of the complex plane. For example, we could define $\sin z = \sin(x + iy)$ to be $\sin(x)$, or $\sin x + i \sin y$, or $\sin x + iy$, etc., for all of these would become the familiar sine function when we return to the real axis by setting $y = 0$. However, we are not interested in letting our imaginations roam wildly. Definitions chosen arbitrarily are not likely to preserve many of the relations such as

$$\sin 2z = 2 \sin z \cos z,$$
$$\sin^2 z + \cos^2 z = 1,$$

etc., We seek NATURAL definitions. Natural definitions will evolve by examining formulas and expressions which reason tells us are probably correct. Only in this way will the new mathematics generated be likely to preserve features of the functions already familiar to us. Perhapse, the new mathematics will even lead to new and undxpected insights into the calculus. In the following sections we will explore ways in which natural definitions can be selected.

2.1 Functions and their graphs

In the study of the real calculus, we usually used the notation $y = f(x)$, with the understanding that x is the independent variable and y is the dependent variable. In our complex calculus, we shall most often write $w = f(z)$, where $z = x+iy$ is the independent variable, and $w = u+iv$ is the dependent variable.(x, y, u and v are all real variables.) Note that graphing $w = f(z)$ involves four real variables, and we shall discuss this difficulty momentarily.

As an example, let us examine the function $w = z^2$. It is most natural to simply replace z by $x+iy$ and square to get

$$w = (x+iy)^2 = x^2 - y^2 + 2xyi .$$

Since $w = u + iv$ we have

$$u = x^2 - y^2 \quad \text{and} \quad v = 2xy.$$

It is sometimes useful to use polar coordinates. Writing $z = r\,e^{i\theta}$, and $w = \rho e^{i\phi}$ we have for the function $w = z^2$

$$\rho e^{i\phi} = r^2\, e^{i2\theta}$$

and thus $\rho = r^2$ and $\phi = 2\theta$.

Problems:

1. Let $z = x+iy$ and $w = u+iv$. Find u and v as functions of x and y. (a) $w = 1/z$, (b) $w = z^2 + 2z$, (c) $w = z^{-2}$, (d) $w = z^3$.

2. Let $z = r e^{i\theta}$, and $w = \rho e^{i\phi}$. Find ρ and ϕ as functions of r and θ. (a) $w = 1/z$, (b) $w = z^3$, (c) $w = \bar{z}$, (d) $w = |z|$.

How can we visualize complex valued functions of a complex variable ? The four dimensional nature of the expression $u+iv = f(x+iy)$ makes this difficult, but nevertheless, good techniques are available.

One method is to plot lines of constant u and constant v directly over the complex z- plane. Then for each value of z, we can estimate u and v and thereby approximate $w = u+iv$.

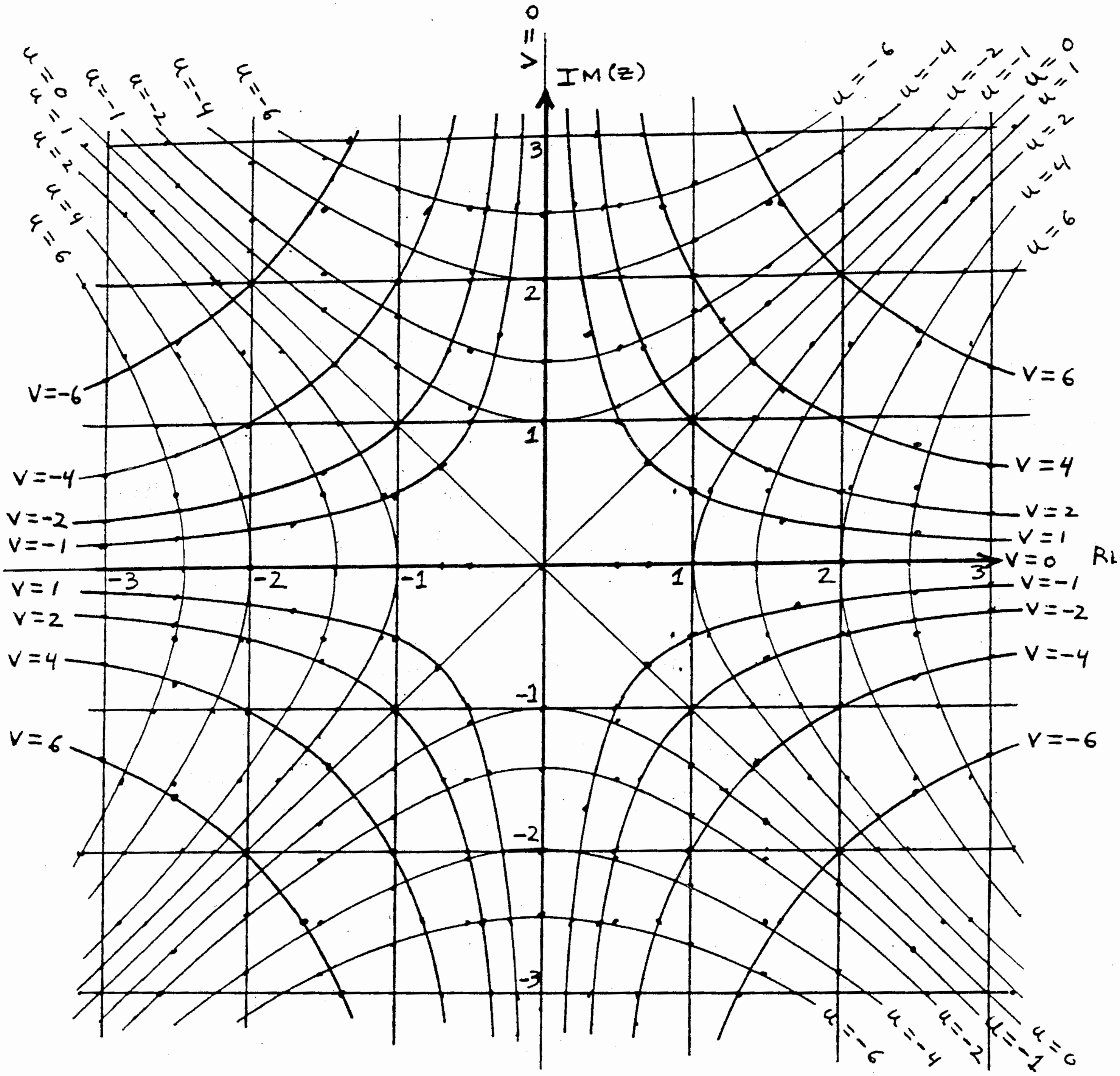
As an example, consider the function $w = z^2$. Previously we showed that $u = x^2-y^2$ and $v = 2xy$. Our previous experience in analytic geometry reveals that the curves u = constant and v = constant are hyperbolas. These "level lines" are shown in Figure 2.1 . This "contour map" of the function $w = z^2$ shows that at $z = 1+i$, $u = 0$, and $v = 2$; thus $w = 2i$.

Problems:

3. Estimate the value of $w = z^2$ from Figure 2.1 at the following points: (a) $z = 2-i$, (b) $z = -1+2i$, (c) $z = 1.5 -2i$, (d) $z = -i$, (e) $z = 2.25 - 1.75i$.

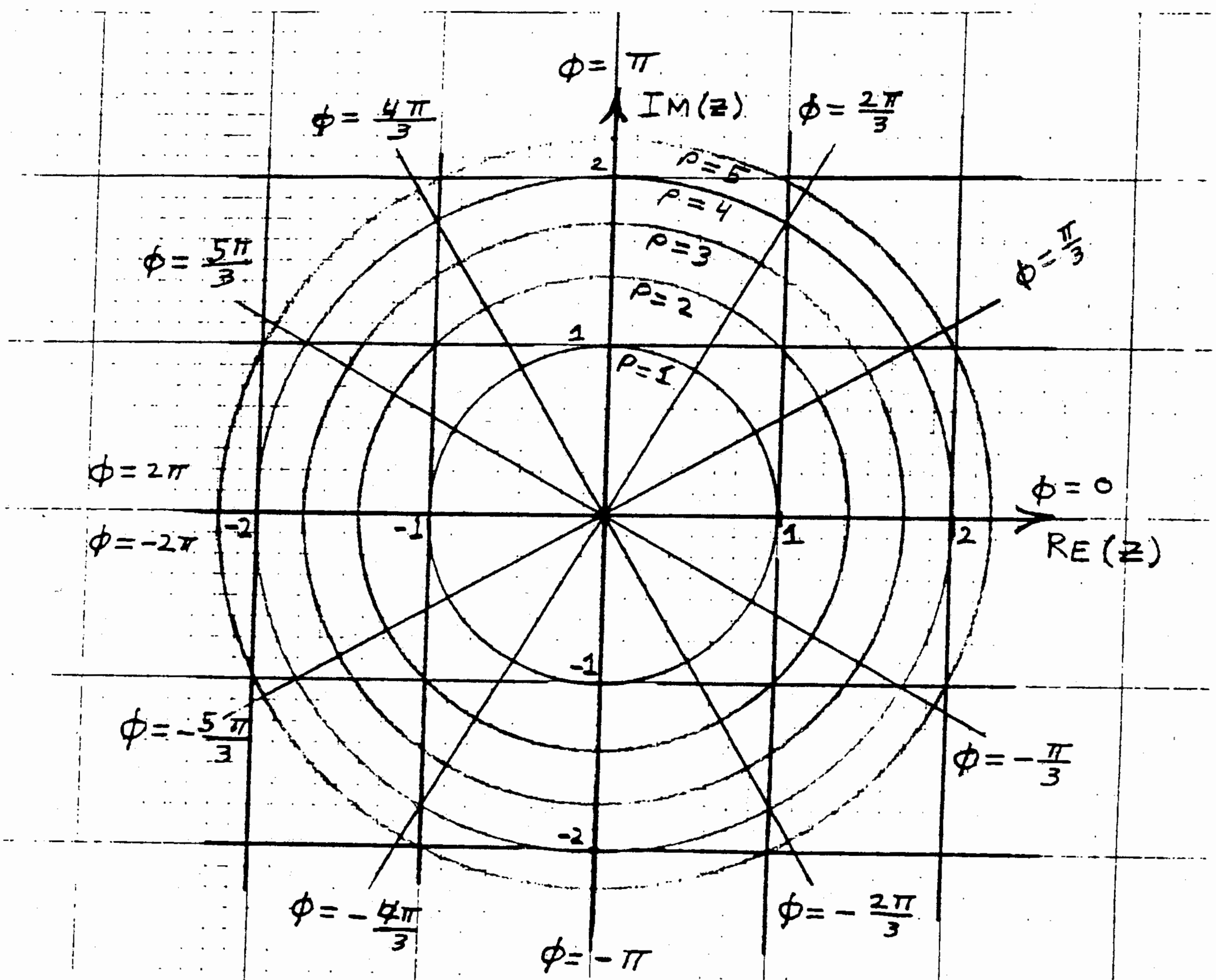
4. Estimate the values of z associated with the following values of w given by the function $w = z^2$. Use Figure 2.1 .
(a) $w = 2i$, (b) $w = -6-6i$, (c) $w = -i$, (d) $w = 2+2i$.

5. Using $u = x^2-y^2$ and $v = 2xy$, sketch the graphs of $u = -4$ and $v = 2$. Check your results by comparing them with Figure 2.1.



<u>Figure 2.1</u> The level lines u = constant and v = constant for the function $w = z^2$ plotted over the complex z plane. ($w = u + iv$, and $z = x + iy$.)

Another "contour map" for $w = z^2$ can be given in which we draw lines of constant modulus of w ($|w| = \rho$) and lines of constant argument of w ($\arg(w) = \phi$) directly over the complex z-plane. We demonstrated previously that $\rho = r^2$ and $\phi = 2\theta$. This contour map is shown in Figure 2.2.



~~Figure 2.2 Level lines for the function $w = z^2$. $w = \rho e^{i\phi}$~~
Lines of constant modulus ρ, and lines of constant argument ϕ are plotted above the complex z-plane.

From Figure 2.2 we can also estimate the values of the function $w = z^2$. As an example, examine the point $z = -1 + 1.75\,i$. Here we see that $\rho = 4$ and $\phi = 4\pi/3$. Thus $w = 4\,e^{i4\pi/3} = -2-2\sqrt{3}\,i$. Look closely at the values of ϕ along the negative real axis. Notice that ϕ approaches the value -2π from below the axis, and 2π from above the axis. This abrupt change in ϕ does not, however cause an abrupt change in the value of $w = \rho e^{i\phi}$ since $e^{-2\pi i} = e^{2\pi i} = 1$.

Problems:

6. From Figure 2.2, estimate the values of the function $w = z^2$ at the following points: (a) $z = 2$, (b) $z = -1.1 + 2i$, (c) $z = 1.5 - 0.9i$.

7. From Figure 2.2, determine the values of z associated with each of the following values of w governed by the equation $w=z^2$. Notice that for each value of w, there are two values of ϕ, one in the range $-2\pi < \phi \leq 0$, and the other in $0 < \phi \leq 2\pi$.
(a) $w = \sqrt{3} + i$, (b) $w = 4\,e^{i2\pi/3}$, (c) $w = -1$.

8. Construct similar "contour maps" for the functions (a) $w = z^3$, and (b) $w = 1/z$, which show level lines for the modulus and the argument of w.

The level lines shown in Figure 2.2 can also be described by means of a three dimensional "relief map " shown in Figure 2.3. In this graphic visualization of the function $w=z^2$, the complex z-plane is the base plane, and the vertical altitudes are the moduli ρ of w. The lines of constant argument ϕ are then drawn on this surface. While it is more difficult to use a relief map to actually estimate values of the function, the relief map provides a vivid picture of the behavior of the function.

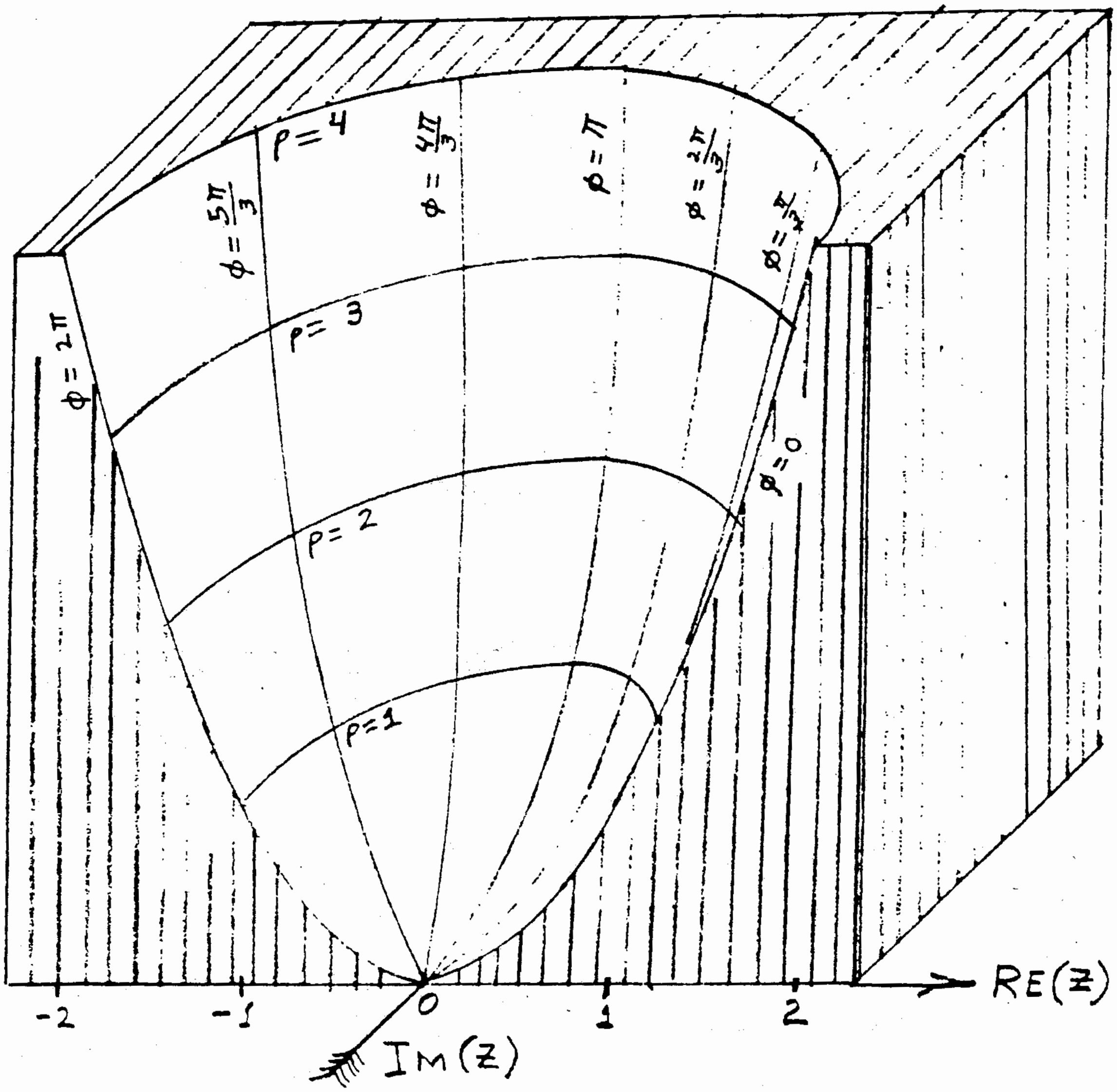


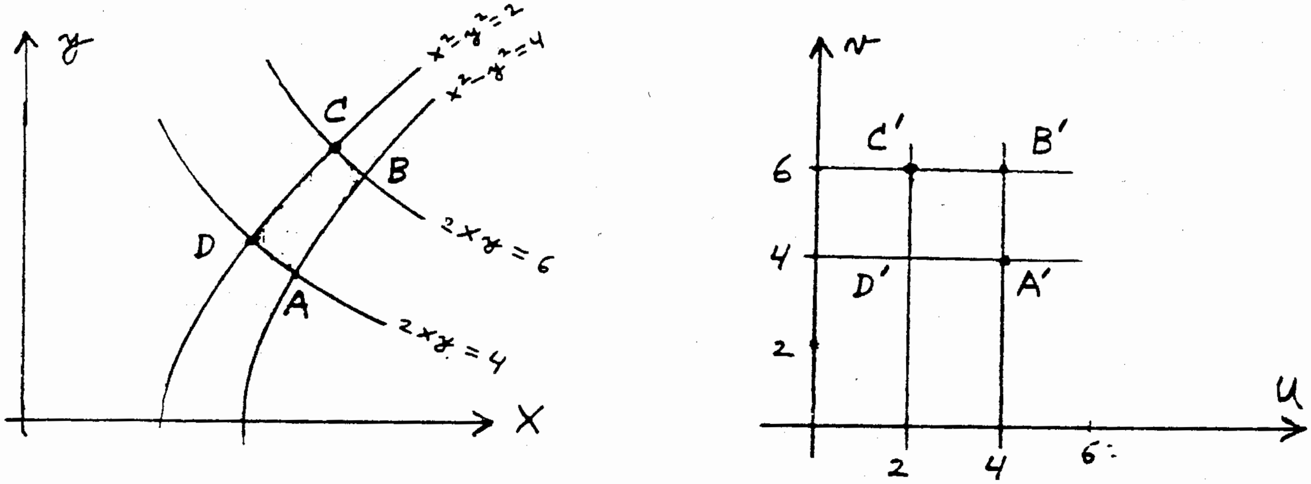
Figure 2.3 Relief map for the function $w = z^2$. The modulus of w is graphed vertically upward over the corresponding points on the complex z-plane. Lines of constant argument ϕ are also shown.

Problem :

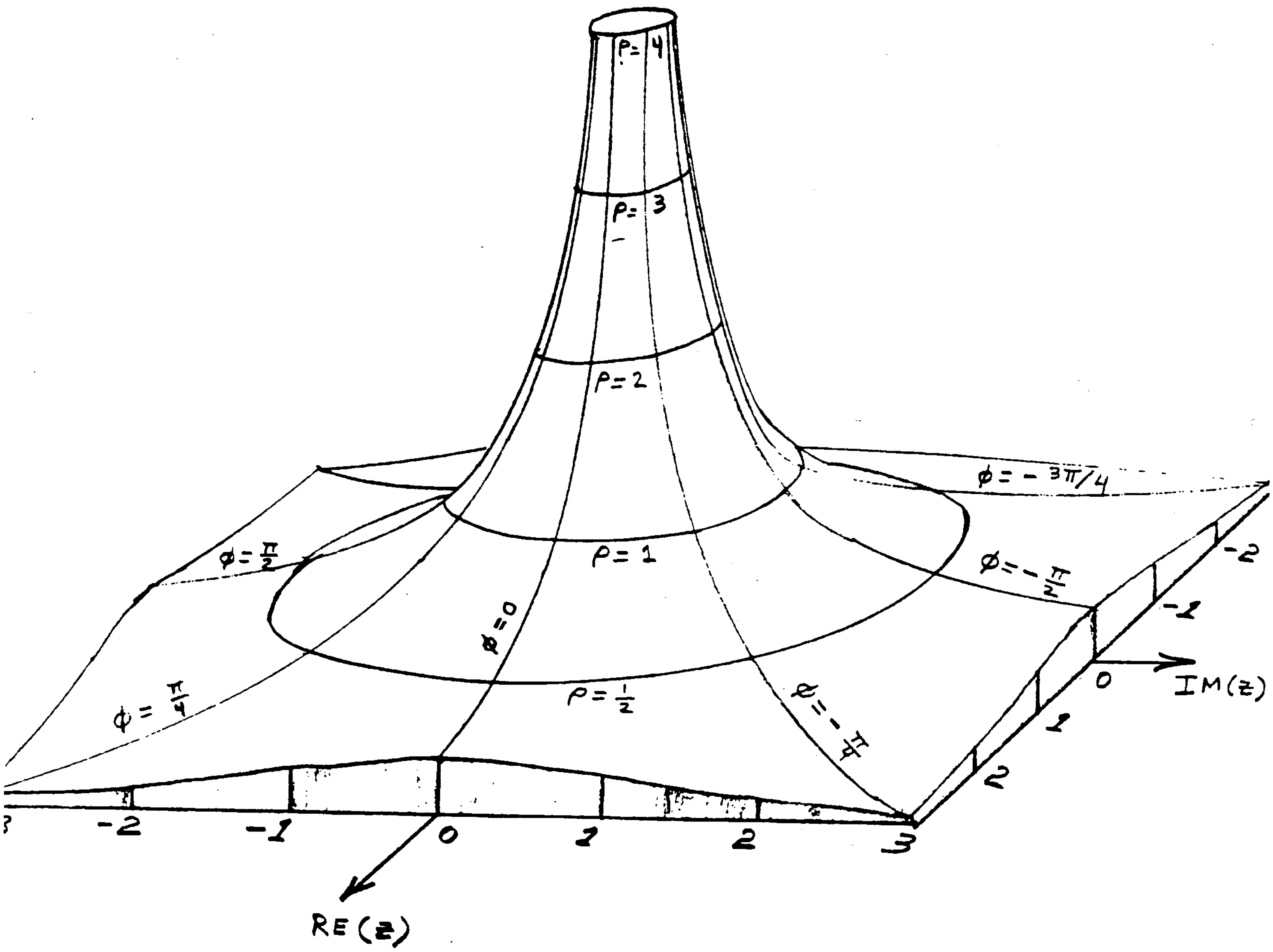
9. Sketch a relief map for the function $w = 1/z$ in which the modulus of w is plotted vertically above the complex z-plane, and the lines of constant argument of w are also drawn on this surface. (See Figure 2.4 for the final relief map.)

It is not surprising that the Relief map for the function $w = 1/z$ shown in Figure 2.4 goes off to infinity at the point $z=0$. Such a point, where the entire relief map is urged to infinity about a particular value of z, is called a "pole" . This is a "singular point" for the function. We will have much to say about singular points later.

We have yet another method for visualizing the behavior of a complex valued function of a complex variable. In this method we reveal certain "mapping properties"of the function $w = f(z)$. That is, we shall determine pairs of regions, one region on the z-plane, and the corresponding region on the w-plane, such that the function $w = f(z)$ maps each point in the z - region onto a point in the w-region. For example, consider the function $w=z^2$. Figure 2.1 reveals that the region in the z-plane bound by the hyperbolas $x^2-y^2 = 2$, $x^2-y^2= 4$, $2xy = 4$ and $2xy = 6$, maps onto the region bound by the lines $u=2, u=4, v=4,$ and $v=6$ in the w-plane.



<u>Figure 2.4</u> Relief map for the function $w = 1/z$ showing the modulus of w plotted vertically over the complex z-plane. Lines of constant argument of w are also plotted on this surface.



Problems:

10. Determine the region in the w-plane obtained when the following region in the z-plane is mapped by the function $w=z^2$.

(a) $0 < y < x$ in the first quadrant.

(b) $1 < x^2 + y^2 < 4$ and $\pi/4 < \arg(z) < \pi/2$.

(c) $0 < \arg(z) < \pi/4$

(d) $0 < \mathrm{Re}(z)$

(e) $0 < xy < 1$ in the first quadrant.

11. Determine the region in the z-plane which corresponds to the following region in the w-plane under the mapping $w = z^2$.

(a) $0 < \arg(w) < \pi/2$, (b) $2<u<4$, $4<v<6$, (c) $-1 < u < 1$, $-1 < v < 1$.

Solving $w = z^2$ for z as a function of w gives $z = \sqrt{w}$. Our experience from working problem 11 reveals that this is a double valued function. That is, under the mapping $z = \sqrt{w}$, two distinct values of z correspond to each given value of w (with the exception of the point $w = 0$). Let us interchange the variables z and w and consider a graphic visualization of the function $w = \sqrt{z}$. Two distinct complex numbers must now be plotted above each point on the z-plane ! How can this be done without much confusion ? We will return to this problem of graphing the function $w = \sqrt{z}$, and we will uncover a new device called the "Riemann-surface" which will aid us in our visualization.

2.2 The exponential function

How should we define the exponential function $w = e^z$? Previously, we obtained a definition for e^{iy} called the Euler formula

$$e^{iy} = \cos y + i \sin y \; .$$

It seems now most natural to write

$$e^z = e^{x+iy} = e^x \, e^{iy}$$

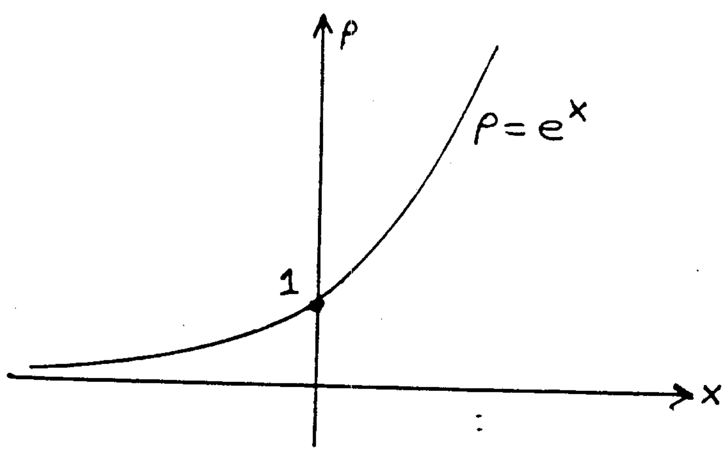
(1) $$\boxed{e^z = e^x(\cos y + i \sin y)} \; .$$

This last expression will be our definition of the exponential function. We will also use the notation $e^z = \exp(z)$. The notation $\exp(z)$ is particularly useful when z is replaced by a large expression. Notice that since e^{iy} has modulus one, the modulus of e^z is

$$\rho = |e^z| = |e^x \; e^{iy}| = |e^x|\cdot|e^{iy}| = |e^x| = e^x \; .$$

Also note that the argument of e^z is simply $\phi = y$. Figure 2.5 is a contour map of e^z showing the lines of constant modulus (ρ = constant) and the lines of constant argument (ϕ = constant). Figure 2.6 gives the corresponding relief map.

An important property of the exponential function is that it is periodic. Recall from the study of trigonometry that the sine function has period 2π, that is to say, $\sin(\theta+2\pi) = \sin \theta$. However, our previous experience in calculus certainly did not reveal that the exponential function was periodic. In fact, the graph of the function $\rho = e^x$ was continually increasing. How then can we say that



the complex exponential function is "periodic"? The answer to this puzzle lies in the fact that e^z repeats itself in the direction of the Imaginary axis of the complex z-plane, and not in the direction of the Real axis. Since we never bothered to look up into the imaginary axis before, it is not surprising that the periodicity of the exponential function escaped us. The period of the exponential fucntion is $2\pi i$. That is to say, $e^{z+2\pi i} = e^z$.

To see this, use (1) and get

$$\begin{aligned} e^{z+2\pi i} &= e^{x+i(y+2\pi)} \\ &= e^x(\cos(y+2\pi) + i\sin(y+2\pi)) \\ &= e^x(\cos y + i \sin y) \\ &= e^z . \end{aligned}$$

This periodicity property of the exponential function is also clear in our two graphic representations. Once we know the values of e^z on any horrizontal strip of the complex z-plane of the form $a < \text{Im}(z) < a + 2\pi$, then we can get all values of e^z by simply placing identical strips directly above and below the original one.

Problems:

12. Let $z=x+iy$ and $\omega = \zeta + i\xi$. Show, using the defining relation (1) that: (a) $e^z e^\omega = e^{z+\omega}$; (b) $e^z/ e^\omega = e^{z-\omega}$; and (c) $(e^z)^N = e^{Nz}$, where $N = 1, 2, 3, 4, \ldots$.

13. Using the function $w = e^z$, and Figure 2.5, map the following regions on the complex z-plane into the complex w-plane:
(a) $0 < y < \pi$; (b) $0 < x$, $0 < y < \pi/2$; (c) $-2 < x < 2$, $2\pi < y < 3\pi$.

14. Find the points on the z-plane which map onto the first quadrant of the w-plane under the mapping $w = e^z$.

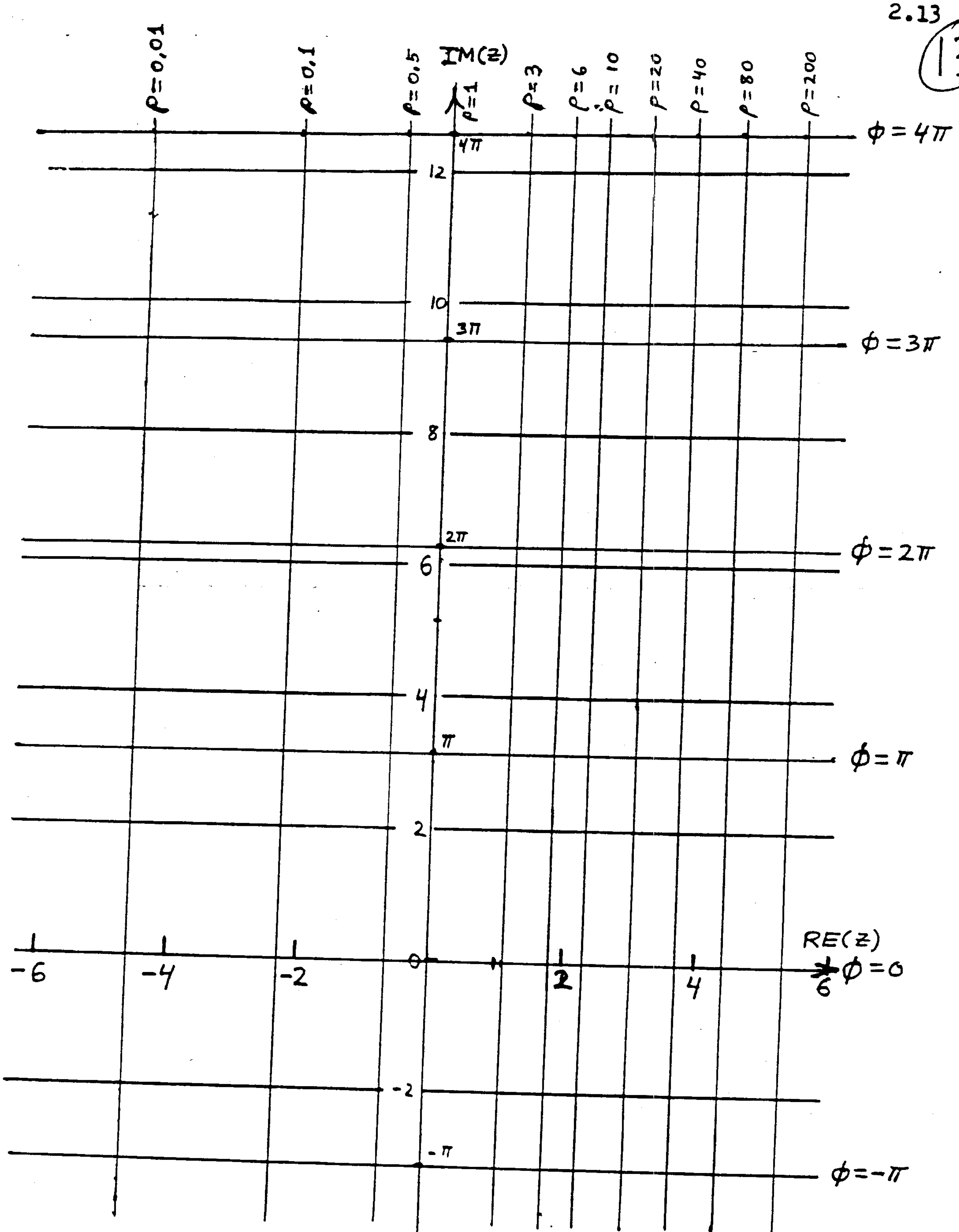
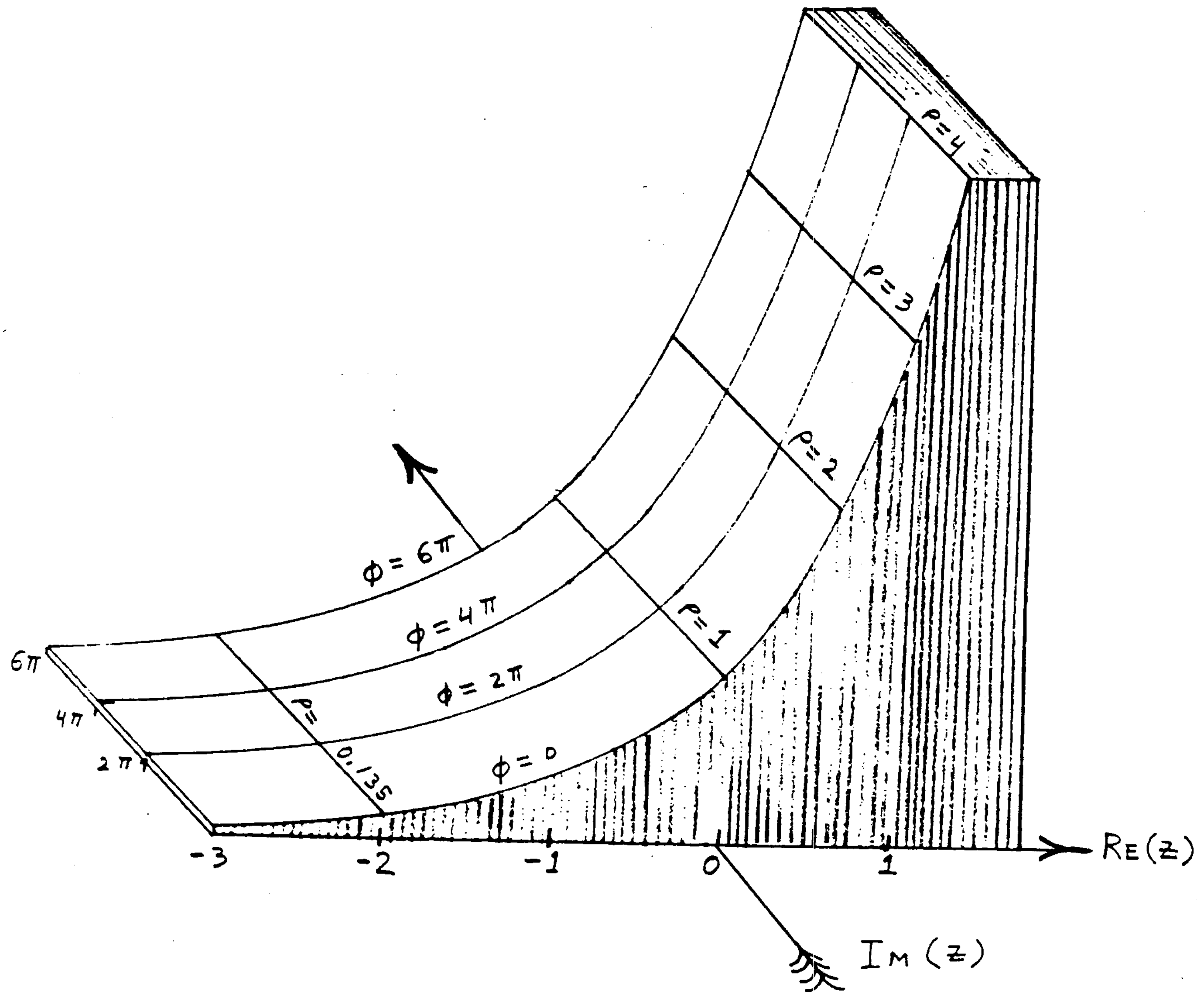


Figure 2.5: Lines of constant modulus and lines of constant argument for the exponential function $w = e^z$.



<u>Figure 2.6:</u> Relief map for the exponential function $w = e^z$, showing the modulus $\rho = e^x$ plotted vertically above the complex z-plane.

2.3 The trigonometric functions

We now seek natural definitions of the trigonometric functions sin z, cos z, tan z, etc., for complex values of z. How can we do this ? We know how to compute e^z, and from Euler's formula we can relate the exponential function to the cosine and the sine

(1) $$e^{i\theta} = \cos\theta + i\sin\theta \,.$$

Replacing θ by $-\theta$, and recalling that $\cos(-\theta) = \cos\theta$, and $\sin(-\theta) = -\sin\theta$ we have

(2) $$e^{-i\theta} = \cos\theta - i\sin\theta \,.$$

Adding (1) and (2) we get

(3) $$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \,,$$

and subtracting (2) from (1) we get

(4) $$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \,.$$

Formulas (3) and (4) are natural expressions because they have evolved by standard manipulations from Euler's formula. Why not replace the real θ in these two expressions by complex. z and get

(5) $$\cos z = \frac{e^{iz} + e^{-iz}}{2} \,, \quad \text{and}$$

(6) $$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad ?$$

Since we know how to compute e^{iz} and e^{-iz} from the previous section, (5) and (6) provide us with definitions for the sine and the cosine which are meaningful for all values of z .

Example:

Compute $\cos i$.

Solution:

From (5) we have

$$\cos i = \frac{e^{ii} + e^{-ii}}{2} = \frac{e^{-1} + e^{1}}{2} = 1.54308 .$$

Values of the $\sin z$ can also be found from the contour map shown in Figure 2.7 . Here the modulus $\rho = |\sin z|$ and the argument $\phi = \arg(\sin z)$ are plotted over the complex z-plane. Figure 2.8 shows the corresponding relief map.

As an example of the use of Figure 2.7, let us find $\sin(1 + 0.5i)$ At the point $x = 1$, $y = 0.5$ we see that ρ is about 1 and ϕ is about $\pi/12$. Therefore

$$\sin(1+0.5i) \gtrsim 1\, e^{i\pi/12} = \cos 15^{\circ} + i \sin 15^{\circ} = 0.97 + i\, 0.26 .$$

Problems:

15. Use equation (6) to compute $\sin(1+i)$. Check your result by using Figure 2.7.

16. Using equations (5) and (6), write expressions for $\tan z$, $\csc z$, $\sec z$, and $\cot z$ in terms of the exponential function.

From (5) we can find the real and imaginary parts of $\cos z$. Recall that we seek realfunctions $u(x,y)$ and $v(x,y)$ such that $\cos z = u + iv$.

17

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$= \frac{e^{-y+ix} + e^{y-ix}}{2}$$

$$= \frac{e^{-y}}{2} (\cos x + i \sin x) + \frac{e^{y}}{2} (\cos x - i \sin x)$$

$$= \frac{e^{y} + e^{-y}}{2} \cos x - i \frac{e^{y} - e^{-y}}{2} \sin x .$$

Recall from the calculus that

$$\cosh y = \frac{e^{y} + e^{-y}}{2}, \quad \text{and}$$

$$\sinh y = \frac{e^{y} - e^{-y}}{2} .$$

Therefore

(7) $\cos z = \cos x \cosh y - i \sin x \sinh y .$

A similar relation for sin z is

(8) $\sin z = \sin x \cosh y + i \cos x \sinh y .$

Problem:

17. Derive (8).

Since our definitions of the trigonometric functions arose in a natural way, it is not unreasonable to expect that the many identities that we learned in trigonometry for real variables remain true for complex variables. This is indeed the case. A sample of such identities includes

(9) $\sin^2 z + \cos^2 z = 1$

(10) $\sin(z+w) = \sin z \cos w + \cos z \sin w$

(11) $\sin 2z = 2 \sin z \cos z .$

To demonstrate (9) simply use the defining relations (5) and (6) to get

$$\sin^2 z + \cos^2 z = \frac{(e^{iz} - e^{-iz})^2}{-4} + \frac{(e^{iz} + e^{-iz})^2}{4}$$

$$= \frac{1}{4} [-(e^{2iz} - 2 + e^{-2iz}) + (e^{2iz} + 2 + e^{-2iz})]$$

$$= 1 .$$

Problem:

18. Derive (11).

Since $\cos z = \sin(z + \pi/2)$, we can use Figure 2.7 to compute $\cos z$ by simply adding $\pi/2 = 1.57\ldots$ to z and then reading off $\sin(z + \pi/2)$. For example, to find $\cos(0.5 + 0.5i)$ we look up $\sin(0.5 + 0.5i + 1.57) = \sin(2.07 + .5i)$ and get approximately $\rho = 1$ and $\phi = -\pi/12$. Therefore

$$\cos(0.5 + 0.5i) \approx 1\, e^{-i\pi/12}$$

$$\approx \cos 15^\circ - i \sin 15^\circ$$

$$\approx 0.97 - i\, 0.26 .$$

Problem:

19. Estimate the value of $\tan(i)$ using Figure 2.7. Check your result using the analytic formula

$$\tan z = -i \frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}} .$$

A glance at Figure 2.8 reveals that all the values of z for which $\sin z = 0$ are on the real axis. Thus we are already familiar with all the zeros of the sine function, which are all the integral multiples of π. In fact, all the zeros of the trigonometric functions occur on the real axis.

Problem:

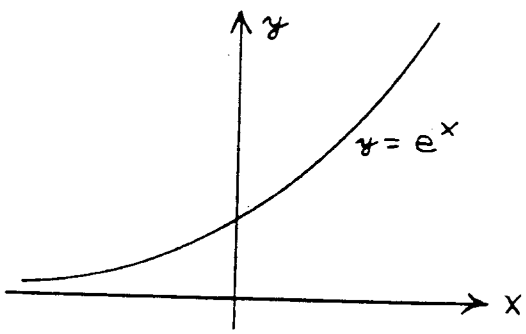
20. Use (8) to find all the zeros of the sine function.

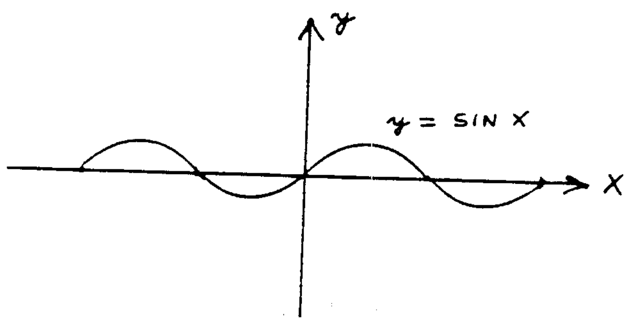
The trigonometric functions also retain their familiar periodicity. However, some features familiar from the real calculus do not carry over into the complex plane. Such is the case for the boundedness of the sine and cosine. In the real calculus we always could rely on the inequalities

$$|\sin x| \le 1 \quad \text{and} \quad |\cos x| \le 1 .$$

However, a glance at Figure 2.8 reveals that these functions grow quite rapidly (in fact, exponentially) as we proceed in either direction along the imaginary axis. In fact, we shall learn later, that the only "natural" functions $w = f(z)$ of a complex variable that are bounded for all z are the constant functions $f(z) \equiv$ a constant .

Finally, we must not fail to recognize a most unexpected feature of our study. In the calculus, the exponential function $y = e^x$ and the sine function $y = \sin x$ seemed quite unrelated.





Their graphs showed no similarity whatsoever. Yet our investigations in the complex plane have revealed that they are intimately related.

The remarkable formula

$$\sin z = \frac{e^{iz} - e^{-iz}}{2}$$

shows this, and moreover, it has made possible the derivation of certain trigonometric identities without appeal to triangles (as in problem 18).

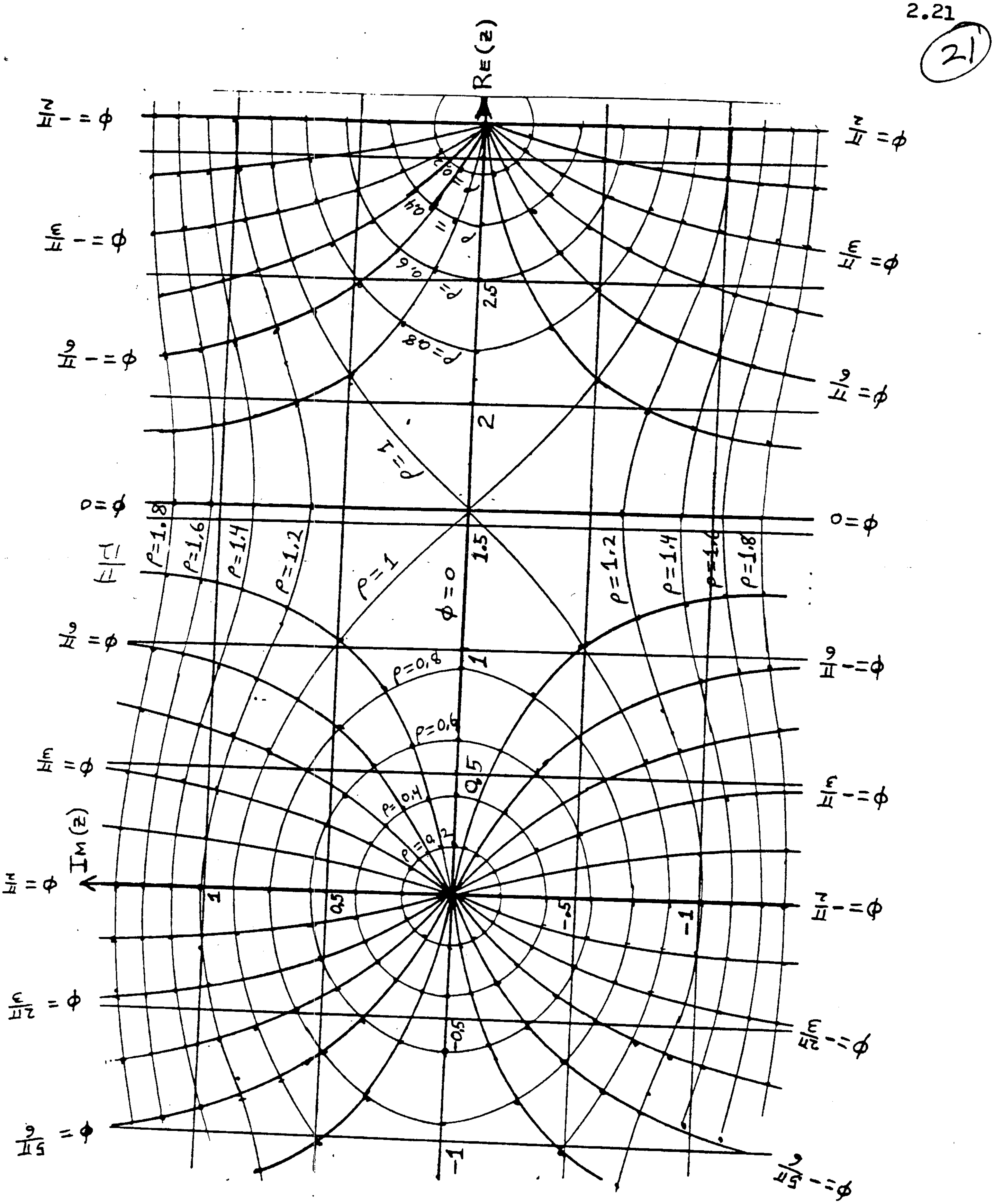


Figure 2.7: Contour map of the function $w = \sin z$. The modulus ρ and the argument ϕ of $\sin z$ are plotted above the z-plane.

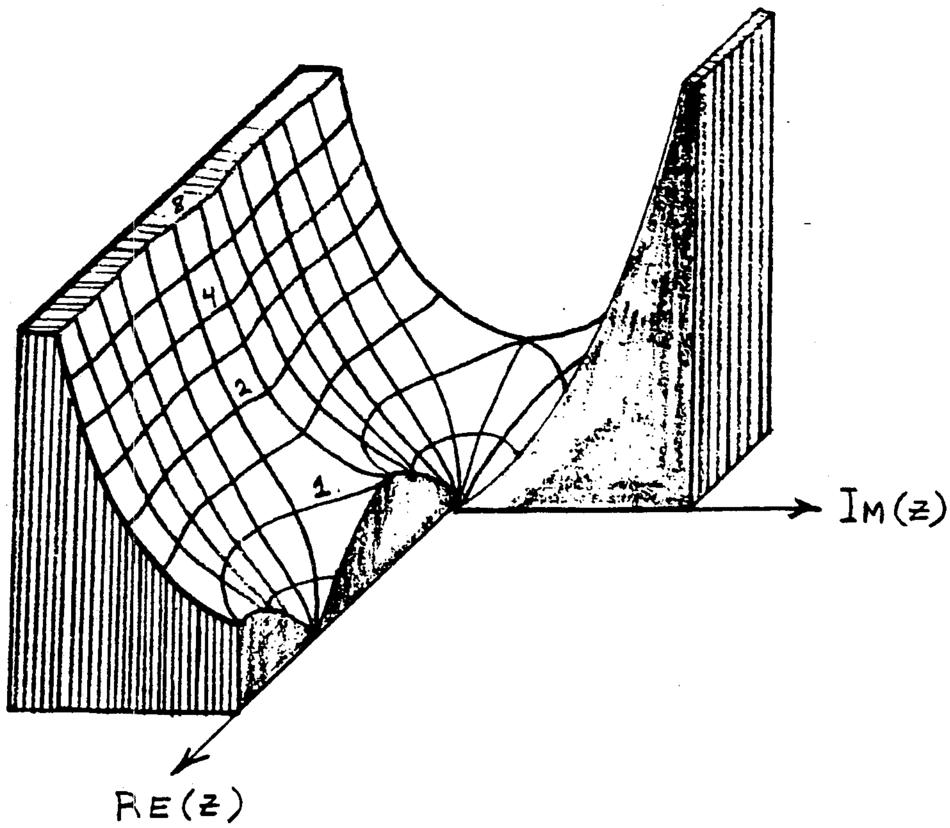


Figure 2.8: Relief map of the function $w = \sin z$. The modulus of the $\sin z$ is plotted vertically above the z-plane to form a surface. Lines of constant argument are also plotted on this surface and appear as lines of greatest descent of the surface.

2.4 The hyperbolic functions

In the real calculus, we encountered the definitions

$$\cosh x = \frac{e^x + e^{-x}}{2} \quad , \text{ and}$$

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad .$$

There is, of course, no difficulty here in replacing real x by complex z since e^z and e^{-z} have already been defined. If we write the definitions we encountered in the previous section alongside of the definitions of the corresponding hyperbolic functions, we notice much similarity:

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \quad , \qquad \cosh z = \frac{e^z + e^{-z}}{2}$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad , \qquad \sinh z = \frac{e^z - e^{-z}}{2} \quad .$$

Replacing z by iz in $\cos z$ we get

$$\cos(iz) = \frac{e^{-z} + e^z}{2}$$

(1) $$\cos(iz) = \cosh z \; .$$

Since we already know how to compute the cosine function, (1) permits us to easily compute the hyperbolic cosine. In a similar manner, replacing z by iz in the $\sin z$ gives

$$\sin(iz) = \frac{e^{-z} - e^z}{2i} = i\,\frac{e^z - e^{-z}}{2}$$

(2) $\sin(iz) = i \sinh z$.

Equation (2) allows us to compute $\sinh z$ when we know $\sin(iz)$.

Example

Find $\cosh(\pi i)$.

Solution

From (1) we see that $\cosh(\pi i) = \cos(-\pi) = -1$.

Example

Prove that $\cosh^2 z - \sinh^2 z = 1$.

Solution

From (1) and (2) we have

$$\cosh^2 z = \cos^2(iz)$$
$$-\sinh^2 z = \sin^2(iz) .$$

Adding these last two relations we have $\cosh^2 z - \sinh^2 z = \cos^2(iz) + \sin^2(iz) = 1$.

The previous example illustrates that not only do the hyperbolic identities resemble the trigonometric identities, they are the very same identities in a new notation ! When we first studied the hyperbolic functions in the real calculus, we were quite surprised to find that functions defined in terms of the exponential could have so many relations which resembled similar relations for the seemingly unrelated trigonometric functions. These relations included

(3) $\sinh 2z = 2 \sinh z \cosh z$

(4) $1 - \tanh^2 z = \operatorname{sech}^2 z$

(5) $\cosh(z + w) = \cosh z \cosh w + \sinh z \sinh w$.

Our recent study of these functions in the complex plane has removed all elements of surprise, and shows that such relations are indeed to be expected.

Problems:

21. Find (a) $\sinh(3\pi i)$, (b) $\cosh(3\pi i)$, (c) $\tanh(3\pi i)$, and (d) $\sinh(1-i)$.

22. Write definitions of the remaining hyperbolic functions in terms of the exponential function.

23. Using (1) and (2), and the appropriate trigonometric identities, derive (3), (4) and (5).

2.5 The function $w = \sqrt{z}$.

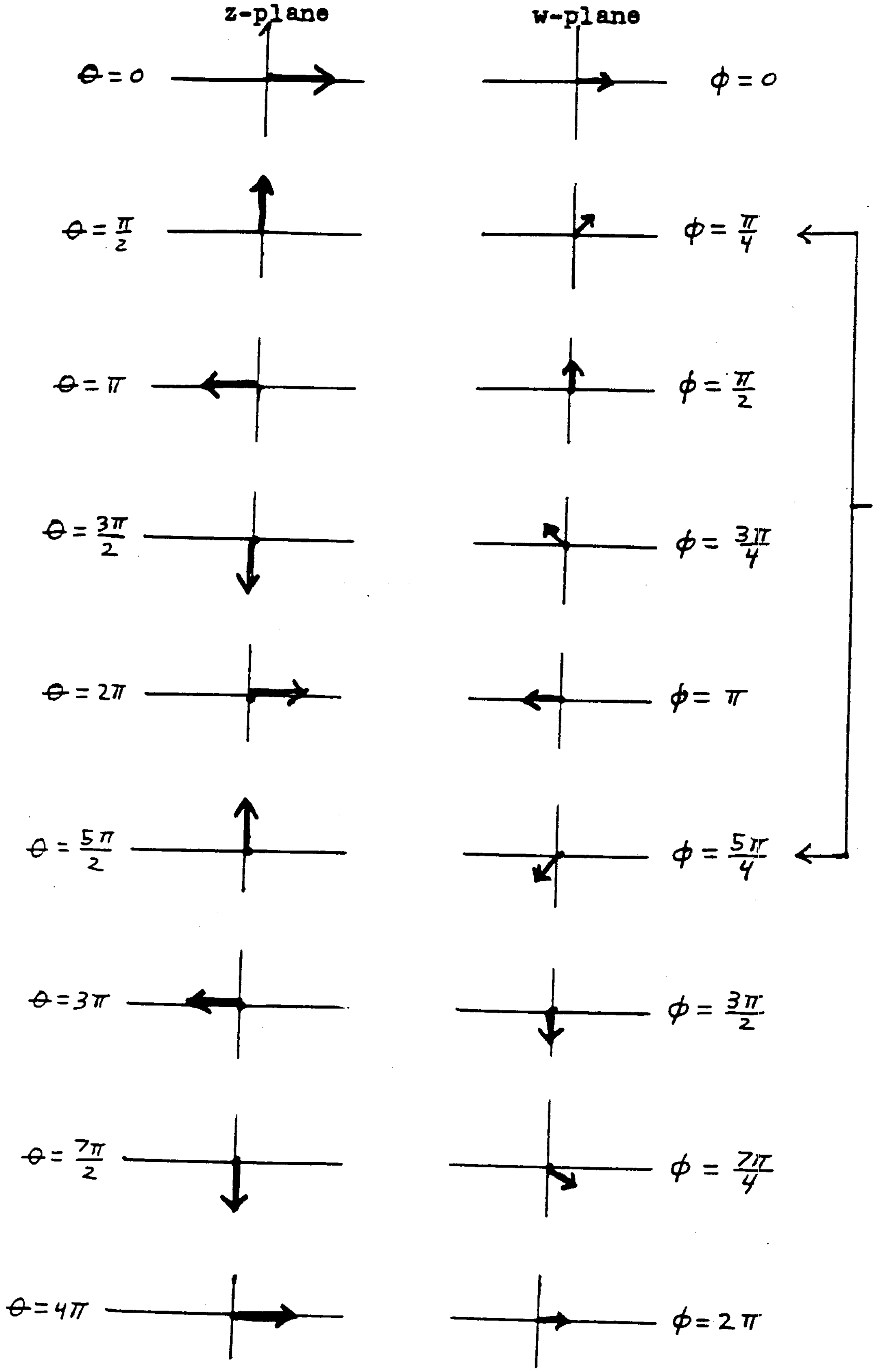
Let us now investigate the function $w = \sqrt{z}$. The analytic investigation of this function is easy when we replace z by its polar form $z = re^{i\theta}$.

$$w = \sqrt{z} = \sqrt{re^{i\theta}} = (re^{i\theta})^{1/2}$$

(1) $$\sqrt{z} = r^{1/2} e^{i\theta/2} .$$

To comprehend the significance of (1), let us imagine that the z-vector is like the hand of a rotating clock. Hold $|z|$ fixed, and let θ vary continuously starting from $\theta = 0$. Simultaneously, as the z-vector rotates, we watch the corresponding w - vector given by (1), $w = \sqrt{z}$ .See Figure 2.9 .

Figure 2.9 $w = \sqrt{z}$ (The z-vector must make two full revolutions before the w-vector makes one full revolution.)



Notice here that the z-vectors are identical, bu the w-vectors are distinct. The w- vectors are 180 degrees out of phase.

An examination of Figure 2.9 reveals the following important features:

(1) The z-vector must negotiate two full revolutions before the w-vector negotiates one full revolution.

(2) To each z-vector there corresponds two distinct w-vectors. (for example, the z-vectors at $\theta = \pi/2$ and $\theta = 5\pi/2$ are identical, while the w-vectors are out of phase by 180 degrees.) We say that the function $w = \sqrt{z}$ is "double valued".

Example

Find all values of $\sqrt{4i}$.

Solution

Write $4i = 4e^{i\pi/2}$ and $4e^{i5\pi/2}$. Now $\sqrt{4i} = (4e^{i\pi/2})^{1/2} = 2e^{i\pi/4}$, and $\sqrt{4i} = (4e^{i5\pi/2})^{1/2} = 2e^{i5\pi/4} = -2e^{i\pi/4}$. Thus $\sqrt{4i}$ equals $\sqrt{2} + \sqrt{2}\, i$ and $-\sqrt{2} - \sqrt{2}\, i$.

Problems

24. Find all values of $\sqrt{2 + 2\sqrt{3}\, i}$.

25. Consider the function $\sqrt[3]{z}$.

(a) Writing $z = re^{i\theta}$, determine a relation similar to (1) for $\sqrt[3]{z}$.

(b) Let the z-vector rotate and examine the corresponding w-vector as in Figure 2.10.

(c) How many revolutions must the z-vector make before the w-vector makes one full revolution ?

(d) How many distinct w-vectors correspond to each z-vector ? If you have one of these w-vectors, how can you immediately write down the others ?

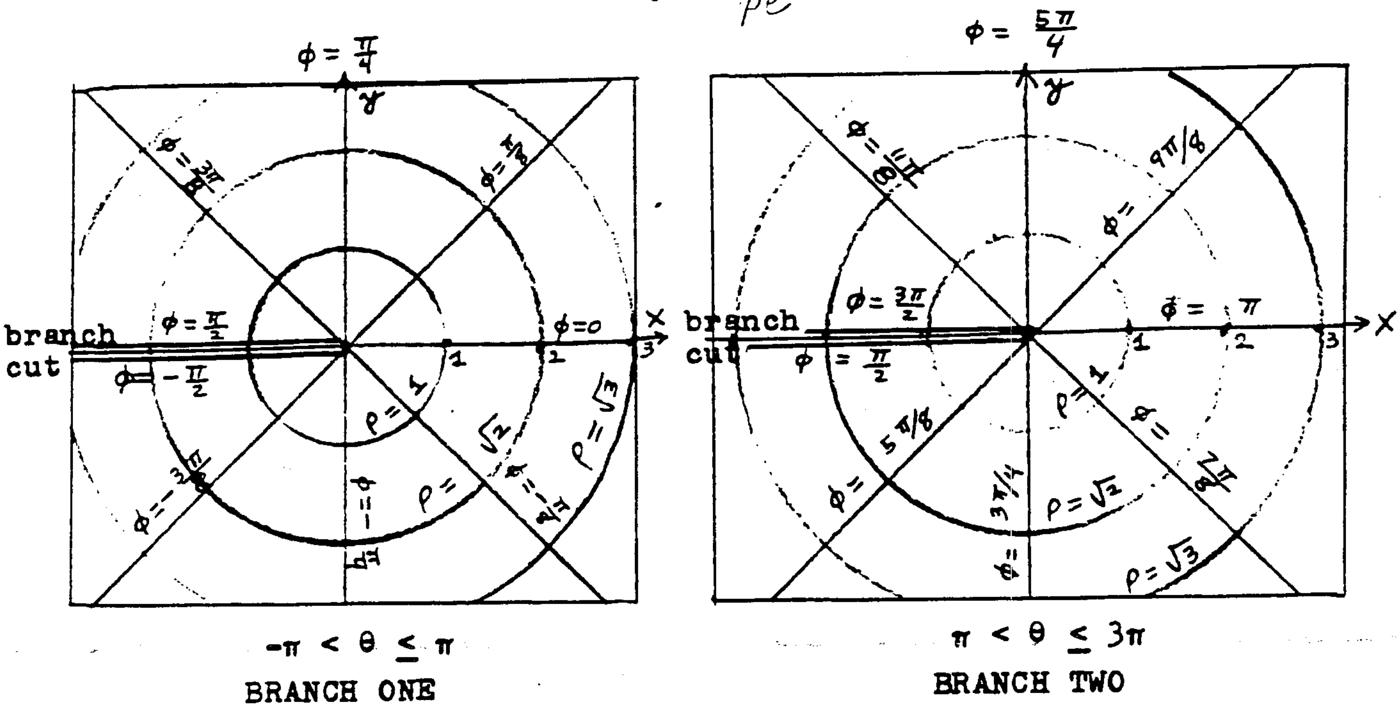
26. Consider the function $\sqrt[N]{z}$, where N is a positive integer. Answer the corresponding questions raised in the previous problem.

The function $w = \sqrt{z}$ is the first function we have encountered which is by nature "multiple-valued". It is convenient to introduce some artificial device which will remove this ambiguity. This idea is not unfamiliar. In the study of trigonometry we encountered the function $\sin^{-1}x$, which has infinitely many values for each x in the domain $-1 \leq x \leq 1$. We called $\text{Sin}^{-1}x$ the "principal value" of the inverse sine function and restricted it to the range $-\pi/2 \leq \text{Sin}^{-1}x \leq \pi/2$ thereby forcing an artificial restriction on the values this function could assume. In this way $\text{Sin}^{-1}x$ becomes single valued. We next introduce a means of removing the ambiguity inherent in our function $w = \sqrt{z}$.

We saw in Figure 2.9 that the z-vector must sweep over the z-plane twice before the w-vector can assume all its possible values. Why not then use two z-planes ? Each z-plane can be used to correspond to "half" of the possible values of w. To be specific, let us assign to $z = re^{i\theta}$ values of θ in the range $-\pi < \theta \leq \pi$ on one plane (called branch one) and assign the values of θ to the range $\pi < \theta \leq 3\pi$ on the second plane (called branch two). Since z executes only one revolution on each of these sheets, only one value of w will correspond to each z, $w = r^{1/2}e^{i\theta/2}$. In Figure 2.10 we plot lines of constant modulus and lines of constant argument for the function $w = \sqrt{z}$ directly over both branches of the function. Notice that on each branch $w = \sqrt{z}$ is single valued. Let us arbitrarily define the "principal value" of $\sqrt{z}$ to be the values obtained on branch one. For example, the principal value of $\sqrt{4i}$ is $2e^{i\pi/4} = \sqrt{2} + \sqrt{2}\ i$. We have therefore, by the introduction of the two "branches" effectively removed the ambiguity in the nature of $\sqrt{z}$.

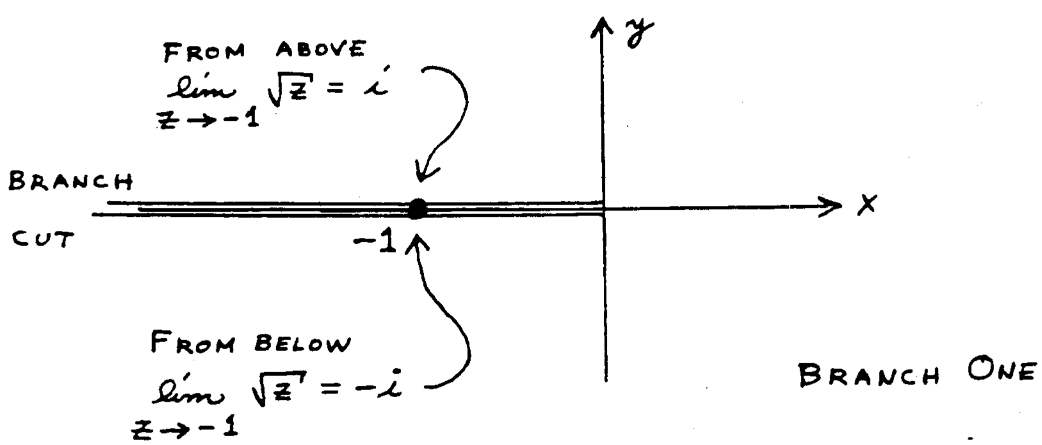
$w = \sqrt{z}$ Level Lines const ρ, const ϕ; $\rho e^{i\phi} = \sqrt{r e^{i\theta}}$

$\rho = \sqrt{r}$, $\phi = \frac{\theta}{2}$



<u>Figure 2.10</u> : Branches for the function $w = \sqrt{z}$. Lines of constant modulus (ρ) and lines of constant argument (ϕ) of $w = \sqrt{z}$ are plotted over each of the branches of the z-plane.

Notice that along the negative real axis of both of our branches in Figure 2.10 , the $\sqrt{z}$ is discontinuous. For example, as we approach the point $-1 = e^{i\pi}$ on the first branch from above, $\sqrt{z}$ approaches $e^{i\pi/2} = i$, but as we approach the same point $-1 = e^{-i\pi}$ from below, $\sqrt{z}$ approaches $e^{-i\pi/2} = -i$.



This line of discontinuities is called a "branch line" or"branch cut" of the function. The two end points of the line are at $z = 0$ and $z = \infty$, and these points are called"branch points" .

We also remark that the choice of the negative real axis as the branch cut was quite arbitrary. We could have choosen the positive real axis, or the positive y-axis, or indeed, any line from $z=0$ to $z = \infty$ as our branch cut. The resulting branches of $\sqrt{z}$ would, of course be different, but the ambiguity in $w = \sqrt{z}$ would again have been removed. While the branch line is arbitrary, the branch points $z = 0$ and $z = \infty$ remain fixed.

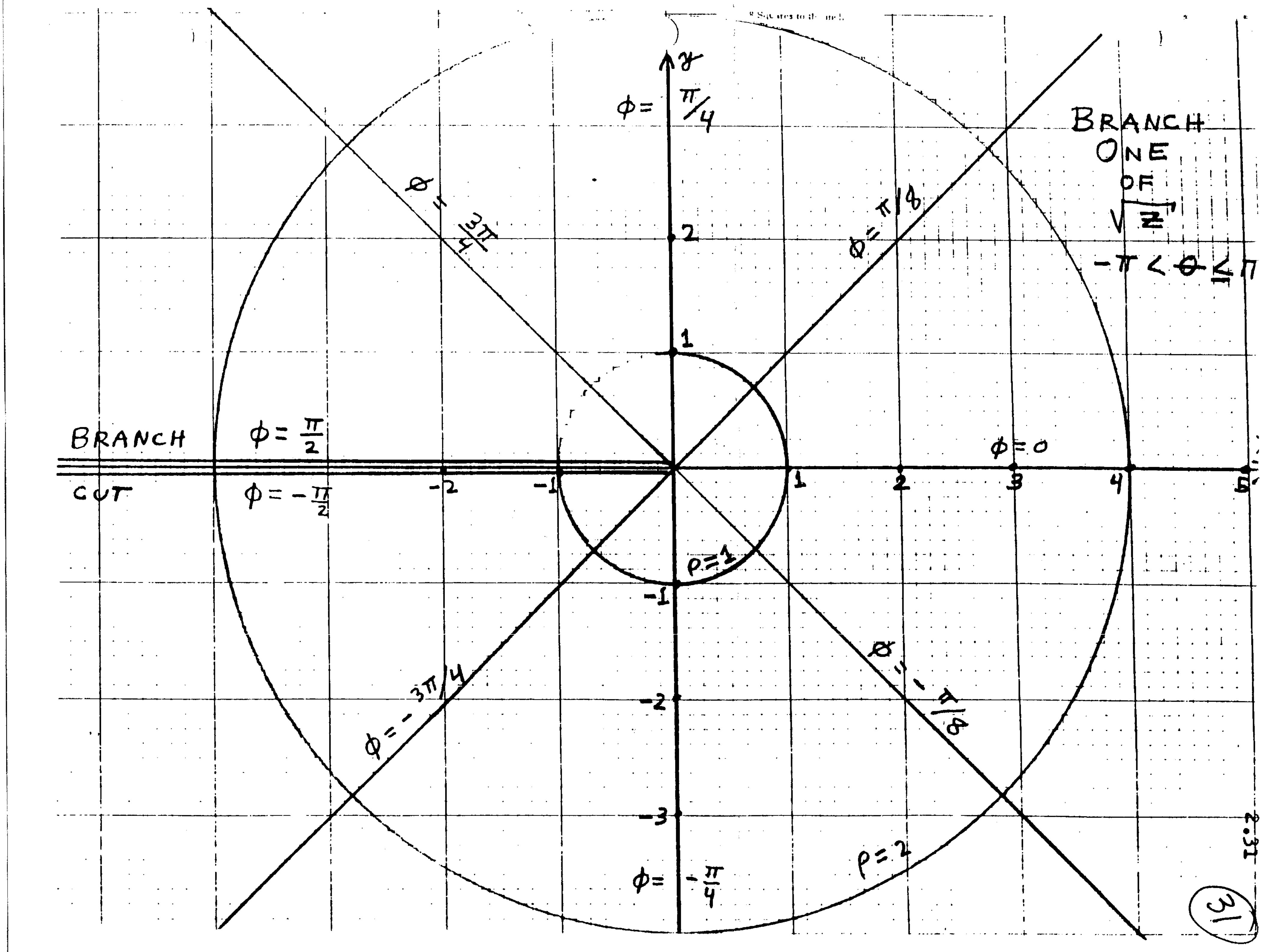
Problems:

27. Introduce new branches for $w = \sqrt{z}$ by taking the negative y-axis as the branch cut. Make a diagram similar to Figure 2.10.

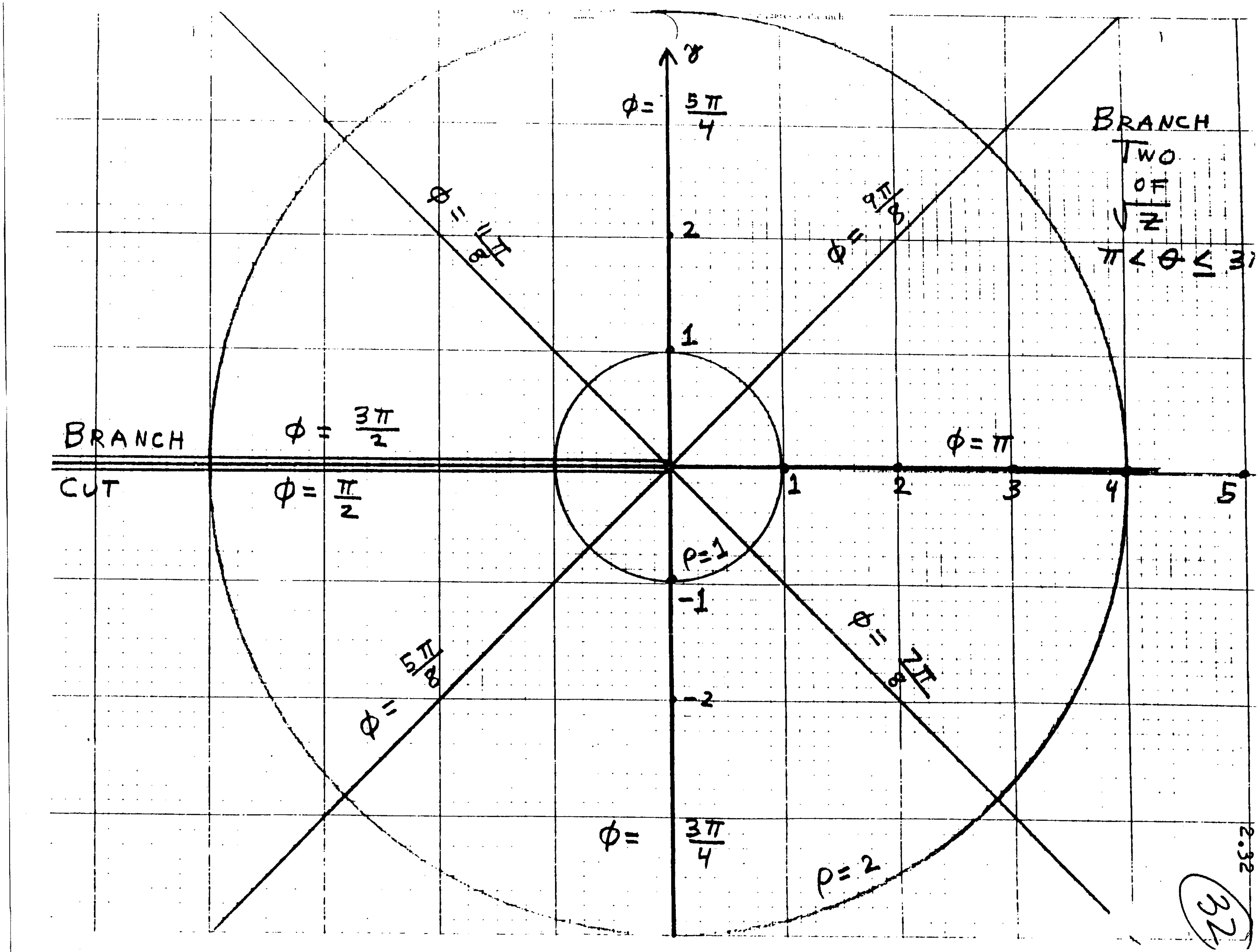
28. Introduce appropriate braches for the function $w = \sqrt[3]{z}$. Select the negative real axis as the branch cut. Make a diagram similar to Figure 2.10.

29. Discuss the problem of introducing branches for the function $w = \sqrt[N]{z}$, where N is a positive integer. How many branches are necessary ?

Besides the introduction of two branches for $w = \sqrt{z}$ to remove the confusion caused by its natural double-valuedness, we have yet another method. In this new method we replace the z-plane by a new surface which is called a "Riemann-surface". The Riemann surface is best understood by actually constructing it. Pages 2.31 and 2.32 show enlarged versions of braches one and two of $\sqrt{z}$ originally described in Figure 2.10. Remove these pages, and actually cut with scissors along the branch line to the branch point at $z = 0$. Lay branch two on top of branch one. Notice that along the top part of the cut on branch one $\phi = \pi/2$,

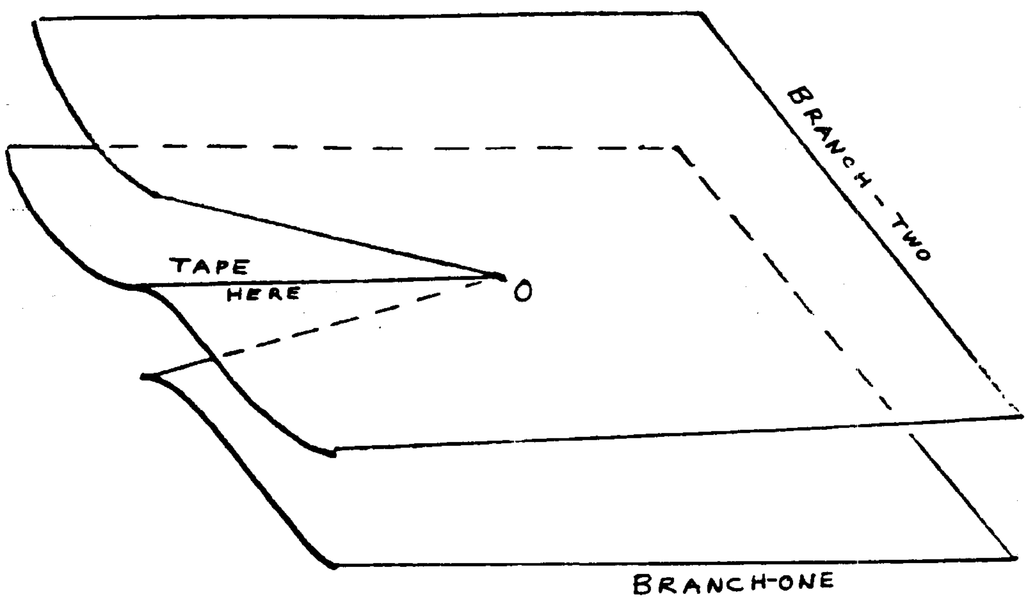


BRANCH ONE OF √z
−π < φ ≤ π
y
φ = π/4
φ = π/8
φ = 3π/4
φ = 0
φ = π/2
φ = −π/2
BRANCH
CUT
φ = −3π/4
φ = −π/8
φ = −π/4
ρ = 1
ρ = 2
1
2
3
4
5
−1
−2
−3
2.31
31

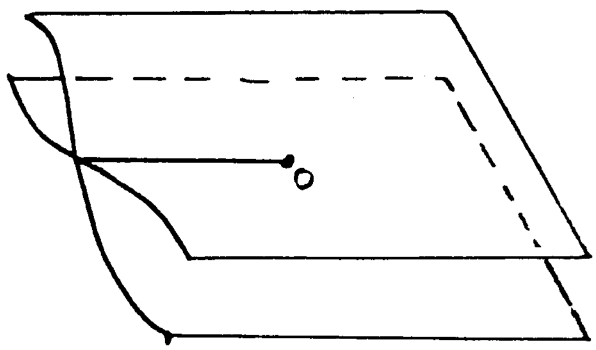


BRANCH TWO OF √Z
π < θ ≤ 3π
γ
φ = 5π/4
φ = 11π/8
φ = 9π/8
φ = 3π/2
φ = π/2
φ = π
BRANCH CUT
ρ = 1
ρ = 2
φ = 5π/8
φ = 7π/8
φ = 3π/4
1 2 3 4 5
2 1 −1 −2
2.32
32

and $\phi = \pi/2$ also on the bottom part of the cut of branch two. Thus ϕ wants to continue from the lower sheet to the upper sheet along this cut. Now actually tape the top edge of the branch cut of the lower sheet to the bottom edge of the branch cut of the upper sheet.



Along the remaining two edges of the branch cuts we have $\phi = -\pi/2$ (branch one) and $\phi = 3\pi/2$ (branch two). Since $e^{-\pi i} = e^{3\pi i}$ we see that the values of $\sqrt{z}$ also join continuously across these edges. We might imagine that these edges are also taped together (although this is physically impossible). This new surface is our "Riemann-surface" for the $w = \sqrt{z}$ function.



The advantage of the Riemann surface over the two separate branches considered previously is that the discontinuities previously seen along the branch cuts are now removed.

Problems:

30. Starting at $z = -1$ on the lower sheet of the Riemann surface for $w = \sqrt{z}$, use a pencil and trace the values of $\sqrt{z}$ encountered as you move along the unit circle $|z| = 1$ in the counterclockwise sense. After making two full revolutions, you should imagine that the starting and ending points are joined together.

31. Make a Riemann surface for $w = \sqrt[3]{z}$.

32. Describe the Riemann surface for $w = \sqrt[N]{z}$, where N is a positive integer.

35

2.6 The natural logarithum

The next function that we wish to explore in the complex plane is the natural logarithum $w = \log z$. How are we to define this function ? The notation $w = \log z$ implies that $z = e^w$. If we write $z = r\,e^{i\theta}$ and $w = u + iv$ we have

$$w = \log z$$
$$z = e^w$$
$$re^{i\theta} = e^{u+iv}$$
$$re^{i\theta} = e^u e^{iv} .$$

This last relation tells us that

(1) $r = e^u$, and

(2) $\theta = v$.

From (1) we have $u = \text{Log } r$, (where we have introduced the notation Log r to mean the familiar real logarithum defined for positive real values of r), Using (2) we get

$$w = \log z = u + iv$$

(3) $w = \text{Log } r + i\theta$.

Relation (3) is our defining relation for the natural logarithum. Since the angle " θ " is not uniquely defined for each z, we see that log z is indeed a multiple valued function.

Example

Find all values of $\log i$.

Solution

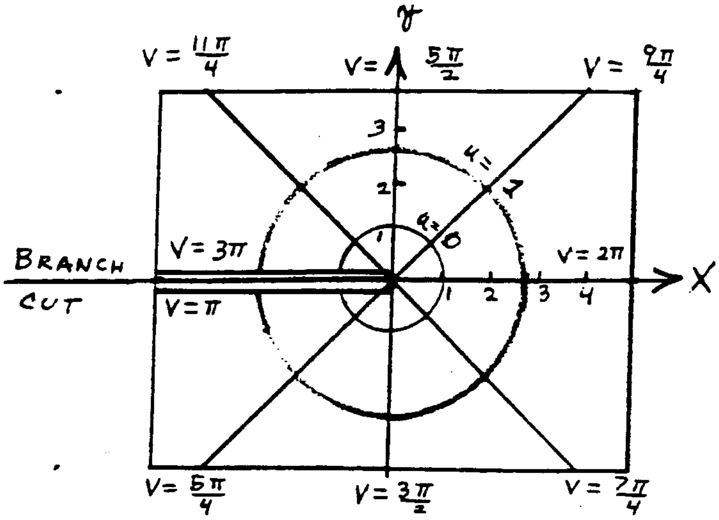
Write i in polar form as $i = 1\,e^{i(\pi/2 + 2\pi n)}$, where $n = 0, \pm1, \pm2, \ldots$. Thus $r = 1$ and $\theta = \pi/2 + 2\pi n$, and (3) gives $\log i = \text{Log } 1 + i(\pi/2 + 2\pi n) = i(\pi/2 + 2\pi n)$.

Problem

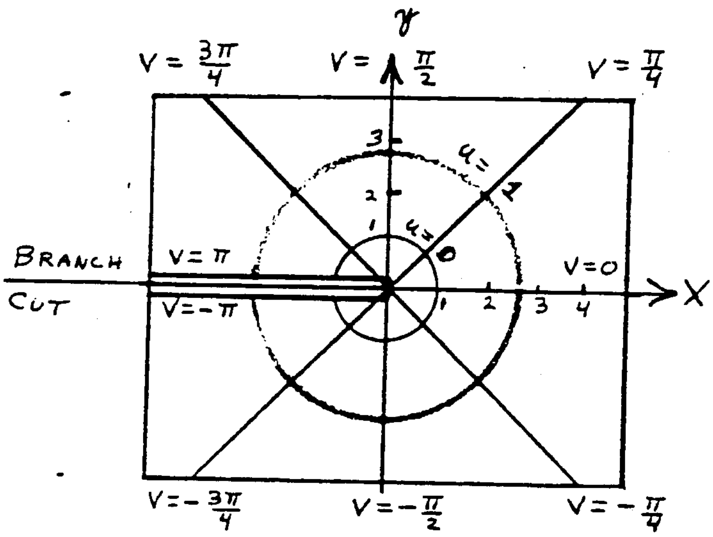
33. Find all values of (a) $\log(-1)$, (b) $\log e$, (c) $\log(1+i)$, (d) $\log 0$.

36

<u>**Figure 2.11**</u> **Branches for the function $w = \log z$. Only three of the infinitely many branches are shown.**

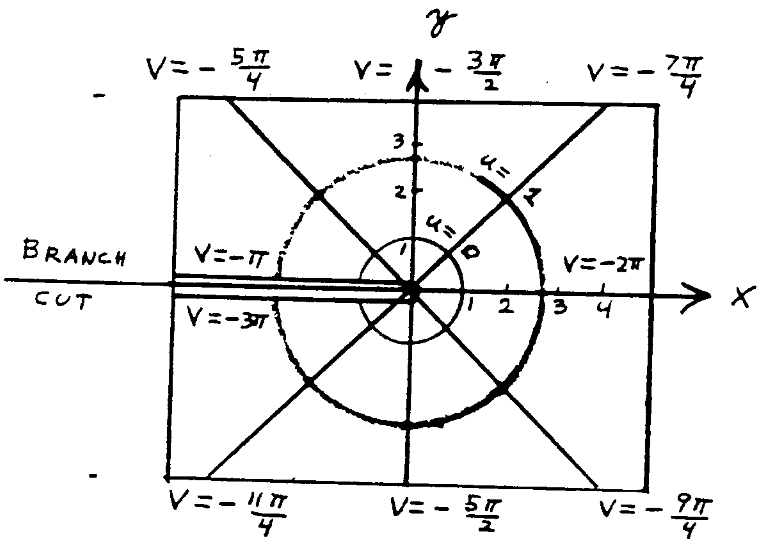


$\pi < v \leq 3\pi$



$-\pi < v \leq \pi$

(Principal Branch, $w = \text{Log } z$)



$-3\pi < v \leq -\pi$

Figure 2.11 shows one method of introducing branches for $w = \log z$. The negative real axis has been selected as the branch cut. Notice that lines of constant u and v have been plotted over each z - plane (in contrast to lines of constant ρ and ϕ used previously).

We notice in particular that :

1. There are infinitely many distinct branches of $w = \log z$, since there are infinitely many values of log z for each z.
2. The branch points are $z = 0$ and $z = \infty$.
3. The branch line is, to a certain degree, arbitrary. We could have selected the negative y-axis as the branch line, or any other line from $z = 0$ to $z = \infty$. (Review problem 27)
3. The function $w = \log z$ is discontinuous at the branch line, because the values of v jump by 2π as we cross the negative x-axis.

We shall arbitrarily call the branch for which $-\pi < v \leq \pi$ the "principal branch" of logarithum of z, and denote it by Log z .

Example

Find (a) Log i and (b) Log(-e) .

Solution

(a) Since we want the "principal value" of the logarithum, we take $i = e^{i\pi/2}$. The value of θ was selected as $\pi/2$ since this value is in the correct range for the principal branch which is $(-\pi, \pi]$. From (3) we have $\text{Log } i = i\pi/2$.

(b) We select $-e = e\, e^{i\pi}$, and get from (3) that $\text{Log}(-e) = \log e + i\pi = 1 + i\pi$.

We recall the following rules for logarithums from study in elementary courses:

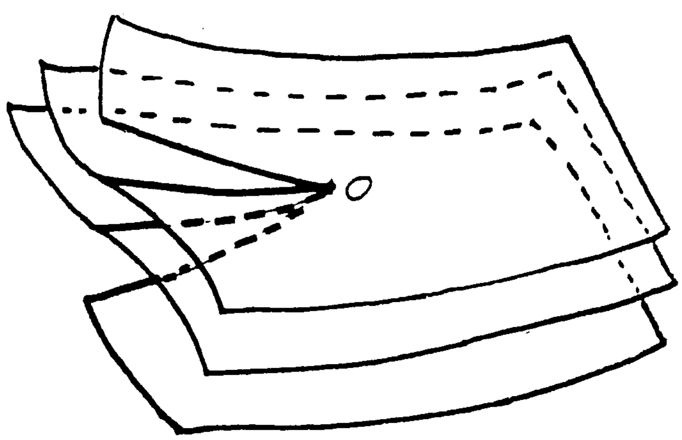
(4) $\log zw = \log z + \log w$

(5) $\log z/w = \log z - \log w$

(6) $\log z^p = p \log z$.

Are these three relations true for complex values of the variables ? Since $\log z$ for complex z was defined in a natural way, we at once anticipate that these relations should be valid. However, when we reflect upon the fact that each of the logarithums in these relations can assume infinitely many values for each fixed z and w , we do indeed anticipate some difficulty. What is true is the following : " For appropriately selected branches of each of the logarithums occuring, the relations (4), (5) and (6) are valid".

We can construct a Riemann surface for $w = \log z$ from the various branches shown in Figure 2.11. The method is the same as that used in the previous section for $w = \sqrt{z}$, only now we have infinitely many sheets for the surface,rather that two. Also, we never join the <u>first</u> and <u>last</u> sheets together (since there is no first or last sheet) !



We will now define z^p, where both z and p are complex numbers. Since by definition

$$c = e^{\log c},$$

we replace c by z^p and get

$$z^p = e^{\log z^p}.$$

Using (6) this last relation becomes

$$(7) \qquad z^p = e^{p \log z}$$

which we will use as our defining relation for z^p.

Example

Find all values of i^i.

Solution

From (7) we get $i^i = e^{i \log i} = e^{i(\pi/2 + 2\pi n)i} = e^{-\pi/2 - 2\pi n}$, where $n = 0, \pm1, \pm2, \ldots$.

We see that z^p has infinitely many values for each z and p. We define the "principal value" of z^p to be

$$(8) \qquad z^p = e^{p \operatorname{Log} z} \quad (\text{ principal value }).$$

Problems:

34. Find (a) Log 1, (b) Log(-1), (c) Log(-ei), (d) Log(1+i) .

35. Find all values of (a) 1^2, (b) $(-1)^2$, (c) $(-ei)^2$, (d) $(1+i)^{1+i}$

36. Find the "principal value" of (a) 1^2, (b) $(-1)^2$, (c) $(-ei)^2$, (d) $(1+i)^{1+i}$.

Review problems for Chapter 2

1. Find the values of each of the following expressions using only the appropriate analytic definitions. Express the results in Cartesian form. (a) $e^{1 + i\pi/4}$, (b) $\sin(\pi/3 + i)$, (c) $\cosh(i\pi/3)$, (d) $\mathrm{Log}(-2 + 2\sqrt{3}\,i)$, (e) $(-2+2\sqrt{3}\,i)^{i}$.

2. Find u and v for $w = (z-1)^3$, where $w = u+iv$.

3. Map the region $\{ z \mid 0 < \arg(z) < \pi/4 \text{ , and } |z| < 2 \}$ onto the w-plane by (a) $w = z^2$, (b) $w = 1/z$, (c) $w = \mathrm{Log}\ z$.

4. Map the region $\{ z \mid 1 < \mathrm{Re}(z) < 2, \text{ and } \pi/2 < \mathrm{Im}(z) < \pi \}$ onto the w plane by $w = e^z$.

5. (a) Describe the Riemann surface for $w = \sqrt{z-1}$. Where are the branch points ? (b) How many sheets are needed to form the Riemann surface for $\sqrt[6]{z}$?

6. Find all values of z such that $e^z = 1 - i$.

SUPPLEMENTARY PROBLEMS

2.1.1 Let $z = x + i y$ and $w = u + i v$. Find u and v as functions of x and y.

(a) $w = 2 z^2 - 3$; (b) $w = (z - 1)(z + 3)$;

(c) $w = z^3 - 3 z^2 + 3 z - 1$, (d) $w = \frac{1}{(z - 1)^3}$.

2.1.2 Let $z = r e^{i\theta}$ and $w = \rho e^{i\phi}$. Find ρ and ϕ as functions of r and θ . (a) $w = \frac{1}{z^2}$; (b) $w = z^5$; (c) $w = 2 z^2 - 3$

(d) $w = \frac{1}{(z - 1)^3}$.

2.1.3 Use Figure 2.1 to estimate the value of z^2 at the following points: (a) $z = 2 + 2 i$, (b) $z = - 1 - 2 i$, (c) $z = 2 i$, (d) $z = - 2 + i$.

2.1.4 Use Figure 2.1 to estimate the values of z associated with the following values of w given by the equation $w = z^2$: (a) $w = - 4$, (b) $w = 2 + 4 i$, (c) $w = - 4 + 4 i$, (d) $w = z + 5 i$.

2.1.5 From Figure 2.2 , estimate the values of the function $w = z^2$ at:

(a) $z = 2i$, (b) $z = 1.75 - i$, (c) $z = 0.7 - 1.2i$.

2.1.6 From Figure 2.2, determine the values of z associated with each of the following values of w governed by the equation $w = z^2$. (For each value of w , there are two values of ϕ , one in the range $-2\pi < \phi \leq 0$, and the other in $0 < \phi \leq 2\pi$.)

(a) $w = 1 - \sqrt{3}\, i$, (b) $w = 4\, e^{-i\pi/3}$, (c) $w = 2$.

2.1.7 Map the following regions onto the w - plane under the mapping $w = z^2$.

(a) $x^2 + y^2 < 4$, $0 < \arg(z) < \frac{\pi}{4}$

(b) $0 < x < y$ in the first quadrant

(c) $\mathrm{Im}(z) < 0$

(d) $\frac{\pi}{4} < \arg(z) < \frac{\pi}{2}$

(e) $0 < xy < 1$ in the third quadrant

2.1.8 Determine the regions in the z - plane which corresponds to the following regions in the w - plane under the mapping $w = z^2$.

(a) $0 < u < 1$, $0 < v < 1$

(b) $0 < \arg(w) < \pi$ (c) $-1 < u < 1$, $4 < v < 6$

2.1.9 Let $w = u + i v = \dfrac{z - 1}{z + 1}$.

Show that $u = \dfrac{x^2 + y^2 - 1}{(x + 1)^2 + y^2}$ and $v = \dfrac{2 y}{(x + 1)^2 + y^2}$.

Also show that the level lines u = constant are circles with center on the real axis passing through $(-1, 0)$, and that the lines v = constant are circles with center on $x = -1$ and passing through $(-1, 0)$. Construct a contour map similar to Figure 2.1 illustrating these level lines over the complex z - plane.

2.1.10 Construct a contour map for the function $w = \frac{z - 1}{z + 1}$ similar to Figure 2.2 in which lines of constant ρ and constant ϕ are sketched over the complex z - plane. (The lines of constant ρ are circles with centers on the real axis. The lines of constant ϕ are circular arcs with centers on the imaginary axis, and which begin and end at the two points $(-1, 0)$ and $(1, 0)$.)

2.1.11 Use the contour map constructed in problem 2.1.9 to demonstrate that any circle in the z - plane maps onto a circle in the w-plane under the mapping $w = \frac{z-1}{z+1}$. (In fact, the general bilinear function $w = \frac{az+b}{cz+d}$ is circular.)

2.2.1 Using the function $w = e^z$ and Figure 2.5 , map the following regions onto the complex w - plane :

(a) $-\pi < y < 0$; (b) $x < 0$, $0 < y < \frac{\pi}{2}$;

(c) $0 < x < 2$, $\pi < y < 2\pi$.

2.2.2 Find the points on the z - plane which map onto the third quadrant of the w - plane under the mapping $w = e^z$.

2.2.3 Prove that $|e^z| = e^x$ and $|e^{iz}| = e^{-y}$.

2.2.4 Prove that there is no value of z such that $e^z = 0$.

2.2.5 Find all values of z such that (a) $e^z = 1$, (b) $e^{2z} = i$,

answers (a) $z = 2\pi n i$, (b) $z = \frac{\pi}{4} + \pi n$, $n = 0, \pm 1, \pm 2, \ldots$.

2.3.1 Compute sin i and check the result with Figure 2.7 .

2.3.2 Prove that (a) $\sin(-z) = -\sin z$, (b) $\cos(-z) = \cos z$.

2.3.3 Prove that $1 + \tan^2 z = \sec^2 z$

2.3.4 Find $u(x, y)$ and $v(x, y)$ such that $w = u + i v = \sin 2z$

2.3.5 Prove that $\cos(z + w) = \cos z \cos w - \sin z \sin w$.

2.4.1 Find (a) $\sinh(-\pi i)$, (b) $\coth(-\pi i)$, (c) $\cosh(-\pi i)$, (d) $\cosh(1 + i)$

2.4.2 Prove that (a) $\sinh(-z) = -\sinh z$, (b) $\cosh(-z) = \cosh z$, (c) $\tanh(-z) = = \tanh z$.

2.4.3 Find all values of z such that $\sinh z = 0$.

2.4.4 Find all values of z such that $\tanh z = 0$.

2.5.1 Find all values of $\sqrt{4\sqrt{3} - 4i}$.

2.5.2 Construct a Riemann surface for the function $\sqrt{z + 3}$.

2.5.3 Construct a Riemann surface for the function $\sqrt[3]{z + 3}$.

2.5.4 Construct a Riemann surface for the function $\sqrt{z^2 - 1}$.

2.6.1 Find (a) Log e , (b) Log e i , (c) Log ($\sqrt{3}$ + i),

(d) Log $(-2\pi i)$.

answers (a) 1 , (b) $1 + \frac{\pi i}{2}$, (c) $\text{Log } 2 + \frac{\pi i}{6}$,

(d) $\text{Log } 2\pi - \frac{\pi i}{2}$.

2.6.2 Find all values of (a) 2^2 , (b) $(-2)^2$, (c) $(ei)^2$.

2.6.3 Find the principal value of (a) 2^2 , (b) $(-2)^2$, (c) $(ei)^2$.

2.6.4 Let $w = \sin^{-1} z$. (a) Show that $e^{2iw} - 2i\ e^{iw} - 1 = 0$.

(b) Show that $\sin^{-1} z = -i \log(iz + \sqrt{1 - z^2})$.

2.6.5 Show that $\cos^{-1} z = -i \log(z + \sqrt{z^2 - 1})$.

2.6.6 Show that $\tan^{-1} z = \frac{1}{2i} \log \left[\frac{1 + iz}{1 - iz}\right]$.

2.6.7 Show that $\sinh^{-1} z = \ln(z + \sqrt{z^2 + 1})$.

2.6.8 There are several places in the following manipulations where the steps are questionable. Find them.

$$
\begin{aligned}
1 &= \sqrt{1} \\
&= \sqrt{(-1)(-1)} \\
&= \sqrt{-1}\,\sqrt{-1} \\
&= i \cdot i \\
&= -1
\end{aligned}
$$

Thus we have $1 = -1$?

48

APPENDIX I

SOLUTIONS TO PROBLEMS

Problems from Chapter 2:

1/ (a) $w = \frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{x - iy}{x^2+y^2}$, Therefore $u = \frac{x}{x^2+y^2}$ and $v = \frac{-y}{x^2+y^2}$.

(b) $w = z^2 + 2z = x^2 - y^2 + 2xy\,i + 2x + 2y\,i$.

Therefore, $u = x^2 - y^2 + 2x$ and $v = 2y(x+1)$.

(c) $w = \frac{1}{z^2} = \frac{1}{x^2-y^2+2xyi} \cdot \frac{x^2-y^2-2xyi}{x^2-y^2-2xyi}$

$$= \frac{x^2-y^2-2xyi}{(x^2-y^2)^2+4x^2y^2} = \frac{x^2-y^2-2xyi}{x^4+2x^2y^2+y^4},$$

Therefore $u = \frac{x^2-y^2}{(x^2+y^2)^2}$ and $v = \frac{-2xy}{(x^2+y^2)^2}$.

(d) $w = z^3 = (x+iy)^3 = x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3$

(Note that the familiar expression from algebra, $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$, was used.)

$w = x^3 + 3x^2y\,i - 3xy^2 - y^3\,i$,

Therefore $u = x^3 - 3xy^2$ and $v = 3x^2y - y^3$.

2/ (a) $w = z^{-1} = r^{-1}e^{-i\theta}$, $\rho = r^{-1}$, $\phi = -\theta$.

(b) $w = z^3 = r^3 e^{i3\theta}$, $\rho = r^3$, $\phi = 3\theta$.

(c) $w = \bar{z} = re^{-i\theta}$, $\rho = r$, $\phi = -\theta$.

(d) $w = |z| = r$, $\rho = r$, $\phi = 0$, 3/ (a) $3-4i$, (b) $-3-4i$, (c) $-1.5-5i$

(d) -1, (e) $2-6i$

4/ (a) $1+i$ and $-1-i$, (b) $-1.4+2.8i$ and $1.4-2.8i$,

(c) $0.7-0.7i$ and $-0.7+0.7i$, (d) $1.6+0.6i$ and

$-1.6-0.6i$

6/ (a) 4, (b) $w = 5e^{i\frac{4\pi}{3}} = -\frac{5}{2}-\frac{5}{2}\sqrt{3}\,i$,

(c) $w = 3e^{-i\frac{\pi}{3}} = +\frac{3}{2}-\frac{3}{2}\sqrt{3}\,i$.

7/ (a) $w = \sqrt{3}+i = 2e^{i\frac{\pi}{6}}$ and $2e^{-i\frac{11\pi}{6}}$.

Thus $z = 1.4+0.4i$ and $-1.4-0.4i$

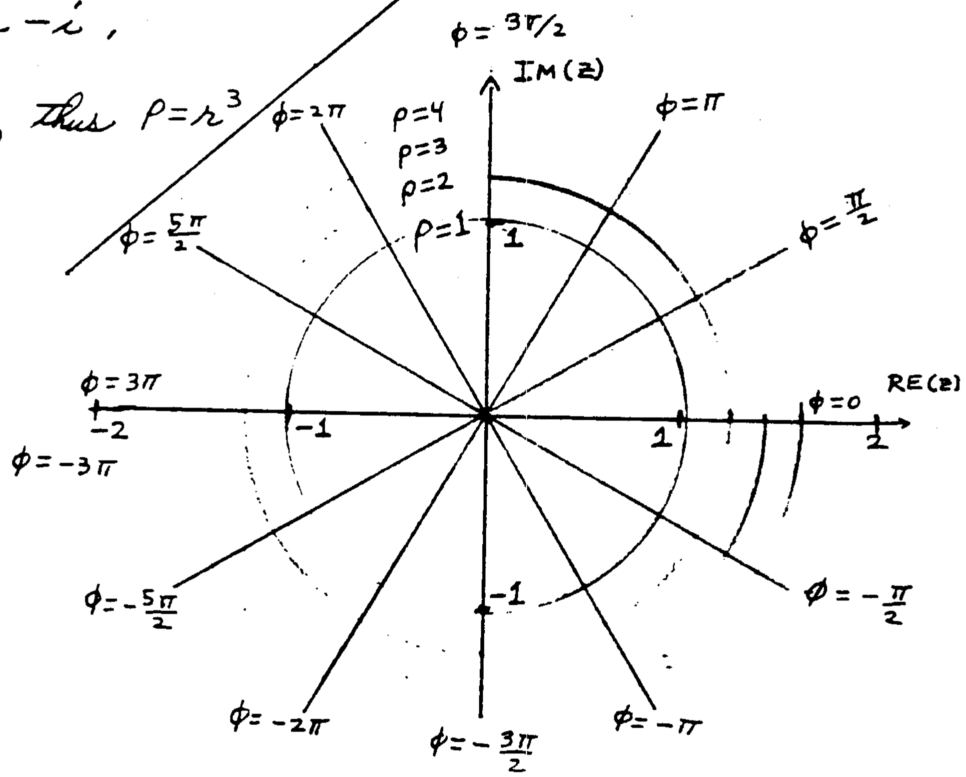
(b) $w = 4e^{i\frac{2\pi}{3}}$ which is also $4e^{-i\frac{4\pi}{3}}$.

Thus $z = 1+1.75i$ and $-1-1.75i$.

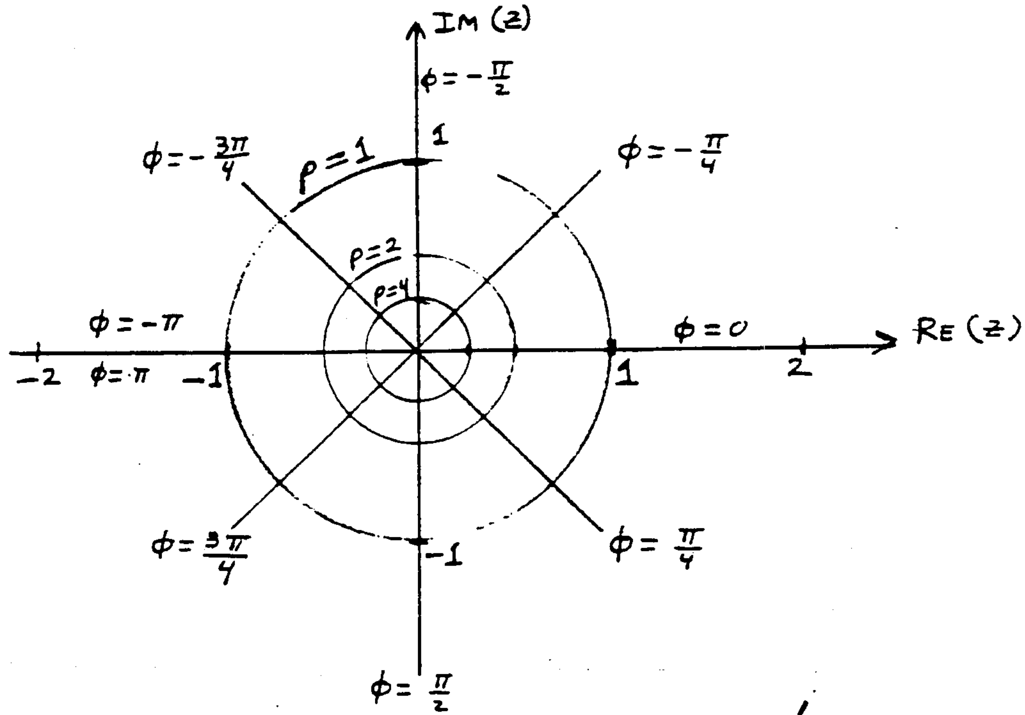
(c) $w=-1 = e^{i\pi}$ and $e^{-i\pi}$. Thus

$z = i$ and $-i$,

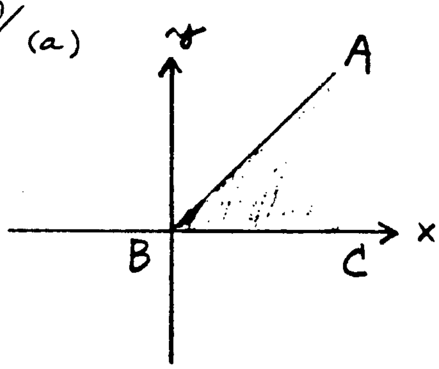
8/ (a) $w = z^3$, thus $\rho = r^3$ and $\phi = 3\theta$.

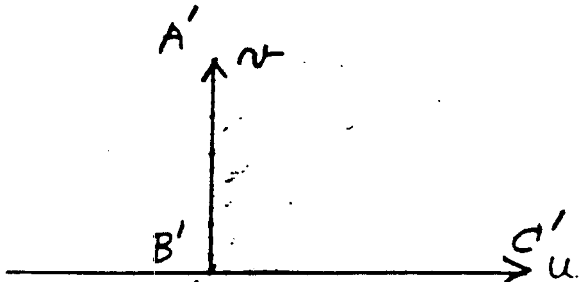


8/ (b) $w = z^{-1}$, therefore $\rho = r^{-1}$ and $\phi = -\theta$.

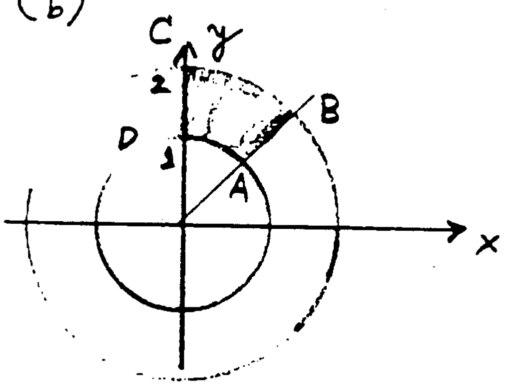


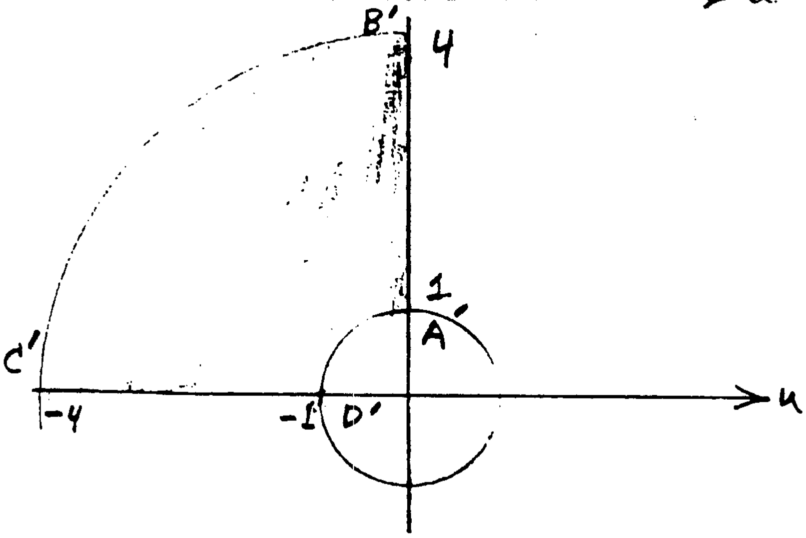
10/ (a)





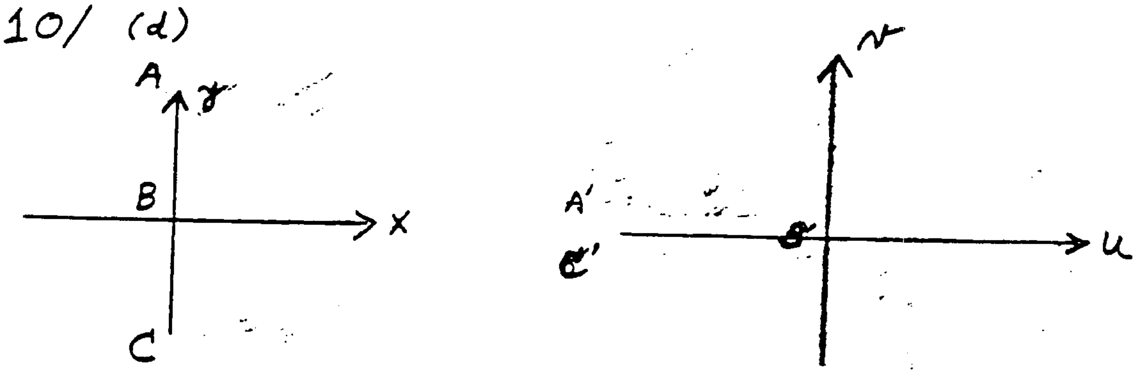
(b)



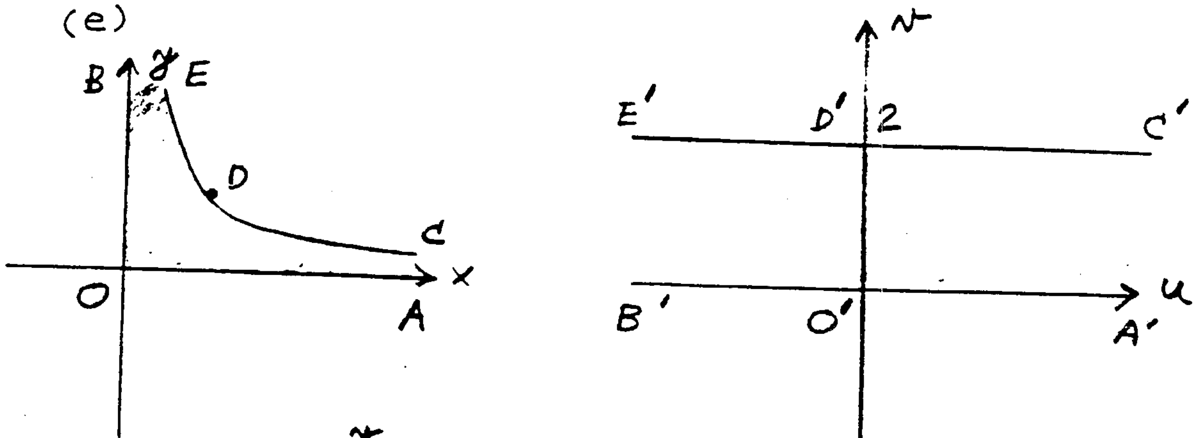


(c) same as (a)

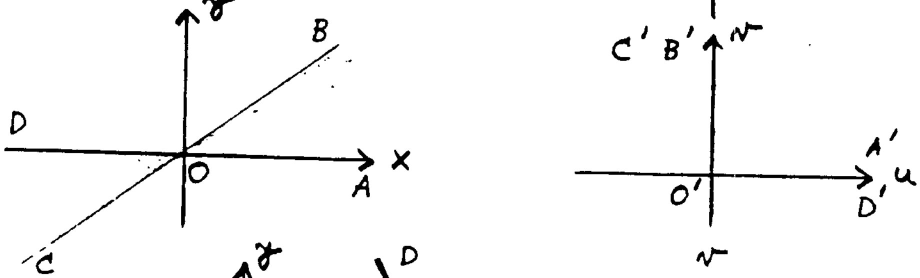
10/ (d)


A
y
B
x
C
v
A'
C'
u

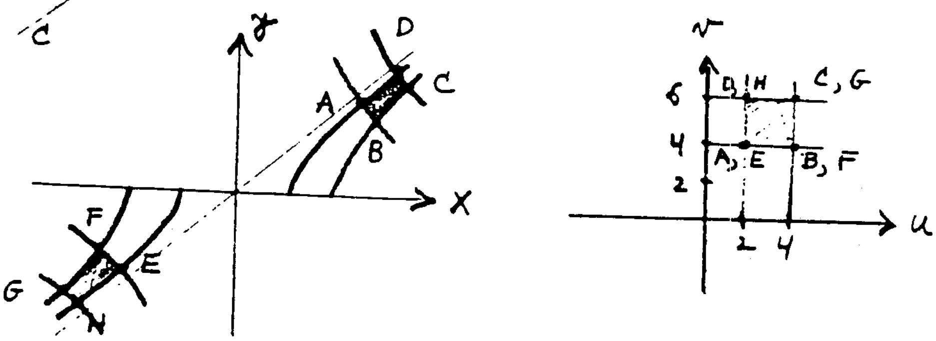
(e)


B
y
E
D
C
x
O
A
v
E'
D'
2
C'
B'
O'
u
A'

11/ (a)


y
B
D
x
O
A
C
C'
B'
v
A'
u
O'
D'

(b)


y
D
A
C
B
x
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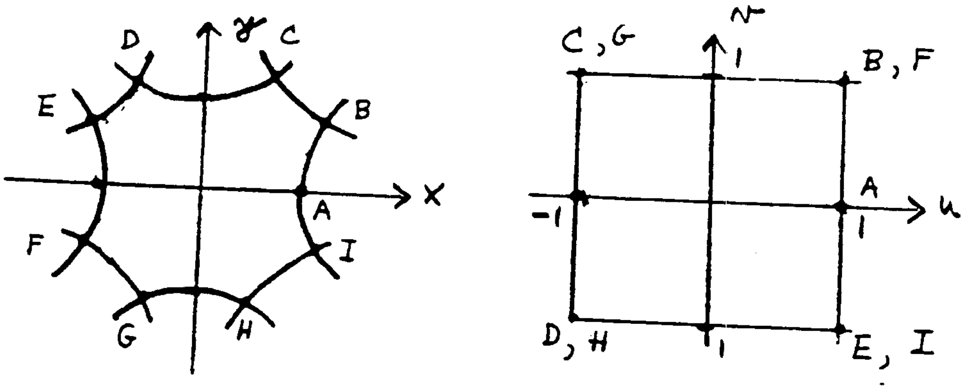
E
G
H
v
6
4
2
D, H
C, G
A, E
B, F
2
4
u

(c)


D
y
C
E
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x
A
F
I
G
H
v
C, G
1
B, F
A
-1
u
1
D, H
1
E, I

12/ (a) $e^z e^w = e^x e^{iy} e^{\xi} e^{i\zeta}$

$= e^x e^{\xi} e^{iy} e^{i\zeta}$

$= e^{x+\xi} (\cos y + i \sin y)(\cos \zeta + i \sin \zeta)$

$= e^{x+\xi} \left[(\cos y \cos \zeta - \sin y \sin \zeta) + i(\cos y \sin \zeta + \sin y \cos \zeta)\right]$

$= e^{x+\xi} \left[\cos(y+\zeta) + i \sin(y+\zeta)\right]$

$= e^{(x+\xi) + i(y+\zeta)} = e^{z+w}$

(b) $$\frac{e^z}{e^w} = \frac{e^x(\cos y + i \sin y)}{e^{\xi}(\cos \zeta + i \sin \zeta)} \cdot \frac{(\cos \zeta - i \sin \zeta)}{(\cos \zeta - i \sin \zeta)}$$

$$= \frac{e^{x-\xi}\left[(\cos y \cos \zeta + \sin y \sin \zeta) + i(\sin y \cos \zeta - \cos y \sin \zeta)\right]}{\cos^2 \zeta + \sin^2 \zeta}$$

$= e^{x-\xi}\left[\cos(y-\zeta) + i \sin(y-\zeta)\right]$

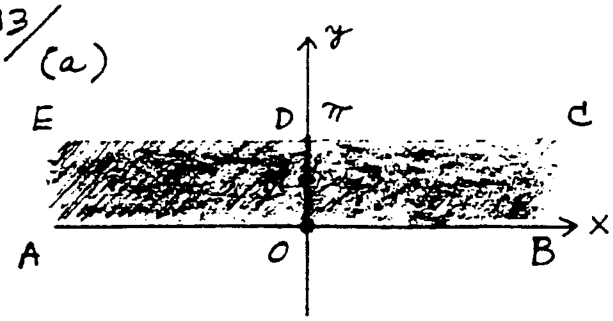
$= e^{(x-\xi)+i(y-\zeta)} = e^{(x+iy)-(\xi+i\zeta)} = e^{z-w}$

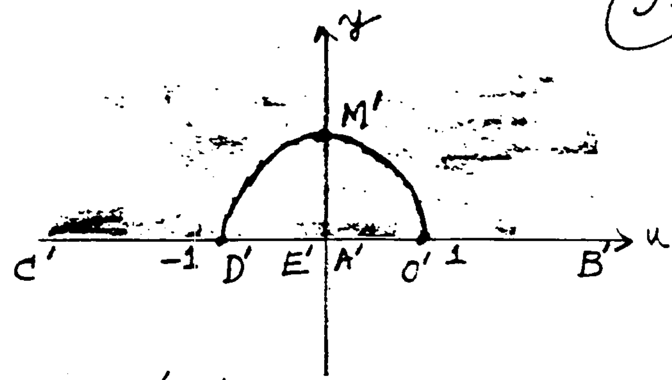
(c) *Use mathematical induction on N, The relation is certainly true for N = 1, Assume it is true for N and try N+1:*

$(e^z)^{N+1} = (e^z)^N e^z = e^{Nz} e^z$.

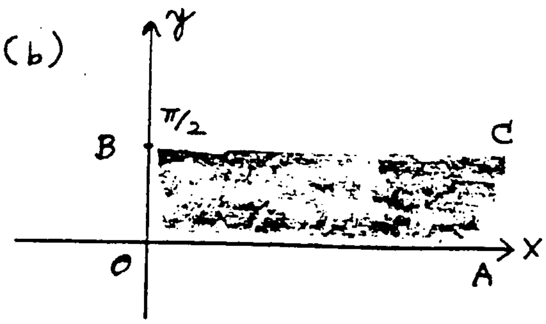
This last relation equals $e^{Nz+z} = e^{z(N+1)}$ *by part (a) of this problem, Therefore the induction is finished.*

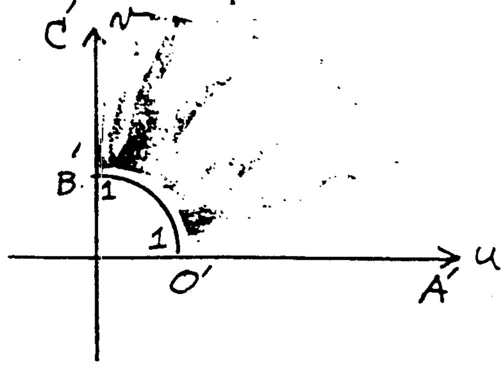
13/(a)


y
E
D
π
C
A
O
B
x

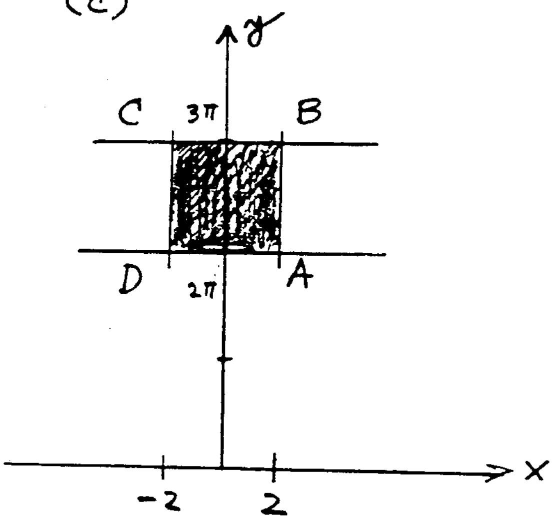

y
M'
C'
-1
D'
E'
A'
O'
1
B'
u

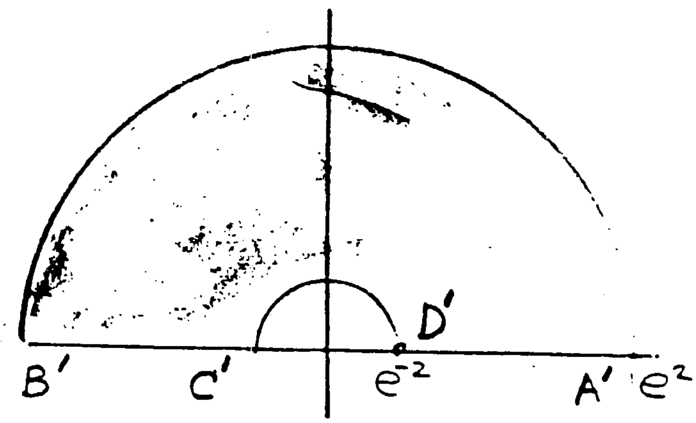
(b)


y
π/2
B
C
O
A
x

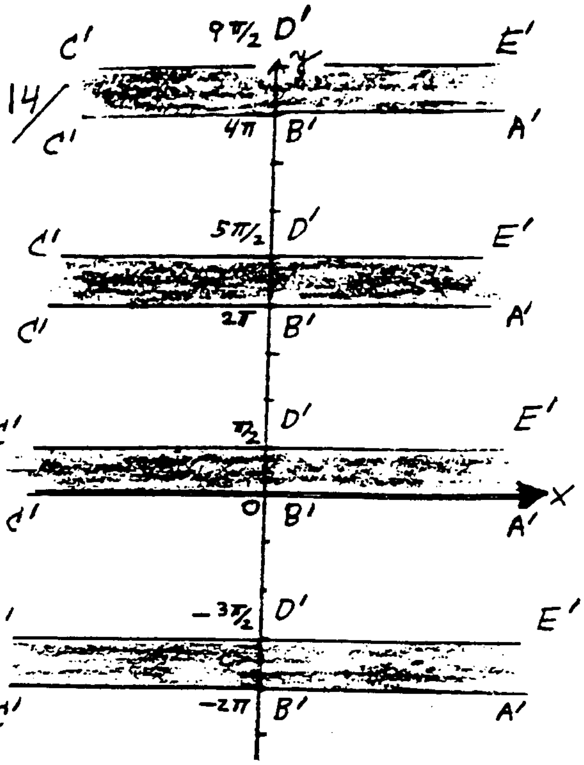

C'
v
B'
1
1
O'
A'
u

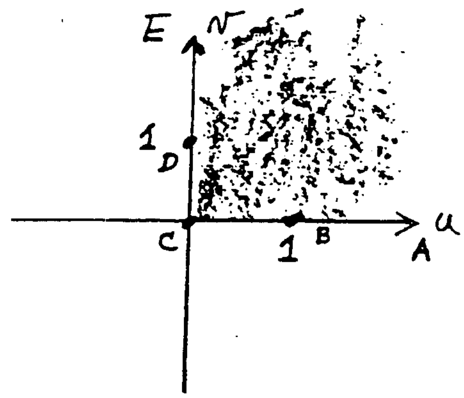
(c)


y
C
3π
B
D
2π
A
-2
2
x


D'
B'
C'
e⁻²
A'
e²

14/





15/ $\sin(1+i) = \dfrac{e^{i(1+i)} - e^{-i(1+i)}}{2i} = \dfrac{e^{i}e^{-1} - e^{-i}e^{1}}{2i}$

Now $e = 2.72$, $e^{-1} = 0.368$,

$e^{i} = \cos 1 + i \sin 1 = 0.54 + i\,0.84$

$e^{-i} = 0.54 - i\,0.84$.

{ USE TABLES OF SINE & COSINE IN RADIANS, OR IN DEGREES WITH 1 RADIAN = 57.3°

Thus

$\sin(1+i) = \dfrac{0.37(0.54 + i\,0.84) - 2.72(0.54 - i\,0.84)}{2i}$

$= 1.3 + 0.64\,i$

FROM FIG. 2.7 WE HAVE $\sin(1+i) \approx 1.5\, e^{i\frac{\pi}{6}}$

$\sin(1+i) \approx 1.5(\cos 30° + i \sin 30°)$

$\approx 1.3 + i\,0.75$

16/ $\tan z = \dfrac{\sin z}{\cos z} = -i \dfrac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}}$,

$\sec z = \dfrac{1}{\cos z} = \dfrac{2}{e^{iz} + e^{-iz}}$, $\csc z = \dfrac{1}{\sin z} = \dfrac{2i}{e^{iz} - e^{-iz}}$

$\cot z = \dfrac{1}{\tan z} = i \dfrac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}}$.

17/ $\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$

$= \frac{1}{2i}(e^{ix-y} - e^{-ix+y})$

$= \frac{1}{2i}\left[e^{-y}(\cos x + i\sin x) - e^{y}(\cos x - i\sin x)\right]$

$= \frac{1}{2i}\left[\cos x\,(e^{-y} - e^{y}) \quad + i\sin x\,(e^{y} + e^{-y})\right]$

$= \sin x \left[\dfrac{e^{y} + e^{-y}}{2}\right] + i\cos x\left[\dfrac{e^{y} - e^{-y}}{2}\right]$

$= \sin x \cosh y + i\cos x \sinh y$,

18/ $\sin 2z = \dfrac{e^{2iz} - e^{-2iz}}{2i}$ ← difference of two squares, which factors as

$= \dfrac{(e^{iz} + e^{-iz})(e^{iz} - e^{-iz})}{2i}$

$= 2\left(\dfrac{e^{iz} + e^{-iz}}{2}\right)\left(\dfrac{e^{iz} - e^{-iz}}{2i}\right)$

$= 2\cos z \sin z$,

19/ From Fig. 27, we have

$$\sin i \cong 1{,}2\, e^{i\frac{\pi}{2}} = 1{,}2\,i$$

$$\cos i = \sin(i + 1{,}57) \approx 1{,}55\, e^{i0} = 1{,}55$$

Thus $\tan i = \dfrac{\sin i}{\cos i} = \dfrac{1{,}2\,i}{1{,}55} = \boxed{0{,}775\,i}$

Using the analytical formula we have

$$\tan i = -i\,\frac{e^{-1} - e^{1}}{e^{-1} + e^{1}} = i\,\frac{2{,}72 - {,}368}{2{,}72 + {,}368}$$

$$= i\,\frac{2{,}35}{3{,}09} = \boxed{0{,}76\,i}$$

20/ $\sin z = 0$ when both

(a) $\sin x \cosh y = 0$ and (b) $\cos x \sinh y = 0$,

Now $\cosh y \geq 1$, and thus (a) is zero only when $\sin x = 0$, which is when $x = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$. Since $\cos x$ is never zero at $x = n\pi$, and $\sinh y$ is zero only when $y = 0$, (b) will be zero only for $y = 0$. Thus the only zeros of $\sin z$ are

$$z = n\pi + i\,0 \qquad n = 0, \pm 1, \pm 2, \ldots.$$

21/ (a) From (2), $\sinh z = -i \sin(iz)$, Thus $\sinh(3\pi i) =$
$-i \sin(-3\pi) = i \sin(3\pi) = i \sin \pi = \boxed{0}$,

(b) From (1), $\cosh z = \cos(iz)$, Thus $\cosh(3\pi i) =$
$\cos(-3\pi) = \cos(3\pi) = \cos \pi = \boxed{-1}$,

(c) $\tanh(3\pi i) = \dfrac{\sinh(3\pi i)}{\cosh(3\pi i)} = \dfrac{0}{-1} = \boxed{0}$,

(d) $\sinh(1-i) = -i \sin(i(1-i)) = -i \sin(1+i)$
$= -i(1.3 + .64i) = \boxed{.64 - 1.3i}$,
(from problem 15)

22/
$$\tanh z = \frac{\sinh z}{\cosh z} = \frac{e^z - e^{-z}}{e^z + e^{-z}}$$

$$\operatorname{sech} z = \frac{1}{\cosh z} = \frac{2}{e^z + e^{-z}}$$

$$\operatorname{csch} z = \frac{1}{\sinh z} = \frac{2}{e^z - e^{-z}}$$

$$\coth z = \frac{\cosh z}{\sinh z} = \frac{e^z + e^{-z}}{e^z - e^{-z}}$$

23/ (3) $\sinh(2z) = -i \sin(2zi)$

$= -i\, 2 \sin(zi) \cos(zi)$

$= 2\left[-i \sin(zi)\right]\left[\cos(zi)\right]$

$= 2 \sinh z \cosh z$,

To derive (4) we need $\tanh z = \frac{\sinh z}{\cosh z} = \frac{-i \sin(iz)}{\cos(iz} =$ $-i \tan(iz)$. Now

$$1 - \tanh^2 z = 1 - (-i \tan(iz))^2 = 1 + \tan^2(iz) =$$

$$\sec^2(iz) = \frac{1}{\cos^2(iz)} = \frac{1}{\cosh^2 z} = \operatorname{sech}^2 z.$$

Finally, we derive (5).

$$\cosh(z+w) = \cos(iz+iw) = \cos iz \cos iw - \sin iz \sin iw$$

$$= (\cos iz)(\cos iw) + (-i \sin iz)(-i \sin iw)$$

$$= \cosh z \cosh w + \sinh z \sinh w,$$

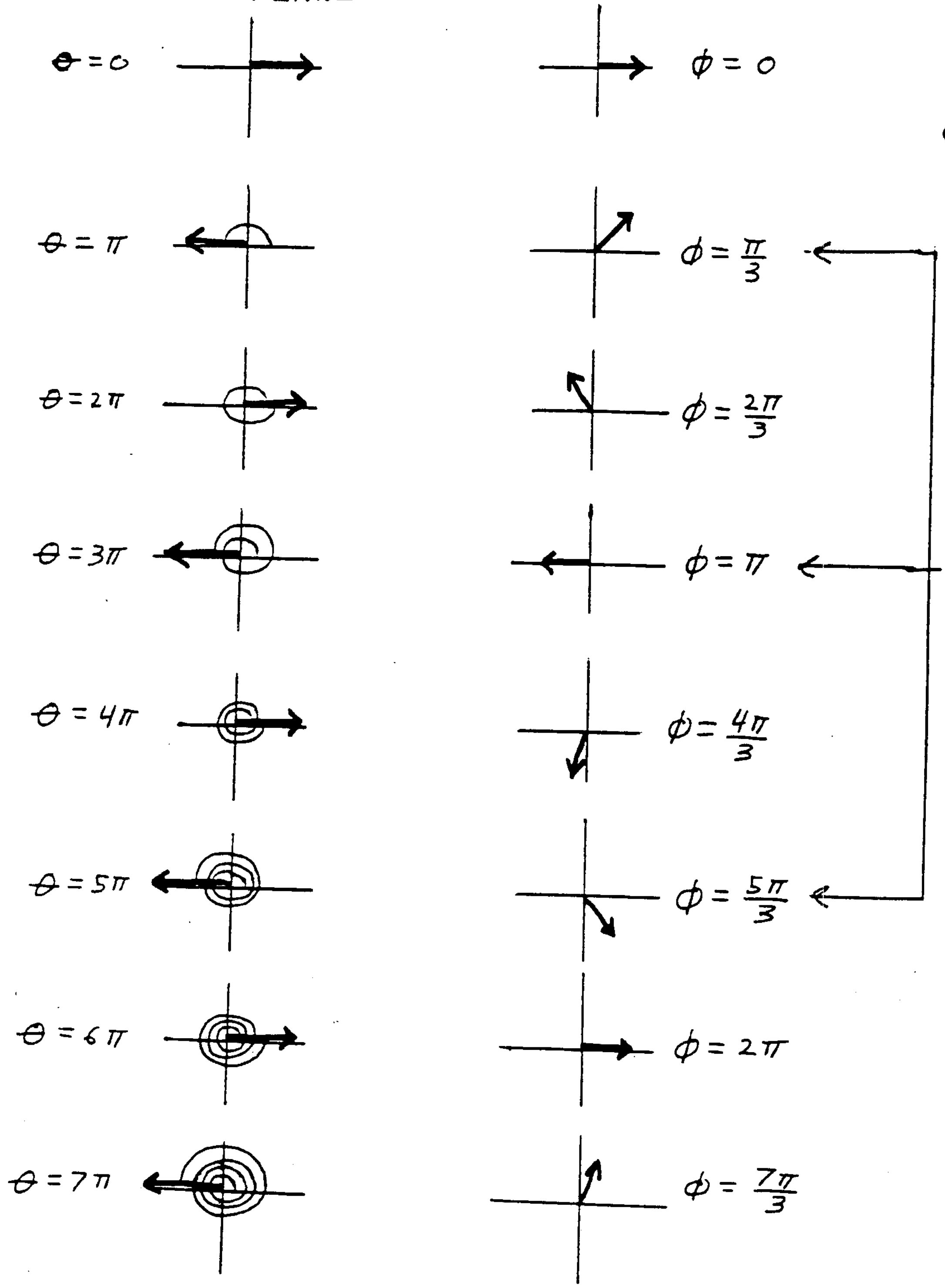
24/ $2 + 2\sqrt{3}\, i = 4 e^{i\frac{\pi}{3}}$, THUS $(2+2\sqrt{3}\, i)^{\frac{1}{2}} = (4 e^{i\frac{\pi}{3}})^{\frac{1}{2}}$ $= 2 e^{i\frac{\pi}{6}} = \boxed{\sqrt{3} + i}$. This is only one value of the square root. The second value is always obtained by multiplying by -1. Thus $\boxed{-\sqrt{3} - i}$ also equals $\sqrt{2 + 2\sqrt{3}\, i}$.

25/(a) $\sqrt[3]{z} = (r e^{i\theta})^{\frac{1}{3}} = r^{\frac{1}{3}} e^{i\frac{\theta}{3}}$

25/(b)

$W = \sqrt[3]{z}$

$\phi = \arg(W) = \frac{\theta}{3}$



NOTE THAT THERE ARE 3-DISTINCT W-VECTORS FOR THE SAME Z-VECTOR. THEIR ANGULAR SEPARATION IS $120° = \frac{2\pi}{3}$.

(c) THE Z-VECTOR MUST MAKE 3 FULL REVOLUTIONS AS THE W-VECTOR MAKES ONE REVOLUTION

(d) THERE ARE 3 DISTINCT W-VECTORS CORRESPONDING TO EACH Z-VECTOR, ALL 3 W-VECTORS HAVE EQUAL ANGULAR SEPARATION ($120° = \frac{2\pi}{3}$). THUS IF $W_0 = \sqrt[3]{z_0}$ IS ONE VALUE, THE OTHER TWO ARE $e^{\frac{2\pi}{3}i} W_0$ AND $-e^{\frac{2\pi}{3}i} W_0$;

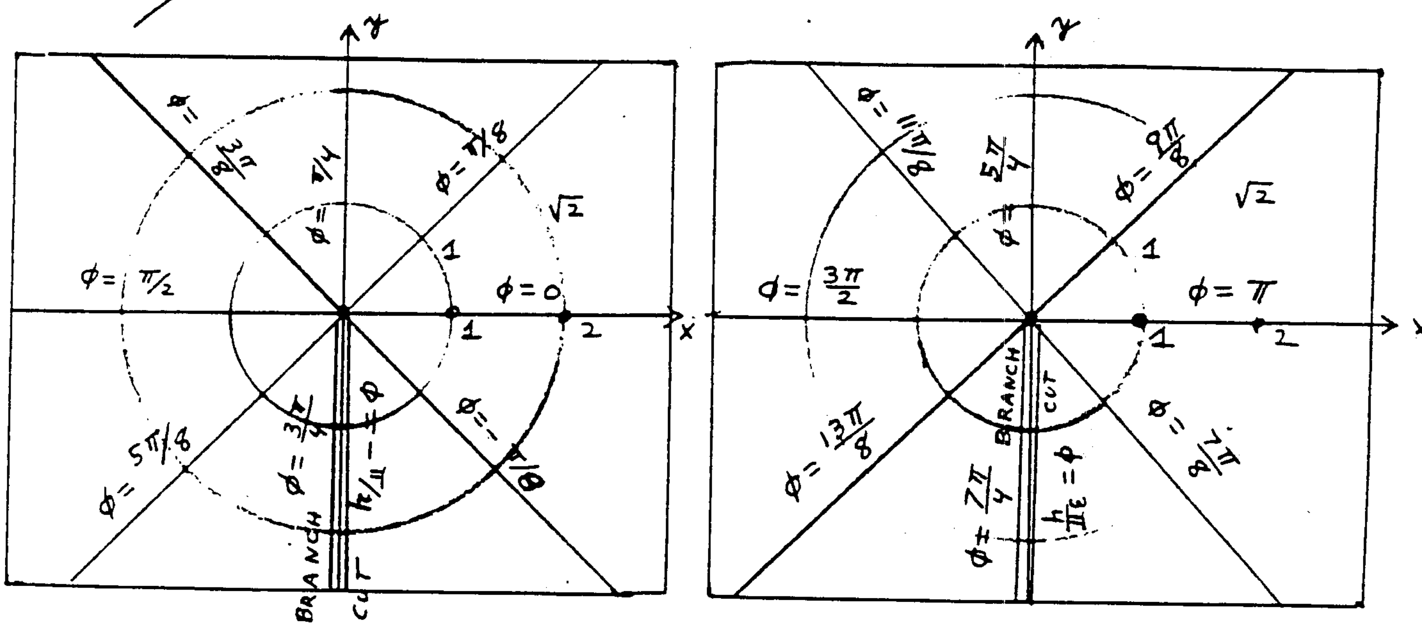
26/ (a) $\sqrt[N]{z} = r^{\frac{1}{N}} e^{i\frac{\theta}{N}}$

(c) THE Z-VECTOR MUST NEGOTIATE N-FULL REVOLUTIONS BEFORE THE W-VECTOR MAKES ONE REVOLUTION.

(d) N DISTINCT W-VECTORS CORRESPOND TO EACH Z-VECTOR. ALL THESE W-VECTORS ARE SPACED EVENLY IN ANGLE, THE SEPARATION BEING $\frac{360^\circ}{N} = \frac{2\pi}{N}$.

IF $W_0 = \sqrt[N]{Z_0}$ IS ONE N-TH ROOT OF Z_0, THEN THE OTHERS ARE $W_0 e^{\frac{2\pi}{N}i}$, $W_0 e^{\frac{4\pi}{N}i}$, $W_0 e^{\frac{6\pi}{N}i}$, ..., $W_0 e^{\frac{2\pi(N-1)}{N}i}$.

27/

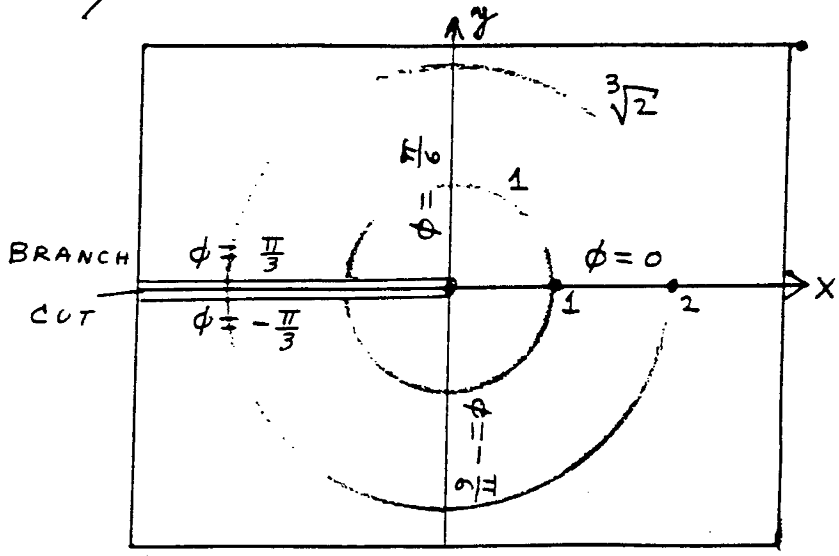


$-\frac{\pi}{2} < \theta \leq \frac{3\pi}{2}$
BRANCH ONE

$\frac{3\pi}{2} < \theta \leq \frac{7\pi}{2}$
BRANCH TWO

ALTERNATIVE BRANCHES FOR $W = \sqrt{Z}$

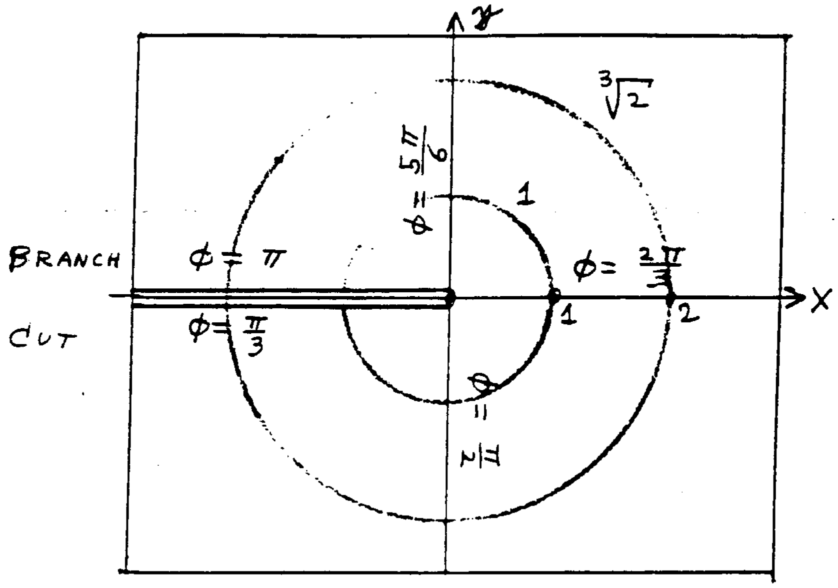
BRANCHES OF THE FUNCTION $w = \sqrt[3]{z}$



BRANCH ONE

$-\pi < \theta \leq \pi$

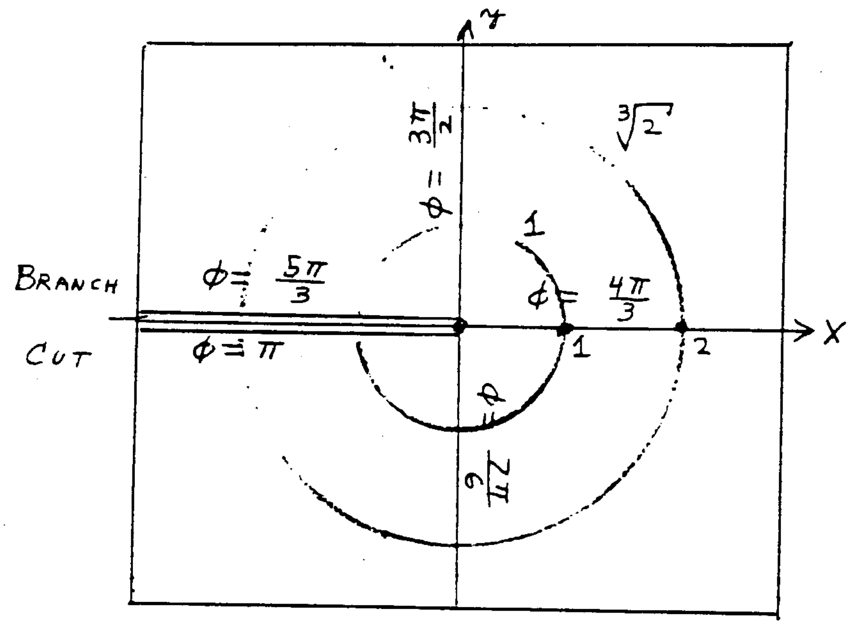
$-\frac{\pi}{3} < \phi \leq \frac{\pi}{3}$



BRANCH TWO

$\pi < \theta \leq 3\pi$

$-\frac{\pi}{3} < \phi \leq \pi$



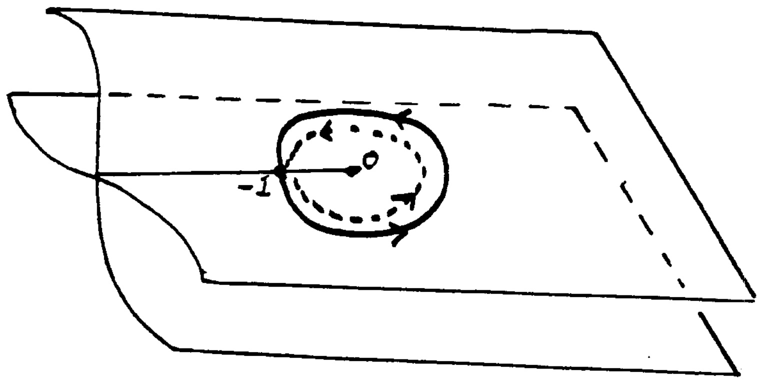
BRANCH THREE

$3\pi < \theta \leq 5\pi$

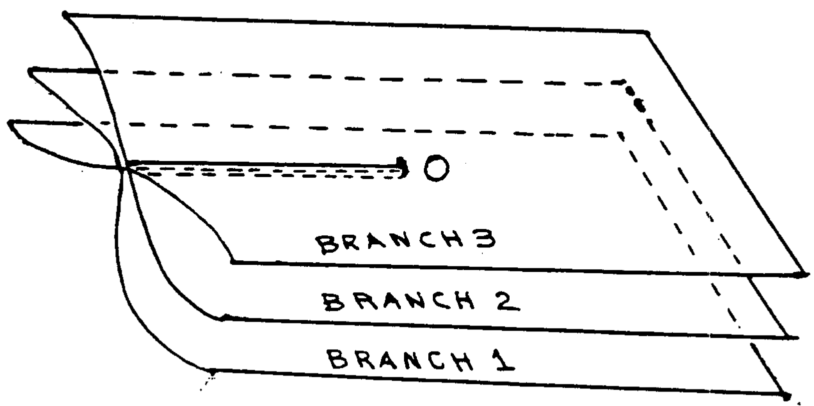
$\pi < \phi \leq \frac{5\pi}{3}$

29/ THERE WILL BE N- BRANCHES,

30/



31/



33/ (a) $-1 = e^{i(\pi+2\pi n)}$, WHERE $n = 0, \pm 1, \pm 2, \ldots$.

THUS WE HAVE FROM (3): $\log(-1) = \text{Log}\, 1 + i(\pi + 2\pi n) =$ $\boxed{i(\pi+2\pi n)}$.

(b) $e = e\, e^{i 2\pi n}$, THUS $\log e = \text{Log}\, e + 2\pi n i, =$

$\boxed{1 + 2\pi n i}$

(c) $1+i = \sqrt{2}\, e^{i(\frac{\pi}{4}+2\pi n)}$, THUS $\log(1+i) =$

$\log\sqrt{2} + i(\frac{\pi}{4}+2\pi n) = \boxed{\frac{1}{2}\text{Log}\, 2 + i(\frac{\pi}{4}+2\pi n)}$.

(d) $\log 0$ IS UNDEFINED. RECALL THAT

$$\lim_{x\to 0^+} \text{Log}\, x = -\infty$$

34/ (a) $1 = e^{i0}$, THUS $\text{Log}\, 1 = \boxed{0}$.

(b) $(-1) = e^{i\pi}$, THUS $\text{Log}(-1) = \boxed{i\pi}$.

(c) $-ei = e\, e^{-i\frac{\pi}{2}}$, Thus $\text{Log}(-ei) = \text{Log}\, e - i\frac{\pi}{2} =$

$\boxed{1 - i\frac{\pi}{2}}$

(d) $(1+i) = \sqrt{2}\, e^{i\frac{\pi}{4}}$, THUS $\text{Log}(1+i) = \text{Log}\sqrt{2} + i\frac{\pi}{4}$

$= \boxed{\frac{1}{2}\text{Log}\, 2 + i\frac{\pi}{4}}$.

35/ (a) $1^2 = e^{2\log 1} = e^{2(2\pi n i)} = e^{4\pi n i} = \boxed{1}$.

(b) $(-1)^2 = e^{2\log(-1)} = \left(\text{using problem 33(a)}\right) = e^{2i(\pi+2\pi n)}$

$= e^{i(2\pi+4\pi n)} = \boxed{1}$

35/ (c) $(-ei)^2 = e^{2\log(-ei)} =$

(using problem 34(c) and adding $2\pi n i$ to the answer) $= e^{2\left[1 - i\left(\frac{\pi}{2} + 2\pi n\right)\right]} =$

$e^2 e^{-i(\pi + 4\pi n)} = \boxed{-e^2}$.

(d) $(1+i)^{1+i} = e^{(1+i)\log(1+i)} =$ (using problem 33(c))

$= e^{(1+i)\left[\frac{1}{2}\mathrm{Log}\,2 + i\left(\frac{\pi}{4} + 2\pi n\right)\right]}$

$= \exp\left[\frac{1}{2}\mathrm{Log}\,2 - \frac{\pi}{4} - 2\pi n + i\left(\frac{1}{2}\mathrm{Log}\,2 + \frac{\pi}{4} + 2\pi n\right)\right]$

$= e^{\frac{1}{2}\mathrm{Log}\,2}\, e^{-\frac{\pi}{4} - 2\pi n}\, e^{\frac{i}{2}\mathrm{Log}\,2}\, e^{i\frac{\pi}{4}}\, e^{i2\pi n}$

$= \boxed{\sqrt{2}\, e^{-\frac{\pi}{4} - 2\pi n}\, e^{i\left(\frac{\pi}{4} + \frac{1}{2}\mathrm{Log}\,2\right)}}$

36/ In this problem we get the very same answers as were obtained in the previous problem, except n is set equal to zero. Thus

(a) $1^2 = \boxed{1}$, (b) $(-1)^2 = \boxed{1}$, (c) $(-ei)^2 = \boxed{-e^2}$,

and (d) $(1+i)^{1+i}_{\text{principal value}} = \boxed{\sqrt{2}\, e^{-\frac{\pi}{4}}\, e^{i\left(\frac{\pi}{4} + \frac{1}{2}\mathrm{Log}\,2\right)}}$

Solutions to Review Problems from Chapter 2

1/ (a) $e^{1+\frac{\pi}{4}i} = e\, e^{\frac{\pi}{4}i} = e\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$= \boxed{\frac{e}{\sqrt{2}} + \frac{e}{\sqrt{2}}\, i}$

(b) $\sin\left(\frac{\pi}{3}+i\right) = \dfrac{e^{i\left(\frac{\pi}{3}+i\right)} - e^{-i\left(\frac{\pi}{3}+i\right)}}{2i} =$

$\dfrac{e^{-1}e^{\frac{\pi}{3}i} - e\,e^{-\frac{\pi}{3}i}}{2i} = \dfrac{e^{-1}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) - e\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)}{2i}$

$= \cos\frac{\pi}{3}\left(\dfrac{e^{-1}-e}{2i}\right) + \sin\frac{\pi}{3}\left(\dfrac{e^{-1}+e}{2}\right)$

$= \sin\frac{\pi}{3}\left(\dfrac{e+e^{-1}}{2}\right) + i\cos\frac{\pi}{3}\left(\dfrac{e-e^{-1}}{2}\right)$

$= \boxed{\frac{\sqrt{3}}{2}\cosh 1 + i\,\frac{\sinh(1)}{2}}$

(c) USING (1) FROM PAGE 2,23 WE HAVE;

$\cosh\left(i\frac{\pi}{3}\right) = \cos\left(i\,i\frac{\pi}{3}\right) = \cos\left(-\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \boxed{\frac{1}{2}}$

(d) $-2+2\sqrt{3}i = 4e^{i\frac{2\pi}{3}}$, WHERE THE ARGUMENT $\frac{2\pi}{3}$ HAS BEEN SELECTED SINCE IT IS IN THE RANGE $(-\pi, \pi]$ FOR THE "PRINCIPAL VALUE" OF THE LOGARITHUM

$\text{Log}\left(-2+2\sqrt{3}i\right) = \boxed{\text{Log}\, 4 + i\,\frac{2\pi}{3}}$

1/(e) $(-2+2\sqrt{3}i)^i = e^{i \log(-2+2\sqrt{3}i)}$,

BUT $\log(-2+2\sqrt{3}i) = \text{Log}(-2+2\sqrt{3}i) + 2\pi n$,

WHERE $n = 0, \pm 1, \pm 2, \ldots$. USING (d) OF THIS PROBLEM WE HAVE

$$\log(-2+2\sqrt{3}i) = \text{Log}\, 4 + i\left(\frac{2\pi}{3} + 2\pi n\right).$$

THUS

$$(-2+2\sqrt{3}i)^i = e^{i\left[\text{Log}\, 4 + i\left(\frac{2\pi}{3}+2\pi n\right)\right]}$$

$$= e^{-\left(\frac{2\pi}{3}+2\pi n\right) + i(\text{Log}\, 4)}$$

$$= e^{-\left(\frac{2\pi}{3}+2\pi n\right)}\left(\cos(\text{Log}\, 4) + i \sin(\text{Log}\, 4)\right)$$

2/ $w = (z-1)^3 = ((x-1)+iy)^3$

$$= (x-1)^3 + 3(x-1)^2(iy) + 3(x-1)(iy)^2 + (iy)^3$$

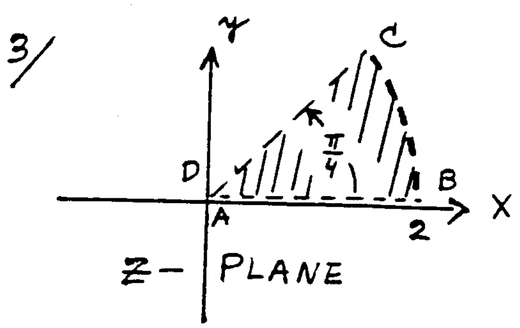
$$= (x-1)^3 + 3(x-1)^2 y i - 3(x-1)y^2 - y^3 i$$

$$= \left[(x-1)^3 - 3(x-1)y^2\right] + i\left[3(x-1)^2 y - y^3\right]$$

$$= (x-1)\left[(x-1)^2 - 3y^2\right] + i\, y\left[3(x-1)^2 - y^2\right]$$

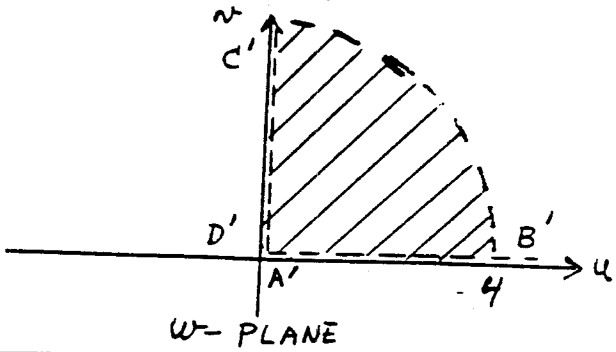
$$\boxed{u = (x-1)\left[(x-1)^2 - 3y^2\right]}, \quad \boxed{v = y\left[3(x-1)^2 - y^2\right]}$$

3/



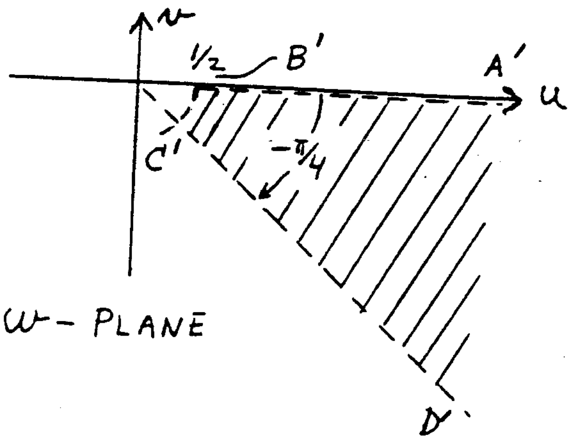
(a) $w = z^2$

$\rho = r^2, \quad \phi = 2\theta$



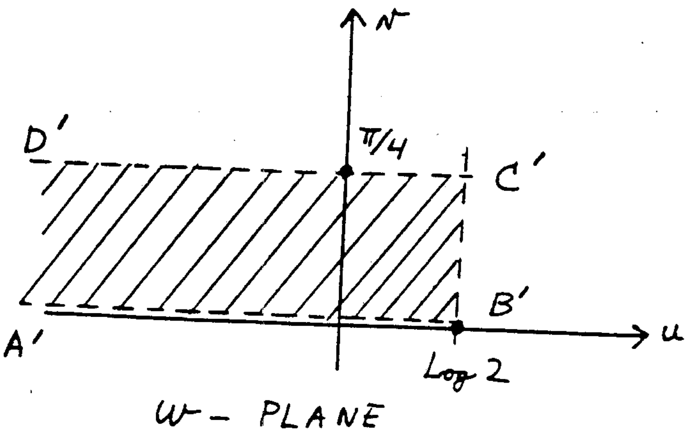
(b) $w = \frac{1}{z}$

$\rho = \frac{1}{r}, \quad \phi = -\theta$

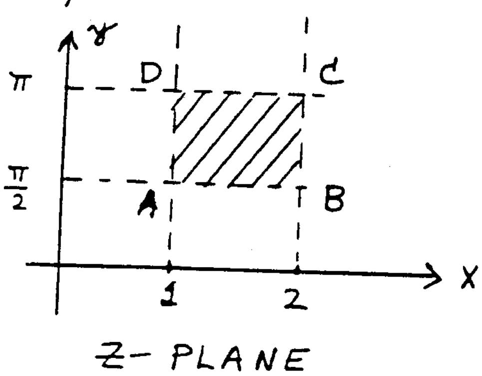


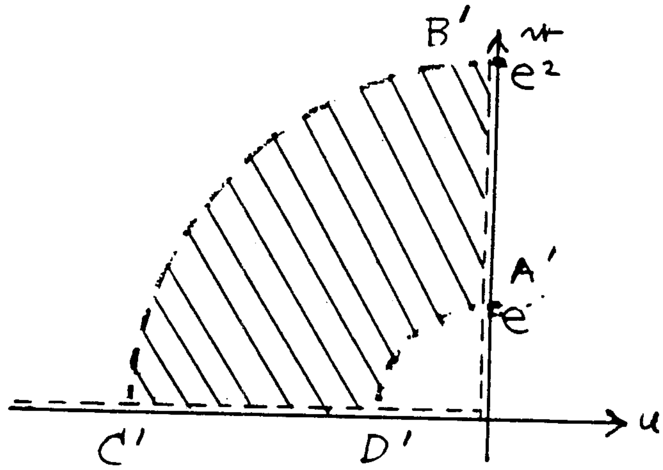
(c) $w = \text{Log}\, z$

$u = \text{Log}\, r, \quad v = \theta$



4/





5/ (a) The Riemann surface for $w = \sqrt{z-1}$ is the same as that for $w = \sqrt{z}$ with the exception that the branch points are now at $z = \infty$ and $z = 1$, (b) Six,

6/ $e^z = 1 - i$ implies that $z = \log(1-i)$

$$z = \log\left(\sqrt{2}\, e^{-i\frac{\pi}{4} + 2\pi n i}\right), \quad n = 0, \pm 1, \pm 2, \ldots,$$

$$= \operatorname{Log}\sqrt{2} - i\left(\frac{\pi}{4} + 2\pi n\right)$$

$$= \boxed{\frac{1}{2}\operatorname{Log} 2 - i\left(\frac{\pi}{4} + 2\pi n\right)}$$