Notice that the area under the rectangles is nearly the area under the curve $y = \log x$ from $x = 3/2$ to $x = N + 1/2$. Thus we have

$$\log N! \approx \int_{3/2}^{N+1/2} \log x \, dx$$

$$\approx x \log x - x \Big|_{3/2}^{N+1/2}$$

$$\approx (N + 1/2) \log(N + 1/2) - (N + 1/2) - \tfrac{3}{2} \log \tfrac{3}{2} + \tfrac{3}{2}$$

(4) $$\approx \log(N + 1/2)^{N + 1/2} + \log\left(\tfrac{2}{3}\right)^{3/2} - N + 1$$

Taking the exponential of both sides of (4) we get

(5) $$N! \approx \left(N + \tfrac{1}{2}\right)^{N+\frac{1}{2}} e^{-N} \left(\tfrac{2}{3}\right)^{3/2} e^{1}$$

Because $\left(N + \frac{1}{2}\right)^{N+\frac{1}{2}} = N^{N + \frac{1}{2}} \left(1 + \frac{1}{2N}\right)^{N + \frac{1}{2}}$, (5) becomes

$$N! \approx N^{N + \frac{1}{2}} e^{-N} \left(\tfrac{2}{3}\right)^{3/2} \left(1 + \tfrac{1}{2N}\right)^{N + \frac{1}{2}} e .$$

Recall that in the calculus we learned that $\lim\limits_{N \to \infty} \left(1 + \frac{x}{N}\right)^{N} = e^{x}$,

and thus for large N, $\left(1 + \frac{1}{2N}\right)^{N} \approx e^{1/2}$. Thus we have

$$N! \approx N^{N + \frac{1}{2}} e^{-N} \left(\tfrac{2}{3}\right)^{3/2} e^{3/2} \left(1 + \tfrac{1}{2N}\right)^{1/2}$$

Now $\left(\frac{2}{3}\right)^{3/2} e^{3/2} = 2.4395 \ldots$, and thus

$$(6) \quad N! \approx (2.4395)\sqrt{N}\; N^{N}\; e^{-N} \; .$$

The correct constant in Stirling's approximation is

$$\sqrt{2\pi} = 2.506628275 \ldots$$

and thus our constant in (6) is off by only 2.7 % . Thus we see that Stirling's approximation can be suggested by comparing the rectangles in Figure 2 to the area under the logarithmic curve.

Review Problems for Chapter 7

1. (a) Find an infinite product representation for $\cosh \pi z$.

   (b) Evaluate the product $\prod_{n=0}^{\infty} (1 + 8(2n+1)^{-2})$

2. Express the most general entire function having zeros at the points $z = i\, n^{3/4}$, where $n = 0, 1, 2, \ldots,$ as an infinite product.

3. Expand $\tanh \pi z$ in partial fractions.

4. Evaluate the integral $\int_0^{\infty} e^{-st}\, t^{3/2}\, dt$.

5. Evaluate the integral $\int_0^1 t^{a-1} (1-t)^{3-a}\, dt$, $0 < a < 4$.

6. Test the integral $\int_1^{\infty} \Gamma(x+1)\, x^{-x}\, dx$ for convergence .

7. Evaluate $\oint_{|z| = 3/2} \Gamma(z)\, dz$

APPENDIX I

SOLUTIONS TO PROBLEMS

Problems from Chapter 7

1/(a) Replace z by $2z$ in (9) and get

$$\boxed{\sin 2z = 2z \prod_{n=1}^{\infty}\left(1 - \frac{4z^2}{n^2\pi^2}\right)}$$

(b) Replace z by $3z$ in (9) $\boxed{\sin 3z = 3z \prod_{n=1}^{\infty}\left(1 - \frac{9z^2}{n^2\pi^2}\right)}$

(c) $\sinh z = -i \sin iz$. Replace z by iz in (9) and get

$$\boxed{\sinh z = z \prod_{n=1}^{\infty}\left(1 + \frac{z^2}{n^2\pi^2}\right)}$$

(d) Replace z by $(2z)$ in problem 1(c) and get

$$\boxed{\sinh 2z = 2z \prod_{n=1}^{\infty}\left(1 + \frac{4z^2}{n^2\pi^2}\right)}$$

(e) Replace z by πz in problem 1(c) and get

$$\boxed{\sinh \pi z = \pi z \prod_{n=1}^{\infty}\left(1 + \frac{z^2}{n^2}\right)}$$

(f) $e^{2z} - 1 = e^{z}(e^{z} - e^{-z}) = 2e^{z}\left[\frac{e^{z} - e^{-z}}{2}\right] = 2e^{z}\sinh z$

$$= \boxed{2e^{z}\, z \prod_{n=1}^{\infty}\left(1 + \frac{z^2}{n^2\pi^2}\right)}$$

(g) $\frac{e^{z} - 1}{z} = \frac{2e^{\frac{z}{2}}}{z}\left(\frac{e^{\frac{z}{2}} - e^{-\frac{z}{2}}}{2}\right) = \frac{2e^{\frac{z}{2}}}{z}\sinh\frac{z}{2}$

$$= \boxed{e^{\frac{z}{2}} \prod_{n=1}^{\infty}\left(1 + \frac{z^2}{4n^2\pi^2}\right)}$$

1/ (Continued)

(h) $e^{az} - \bar{e}^{bz} = e^{\frac{a-b}{2}z}\left(e^{\frac{a+b}{2}z} - e^{-\frac{a+b}{2}z}\right)$

$$= 2e^{\frac{a-b}{2}z} \sinh \frac{a+b}{2} z$$

$$= 2e^{\frac{a-b}{2}z}\left(\frac{a+b}{2}z\right)\prod_{n=1}^{\infty}\left(1+\frac{(a+b)^2 z^2}{4\pi^2 n^2}\right)$$

$$= \boxed{(a+b)e^{\frac{a-b}{2}z}\, z\prod_{n=1}^{\infty}\left(1+\frac{(a+b)^2 z^2}{4\pi^2 n^2}\right)}$$

2/ (a) $\cos z$ has zeros at $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \pm\frac{7\pi}{2}, \cdots$

i.e. $\pm\frac{(2n+1)\pi}{2}$, where $n = 0, 1, 2, \cdots$,

Then we effect

$$\cos z = g(z)\prod_{n=0}^{\infty}\left(1-\frac{z^2}{\left[\frac{(2n+1)\pi}{2}\right]^2}\right)$$

$$= g(z)\prod_{n=0}^{\infty}\left(1-\frac{2^2 z^2}{(2n+1)^2\pi^2}\right)$$

When z is very small, $\cos z \approx 1$ and

$1-\frac{4z^2}{\pi^2(2n+1)^2} \approx 1$, Therefore $1 \approx g(0)\cdot 1$

Thus $g(0) = 1$, and as before we guess $g(z) \equiv 1$,

We have

$$\boxed{\cos z = \prod_{n=0}^{\infty}\left(1-\frac{4z^2}{\pi^2(2n+1)^2}\right)}$$

2/ (Continued)

(b) Since $\cosh z = \cos iz$ we replace z by iz in problem 2(a) to get

$$\boxed{\cosh z = \prod_{n=0}^{\infty}\left(1 + \frac{4z^2}{\pi^2(2n+1)^2}\right)}$$

(c) $$e^{i2\pi z}+1 = 2e^{i\pi z}\left(\frac{e^{i\pi z}+e^{-i\pi z}}{2}\right)$$

$$= 2e^{i\pi z}\cos\pi z$$

$$= \boxed{2e^{i\pi z}\prod_{n=0}^{\infty}\left(1 - \frac{4z^2}{(2n+1)^2}\right)}$$

(d) $$\boxed{z\prod_{n=0}^{\infty}\left(1 - \frac{4z^2}{(2n+1)^2}\right)}$$

(e) $$e^{az}+e^{-bz} = e^{\frac{a-b}{2}z}\left(e^{\frac{a+b}{2}z}+e^{-\frac{a+b}{2}z}\right)$$

$$= 2e^{\frac{a-b}{2}z}\cosh\left(\frac{a+b}{2}z\right)$$

Now use problem 2(b) and get

$$e^{az}+e^{-bz} = \boxed{2e^{\frac{a-b}{2}z}\prod_{n=0}^{\infty}\left(1 + \frac{(a+b)^2z^2}{\pi^2(2n+1)^2}\right)}$$

3/ (a) Set $z=1$ in problem 1(e) and get $\boxed{\frac{\sinh\pi}{\pi}}$.

(b) Set $z=\frac{\pi}{2}$ in problem 2(b) and get $\boxed{\cosh\frac{\pi}{2}}$.

(c) Set $z=\pi$ in problem 2(a) and get $\cos\pi = \boxed{-1}$.

4/ (a) Since $\lim_{n\to\infty} n^2 \neq 0$, the product diverges,

(b) Since $\sum |n^{-3}| = \sum n^{-3}$ converges, the product converges.

(c) Since $\sum |e^{-n}| = \sum e^{-n}$ converges, the product converges,

(d) Since $\lim_{n\to\infty} e^{n} \neq 0$, the product diverges,

(e) Since $\sum \frac{1}{\log n}$ diverges, the product diverges,

(f) Since $\sum \left|\frac{1}{n^n}\right|$ converges, the product converges,

5/ All these products converge for all z since the series $\sum |u_n|$ in each case behaves like $\sum n^{-2}$ which converges,

6/ Since

$$\boxed{e^{h(z)} \prod_{n=1}^{\infty} \left(1-\frac{z}{n^2}\right)^2}$$

converges for all z $\left(\sum \left|\frac{z}{n^2}\right| = |z| \sum n^{-2} \text{ converges}\right)$, no exponential convergence factors are required, Here $h(z)$ is an arbitrary entire function,

7/ $e^{h(z)} z \prod_{n=1}^{\infty} \left(1-\frac{z}{in}\right)^3$ does not converge because $\sum \frac{z}{in}$ diverges, The factor $e^{\frac{z}{in}} \approx 1+\frac{z}{in}$ will remove the $\frac{1}{n}$ type term since $\left(1-\frac{z}{in}\right)e^{\frac{z}{in}} \approx \left(1-\frac{z}{in}\right)\left(1+\frac{z}{in}\right) = \left(1+\frac{z^2}{n^2}\right)$, Thus

$$\boxed{e^{h(z)} z \prod_{n=1}^{\infty} \left[\left(1-\frac{z}{in}\right)e^{\frac{z}{in}}\right]^3}$$

converges,

8/ The product $\prod_{n=1}^{\infty}\left(1-\frac{z}{n^{2/3}}\right)^{2}$ diverges since $\sum \frac{z}{n^{2/3}}$ diverges. We try a convergence factor of the type $e^{\frac{z}{n^{2/3}}} = 1 + \frac{z}{n^{2/3}} + \frac{z^2}{2n^{4/3}} + \cdots$

Thus $\left(1-\frac{z}{n^{2/3}}\right)e^{\frac{z}{n^{2/3}}}$ behaves like

$$\begin{array}{l} 1+\frac{z}{n^{2/3}}+\frac{z^2}{2n^{4/3}} \\ 1-\frac{z}{n^{2/3}} \\ \hline 1+\frac{z}{n^{2/3}}+\frac{z^2}{2n^{4/3}} \\ \quad -\frac{z}{n^{2/3}}-\frac{z^2}{n^{4/3}}-\cdots \\ \hline 1 \qquad -\frac{z^2}{2n^{4/3}} \end{array}$$

for large n, Since $\sum \frac{1}{n^{4/3}}$ converges, we have

$$\boxed{e^{h(z)}\prod_{n=1}^{\infty}\left[\left(1-\frac{z}{n^{2/3}}\right)e^{\frac{z}{n^{2/3}}}\right]^{2}}$$

where $h(z)$ is an arbitrary entire function,

9/ The product $\prod_{n=1}^{\infty}\left(1-\frac{z}{n^{1/3}}\right)$ diverges since $\sum \frac{z}{n^{1/3}}$ diverges. Try the convergence factor.

$$e^{\frac{z}{n^{1/3}}} = 1 + \frac{z}{n^{1/3}} + \frac{z^2}{2n^{2/3}} + \cdots$$

Thus the product $\left(1-\frac{z}{n^{1/3}}\right)e^{\frac{z}{n^{1/3}}}$, for large n, behaves like

9/ (continued)

$$\begin{array}{l}
1 + \frac{z}{n^{1/3}} + \frac{z^2}{2n^{2/3}} + \frac{z^3}{6n} + \cdots \\
1 - \frac{z}{n^{1/3}} \\
\hline
1 + \frac{z}{n^{1/3}} + \frac{z^2}{2n^{2/3}} + \frac{z^3}{6n} + \cdots \\
\quad - \frac{z}{n^{1/3}} - \frac{z^2}{n^{2/3}} - \frac{z^3}{2n} - \cdots \\
\hline
1 \qquad - \frac{z^2}{2n^{2/3}} - \frac{z^3}{3n} - \cdots
\end{array}$$

Since $\sum \frac{z^2}{2n^{2/3}}$ diverges also, the product still diverges, Now try the additional convergence factor

$$e^{\frac{z^2}{2n^{2/3}}} = 1 + \frac{z^2}{2n^{2/3}} + \frac{z^4}{8n^{4/3}} + \cdots$$

Thus the factor $\left(1 - \frac{z}{n^{1/3}}\right) e^{\frac{z}{n^{1/3}}} e^{\frac{z^2}{2n^{2/3}}}$ behaves like

$$\begin{array}{l}
1 - \frac{z^2}{2n^{2/3}} - \frac{z^3}{3n} - \\
1 + \frac{z^2}{2n^{2/3}} + \frac{z^4}{8n^{4/3}} \\
\hline
1 - \frac{z^2}{2n^{2/3}} - \frac{z^3}{3n} \\
\quad + \frac{z^2}{2n^{2/3}} \qquad\qquad - \frac{z^4}{4n^{4/3}} - \frac{z^5}{6n^{5/3}} \\
\qquad\qquad\qquad\qquad + \frac{z^4}{8n^{4/3}} \qquad\qquad - \frac{z^6}{16n^2} \\
\hline
1 \qquad\qquad - \frac{z^3}{3n} - \frac{z^4}{8n^{4/3}} - \cdots
\end{array}$$

9/ (continued)

This last factor behaves like $1 - \frac{z^3}{3n}$ for large n which is still too slow because $\sum \frac{z^3}{3n}$ diverges. One more convergence factor $e^{\frac{z^3}{3n}}$ will remove the $\frac{1}{n}$ type term and replace it with a term of the type $\frac{1}{n^{4/3}}$ which is sufficient to insure convergence of the infinite product. Thus we have

$$\boxed{e^{h(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{n^{1/3}}\right) e^{\frac{z}{n^{1/3}} + \frac{z^2}{2n^{2/3}} + \frac{z^3}{3n}}}$$

10/ Looking at the solution to problem 2(a), we anticipate

$$\cos z = \left[\prod_{n=0}^{\infty}\left(1 - \frac{2z}{\pi(2n+1)}\right)\right]\left[\prod_{n=0}^{\infty}\left(1 + \frac{2z}{\pi(2n+1)}\right)\right].$$

Neither of these products converge because $\sum \frac{2z}{\pi(2n+1)}$ diverges at the same rate as $\sum \frac{1}{n}$, Convergence factors $e^{\pm\frac{2z}{\pi(2n+1)}}$ are required,

$$\boxed{\cos z = \left[\prod_{n=0}^{\infty}\left(1 - \frac{2z}{\pi(2n+1)}\right) e^{\frac{2z}{\pi(2n+1)}}\right] \cdot \left[\prod_{n=0}^{\infty}\left(1 + \frac{2z}{\pi(2n+1)}\right) e^{-\frac{2z}{\pi(2n+1)}}\right]}$$

11/(a)

$$\frac{\sinh \pi z}{\pi} = z \prod_{n=1}^{\infty}\left(1+\frac{z^2}{n^2}\right)$$

$$\frac{\pi \cosh \pi z}{\sinh \pi z} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{\frac{2z}{n^2}}{1+\frac{z^2}{n^2}}$$

$$\boxed{\pi \coth \pi z = \frac{1}{z} + 2z \sum_{n=1}^{\infty} \frac{1}{z^2+n^2}}$$

(b)

$$\cos z = \prod_{n=0}^{\infty}\left(1 - \frac{4z^2}{\pi^2(2n+1)^2}\right)$$

$$-\frac{\sin z}{\cos z} = \sum_{n=0}^{\infty} \frac{\frac{-8z}{\pi^2(2n+1)^2}}{1-\frac{4z^2}{\pi^2(2n+1)^2}}$$

$$\boxed{-\tan z = 8z \sum_{n=0}^{\infty} \frac{1}{4z^2-\pi^2(2n+1)^2}}$$

(c)

$$\cosh z = \prod_{n=0}^{\infty}\left(1 + \frac{4z^2}{\pi^2(2n+1)^2}\right)$$

$$\frac{\sinh z}{\cosh z} = \sum_{n=0}^{\infty} \frac{\frac{8z}{\pi^2(2n+1)^2}}{1+\frac{4z^2}{\pi^2(2n+1)^2}}$$

$$\boxed{\tanh z = 8z \sum_{n=0}^{\infty} \frac{1}{4z^2+\pi^2(2n+1)^2}}$$

(d)

$$\cos z = \left[\prod_{n=0}^{\infty}\left(1-\frac{2z}{\pi(2n+1)}\right)e^{\frac{2z}{\pi(2n+1)}}\right]\left[\prod_{n=0}^{\infty}\left(1+\frac{2z}{\pi(2n+1)}\right)e^{-\frac{2z}{\pi(2n+1)}}\right]$$

$$\frac{-\sin z}{\cos z} = \sum_{n=0}^{\infty}\left[\frac{\frac{-2}{\pi(2n+1)}}{1-\frac{2z}{\pi(2n+1)}} + \frac{\frac{2}{\pi(2n+1)}\,e^{\frac{2z}{\pi(2n+1)}}}{e^{\frac{2z}{\pi(2n+1)}}}\right]$$

$$+ \sum_{n=0}^{\infty}\left[\frac{\frac{2}{\pi(2n+1)}}{1+\frac{2z}{\pi(2n+1)}} + \frac{\frac{-2}{\pi(2n+1)}\,e^{-\frac{2z}{\pi(2n+1)}}}{e^{-\frac{2z}{\pi(2n+1)}}}\right]$$

11/ (d) (continued)

$$\boxed{\begin{aligned}-\tan z = {} & 2\sum_{n=0}^{\infty}\left[\frac{1}{2z-\pi(2n+1)}+\frac{1}{\pi(2n+1)}\right]\\ & +2\sum_{n=0}^{\infty}\left[\frac{1}{2z+\pi(2n+1)}-\frac{1}{\pi(2n+1)}\right]\end{aligned}}$$

12/

$$\boxed{f(z) = g(z) + \sum_{n=-\infty}^{\infty}\frac{1}{(z-n)^2}}$$

No alteration is needed since $\Sigma \frac{1}{n^2}$ converges.

13/ The function $f(z) = g(z) + \sum\limits_{n=-\infty}^{\infty} \frac{(-1)^n}{z-n}$

requires alteration since $\Sigma \frac{1}{n}$ diverges.

$$\frac{1}{z-n} = -\frac{1}{n}\left(\frac{1}{1-\frac{z}{n}}\right) = -\frac{1}{n}\left(1+\frac{z}{n}+\cdots\right) = -\frac{1}{n}-\frac{z}{n^2}-\cdots$$

Thus since $\frac{1}{z-n}+\frac{1}{n} \approx \frac{z}{n^2}$ we have

$$\boxed{f(z) = g(z) + \frac{1}{z} + \sum_{n=-\infty}^{\infty}{}' \left[\frac{(-1)^n}{z-n}+\frac{(-1)^n}{n}\right]}$$

where $g(z)$ is entire.

14/ $\csc \pi z = \frac{1}{\sin \pi z}$ has simple poles at $z=n$ where n is any integer with residue $\frac{(-1)^n}{\pi}$.

From problem 13 we have

$$\csc \pi z = g(z) + \frac{1}{\pi z} + \frac{1}{\pi}\sum_{n=-\infty}^{\infty}{}' \left[\frac{(-1)^n}{z-n}+\frac{(-1)^n}{n}\right].$$

The function $g(z) \equiv 0$, although we will not demonstrate this.

$$\boxed{\pi \csc \pi z = \frac{1}{z} + \sum_{n=-\infty}^{\infty}{}' (-1)^n\left[\frac{1}{z-n}+\frac{1}{n}\right]}$$

15/ $f(z) = g(z) + \sum_{n=1}^{\infty} \frac{1}{z - n^{1/3}}$ does not converge since it behaves like $\sum n^{-1/3}$,

$$\frac{1}{z - n^{1/3}} = -\frac{1}{n^{1/3}}\left(\frac{1}{1 - \frac{z}{n^{1/3}}}\right) =$$

$$= -\frac{1}{n^{1/3}}\left(1 + \frac{z}{n^{1/3}} + \frac{z^2}{n^{2/3}} + \frac{z^3}{n} + \cdots\right)$$

$$= -\frac{1}{n^{1/3}} - \frac{z}{n^{2/3}} - \frac{z^2}{n} - \frac{z^3}{n^{4/3}} - \cdots$$

Thus for large n we have

$$\frac{1}{z - n^{1/3}} + \frac{1}{n^{1/3}} + \frac{z}{n^{2/3}} + \frac{z^2}{n} \approx -\frac{z^3}{n^{4/3}}$$

Since $\sum n^{-4/3}$ converges we have

$$\boxed{f(z) = g(z) + \sum_{n=1}^{\infty}\left[\frac{1}{z - n^{1/3}} + \frac{1}{n^{1/3}} + \frac{z}{n^{2/3}} + \frac{z^2}{n}\right]}$$

where $g(z)$ is an arbitrary entire function,

16/(a) at $z = -2+3i$ on FIG 3, we have $\rho = 0{,}001$ and $\phi = \frac{\pi}{2}$. Thus $\Gamma(-2+3i) \approx 0{,}001\, \rho^{i\pi/2} \approx \boxed{0{,}001\, i}$

(b) at $z = 4 - i$ on FIG 3, we have $\rho = 5$ and $\phi = -\frac{\pi}{2}$, Thus $\Gamma(4-i) \approx 5\, e^{-i\pi/2} \approx \boxed{-5i}$

(c) Using Fig 1, we have $\Gamma(-\frac{1}{2}) \approx \boxed{-3{,}6}$

17/ Since $z! = \Gamma(z+1)$, the answers to (a), (b) and (c) are the same as those of the previous problem.

(d) Using Figure 1 we have $\left(-\frac{5}{2}\right)! = \Gamma\left(-\frac{5}{2}+1\right) = \Gamma\left(-\frac{3}{2}\right)$

$\approx \boxed{2.4}$

18/(a) Set $z = 2$ in (1) and get $\Gamma(2) = \int_0^\infty e^{-t} t \, dt$.

Integrating by parts we have

$$\Gamma(2) = \int_0^\infty t \, d(-e^{-t}) = -te^{-t}\Big|_0^\infty + \int_0^\infty e^{-t} dt = 0 + 1.$$

(b) Set $z = 3$ in (1) and get $\Gamma(3) = \int_0^\infty e^{-t} t^2 dt$

$$\Gamma(3) = \int_0^\infty t^2 d(-e^{-t})$$

$$= -t^2 e^{-t}\Big|_0^\infty + 2\int_0^\infty t \, e^{-t} dt$$

$$= 0 \quad + 2\,\Gamma(2) \quad = \quad 2$$

(c) Set $z = 4$ in (1) and get $\Gamma(4) = \int_0^\infty e^{-t} t^3 dt$

$$= \int_0^\infty t^3 \, d(-e^{-t}) = -t^3 e^{-t} + 3\int_0^\infty e^{-t} t^2 dt$$

$$= \quad 0 + 3\;\Gamma(3) \quad = \quad 3 \cdot 2 = 6.$$

19/ Replace z by $z+1$ in (1) to get

$$\Gamma(z+1) = \int_0^\infty e^{-t} t^z dt = \int_0^\infty t^z d(-e^{-t}) =$$

$$-t^z e^{-t}\Big|_0^\infty + z\int_0^\infty e^{-t} t^{z-1} dt = 0 + z\,\Gamma(z),$$

20/(a) $\Gamma\left(-\frac{1}{2}\right) = \dfrac{\left(-\frac{1}{2}\right)\Gamma\left(-\frac{1}{2}\right)}{\left(-\frac{1}{2}\right)} = \dfrac{\Gamma\left(1-\frac{1}{2}\right)}{\left(-\frac{1}{2}\right)} = -2\,\Gamma\left(\frac{1}{2}\right) = \boxed{-2\sqrt{\pi}}$

(b) $\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \boxed{\frac{\sqrt{\pi}}{2}}$

(c) $\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{2}\,\frac{\sqrt{\pi}}{2} = \boxed{\frac{3\sqrt{\pi}}{2^2}}$

(d) $\Gamma\left(\frac{7}{2}\right) = \frac{5}{2}\Gamma\left(\frac{5}{2}\right) = \frac{5}{2} \cdot \frac{3\sqrt{\pi}}{2^2} = \boxed{\frac{5 \cdot 3\sqrt{\pi}}{2^3}}$

(e) $\Gamma\left(\frac{9}{2}\right) = \frac{7}{2}\Gamma\left(\frac{7}{2}\right) = \frac{7}{2} \cdot \frac{5 \cdot 3\sqrt{\pi}}{2^3} = \boxed{\frac{7 \cdot 5 \cdot 3\sqrt{\pi}}{2^4}}$

21/ AN EXAMINATION OF THE RESULTS OF THE PREVIOUS PROBLEMS SHOWS THAT FOR (b), $n=1$; (c) $n=2$; (d) $n=3$; (e) $n=4$;

$$\Gamma\left(\frac{2n+1}{2}\right) = \frac{1\cdot 3\cdot 5\cdots(2n-1)\sqrt{\pi}}{2^n}$$

$$= \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdots(2n-2)(2n-1)\sqrt{\pi}}{2\cdot 4\cdot 6\cdots(2n-2)\quad 2^n}$$

$$= \frac{(2n-1)!\,\sqrt{\pi}}{(2\cdot 1)\cdot(2\cdot 2)\cdot(2\cdot 3)\cdots(2\cdot(n-1))\quad 2^n}$$

$$= \frac{(2n-1)!\,\sqrt{\pi}}{2^{n-1}(1\cdot 2\cdot 3\cdots(n-1))\quad 2^n}$$

$$= \boxed{\frac{(2n-1)!\,\sqrt{\pi}}{2^{2n-1}\,(n-1)!}}$$

22/ (a) $$\Gamma(z) = \frac{\Gamma(z)\,z\,(z+1)\,(z+2)}{z\,(z+1)(z+2)} = \frac{\Gamma(z+1)\,(z+1)(z+2)}{z\,(z+1)\,(z+2)}$$

$$= \frac{\Gamma(z+2)(z+2)}{z\,(z+1)(z+2)} = \frac{\Gamma(z+3)}{z\,(z+1)(z+2)}$$

as z nears -2 this last expression becomes

$$\Gamma(z)\Big|_{z \text{ near } -2} \approx \frac{\Gamma(1)}{-2(-1)\,(z+2)} = \frac{1}{2\,(z+2)}$$

Residue at -2 = $\boxed{\frac{1}{2}}$

(b) $$\Gamma(z) = \frac{\Gamma(z+3)}{z(z+1)(z+2)}\,\frac{(z+3)}{(z+3)} = \frac{\Gamma(z+4)}{z\,(z+1)(z+2)\,(z+3)}$$

as z nears -3 we have

$$\Gamma(z)\Big|_{z \text{ near } -3} \approx \frac{\Gamma(1)}{(-3)(-2)(-1)\,(z+3)} = \frac{(-1)^3}{3!\,(z+3)}$$

Residue at -3 = $\boxed{\frac{(-1)^3}{3!}}$

22/ (continued)

(c) $$\Gamma(z) = \frac{\Gamma(z+n+1)}{z(z+1)(z+2)\cdots(z+n)}$$

$$\Gamma(z)\Big|_{\substack{\text{near}\\ z=-n}} \approx \frac{\Gamma(1)}{(-n)(1-n)(2-n)\cdots(-1)\ (z+n)} = \frac{(-1)^n}{n!\ (z+n)}$$

Thus residue at $z=-n$ $= \boxed{\dfrac{(-1)^n}{n!}}$

23/ Set $t = ua$ and get $I = \int_0^1 a^2u^2(a-au)^{-\frac{1}{2}}\,a\,du$

$$I = a^2a^{-\frac{1}{2}}a\int_0^1 u^2(1-u)^{-\frac{1}{2}}du = a^{5/2}\,B(3,\tfrac{1}{2})$$

$$= a^{5/2}\,\frac{\Gamma(3)\Gamma(\frac{1}{2})}{\Gamma(\frac{7}{2})} = a^{5/2}\,\frac{2(\sqrt{\pi})}{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}(\sqrt{\pi})} = \boxed{\frac{16\,a^{5/2}}{15}}$$

24/ Set $t = ua$ and get $I = \int_0^1 a^{1/2}u^{1/2}(a-au)^{-1/2}\,a\,du$

$$I = a\int_0^1 u^{1/2}(1-u)^{-1/2}du = a\,B(\tfrac{3}{2},\tfrac{1}{2})$$

$$= a\,\frac{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})}{\Gamma(2)} = a\,\frac{(\frac{\sqrt{\pi}}{2})(\sqrt{\pi})}{1} = \boxed{\frac{\pi a}{2}}$$

25/ $t = \sin^2\theta$, $dt = 2\sin\theta\cos\theta\,d\theta$, $1-t = 1-\sin^2\theta = \cos^2\theta$

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt = \int_0^{\pi/2}(\sin\theta)^{2x-2}(\cos\theta)^{2y-2}(2\sin\theta\cos\theta\,d\theta)$$

$$= 2\int_0^{\pi/2}(\sin\theta)^{2x-1}(\cos\theta)^{2y-1}d\theta$$

Set $2x-1 = p$ and $2y-1 = q$ $\Rightarrow$ $x = \frac{p+1}{2}$, $y = \frac{q+1}{2}$

Therefore

$$\int_0^{\pi/2}(\sin\theta)^p(\cos\theta)^q d\theta = \frac{1}{2}B\left(\frac{p+1}{2},\frac{q+1}{2}\right) = \boxed{\frac{\Gamma(\frac{p+1}{2})\Gamma(\frac{q+1}{2})}{2\Gamma(\frac{p+q+2}{2})}}$$

28/ For $z=2$, we multiply 1,919 by $\left(1+\frac{1}{12z}\right)$ to get

$$2! \approx (1,919)\left(1+\frac{1}{24}\right) = \boxed{1,99896}$$

For $z=3$, we multiply 5,836 by $\left(1+\frac{1}{12z}\right)$ to get

$$3! \approx (5,836)\left(1+\frac{1}{36}\right) = \boxed{5,9981}$$

For $z=4$, we multiply 23,506 by $\left(1+\frac{1}{12z}\right)$ to get

$$4! \approx (23,506)\left(1+\frac{1}{48}\right) = \boxed{23,9957}$$

29/ $\Gamma(1+z) \approx z^{z} e^{-z} \sqrt{2\pi z} = e^{z \log z} e^{-z} \sqrt{2\pi z}$

$$= e^{4i\left[\log 4 + \frac{\pi}{2}i\right]} e^{-4i} \sqrt{2\pi i 4}$$

$$= 2e^{-2\pi} e^{i(4\log 4 - 4)} \sqrt{2\pi}\, e^{i\frac{\pi}{4}}$$

$$= 2\sqrt{2\pi}\, e^{-2\pi} e^{i\left(4\log 4 - 4 + \frac{\pi}{4}\right)}$$

$$= \boxed{0.00936\, e^{i(2,33)}} \longleftarrow \text{check this}$$

Looking at Fig. 3, of Section 7,5 we see that

$$\Gamma(1+4i) \approx 0.01\, e^{i\frac{3\pi}{4}} = \boxed{0.01\, e^{i(2,36)}}.$$

30/ $\Gamma(z+1) = \lim\limits_{N\to\infty} \dfrac{N^{z+1}\, N!}{(z+1)(z+2)\cdots(z+1+N)}$

$$= z \lim_{N\to\infty} \frac{N!\, N^{z}}{z(z+1)(z+2)\cdots(z+N)} \cdot \frac{N}{(z+N+1)}$$

$$= z\,\Gamma(z) \cdot \lim_{N\to\infty} \frac{N}{(N+z+1)}$$

$$= z\,\Gamma(z)$$

Solutions to Review Problems from Chapter 7

1 (a) Since $\cos z = \prod_{n=0}^{\infty}\left(1 - \frac{4z^2}{\pi^2(2n+1)^2}\right)$, (see Problem 2(a)),

and since $\cosh z = \cos iz$ we replace z by $i\pi z$ this product to get

$$\boxed{\cosh \pi z = \prod_{n=0}^{\infty}\left(1 + \frac{4z^2}{(2n+1)^2}\right)}.$$

(b) Set $z = \sqrt{2}$ and get $\boxed{\cosh\sqrt{2}}$.

2/ We first try $g(z) z \prod_{n=1}^{\infty}\left(1 - \frac{z}{i n^{3/4}}\right)$,

but this product diverges since $\sum n^{-3/4}$ diverges. Try the exponential convergence factor

$$e^{\frac{z}{in^{3/4}}} \approx 1 + \frac{z}{i n^{3/4}} - \frac{z^2}{2n^{3/2}}$$

$$\begin{array}{rlll} & 1 - \frac{z}{in^{3/4}} & & \\ \hline & 1 + \frac{z}{in^{3/4}} & - \frac{z^2}{2n^{3/2}} & + \cdots \\ & \quad - \frac{z}{in^{3/4}} & + \frac{z^2}{n^{3/2}} & + \cdots \\ \hline \left(1 - \frac{z}{in^{3/4}}\right)e^{\frac{z}{in^{3/4}}} \approx & 1 & + \frac{z^2}{n^{3/2}} & \end{array}$$

Since $\sum n^{-3/2}$ converges, the desired product is

$$\boxed{g(z) z \prod_{n=0}^{\infty}\left(1 - \frac{z}{i n^{3/4}}\right)e^{\frac{z}{in^{3/4}}}},$$

where $g(z) = e^{h(z)}$, $h(z)$ any entire function.

3/ Using the product for cosh πz in review problem 1(a), we have

$$\frac{\frac{d}{dz}(\cosh \pi z)}{\cosh \pi z} = \sum_{n=0}^{\infty} \frac{\frac{d}{dz}\left(1 + \frac{4z^2}{(2n+1)^2}\right)}{1 + \frac{4z^2}{(2n+1)^2}}$$

$$\frac{\pi \sinh \pi z}{\cosh \pi z} = \sum_{n=0}^{\infty} \frac{\frac{8z}{(2n+1)^2}}{1 + \frac{4z^2}{(2n+1)^2}}$$

$$\boxed{\tanh \pi z = \frac{8z}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2 + 4z^2}}$$

4/ Set $st = u$, $du = s\,dt$ and get

$$I = \int_0^{\infty} e^{-u} \left(\frac{u}{s}\right)^{3/2} \frac{du}{s} = \frac{1}{s^{5/2}} \Gamma\left(\frac{5}{2}\right)$$

$$= s^{-5/2} \left(\frac{3}{2}\right) \Gamma\left(\frac{3}{2}\right) = s^{-5/2} \left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)$$

$$= \boxed{\frac{3\sqrt{\pi}}{4} s^{-5/2}}$$

5/ $$\int_0^1 t^{a-1} (1-t)^{4-a-1} dt = \frac{\Gamma(a)\Gamma(4-a)}{\Gamma(a+4-a)}$$

$$= \frac{\Gamma(a)\Gamma(3-a)(3-a)}{\Gamma(4)} = \frac{\Gamma(a)\Gamma(2-a)(2-a)(3-a)}{3!}$$

$$= \frac{\Gamma(a)\Gamma(1-a)(1-a)(2-a)(3-a)}{6} = \boxed{\frac{(1-a)(2-a)(3-a)}{6} \frac{\pi}{\sin \pi a}}$$

6/ For large x, $\Gamma(x+1) \approx x^{x} e^{-x}\sqrt{2\pi x}$, thus $\Gamma(x+1)\, x^{-x} \approx e^{-x}\sqrt{2\pi x}$ for large x, Since e^{-x} dominates $\sqrt{x}$ for large x, the integral converges.

7/

$$\oint_{|z| = 3/2} \Gamma(z)\, dz = 2\pi i \left[\text{Res}\left(\Gamma(z), 0\right) + \text{Res}\left(\Gamma(z), -1\right) \right]$$

$$= 2\pi i \left[1 + \frac{(-1)^{1}}{1!} \right] = \boxed{0}.$$

APPENDIX II

ANSWERS TO CONJECTURES

Chapter 7

1. The infinite product itself $\prod_{n=1}^{\infty}(1 - \frac{z}{r_n})^{m_n}$ gives no hint of having any singularities. In fact, if this product converges, it represents an entire function. The function g(z) is therefore, iself an entire function.

2. The most general form for an entire function without zeros, g(z), is $e^{h(z)}$, where h(z) is any entire function. Thus we can replace g(z) in (5) by $e^{h(z)}$ and get

$$(1) \qquad f(z) = e^{h(z)} \prod_{n=0}^{\infty}(1 - \frac{z}{r_n})^{m_n}$$

where h(z) is an arbitrary entire function. This formula (1) is the most general form for an entire function f(z) having zeros at r_n of multiplicity m_n, n=1, 2, ... , provided the r_n tend to infinity fast enough to make (1) converge. In Conjecture 4 we will modify (1) so that it always converges.

3. The n^{th} term test for infinite products

A necessary (but not sufficient) condition for the convergence of the infinite product

$$\prod_{n=0}^{\infty}(1 + u_n)$$

is that

$$\lim_{n \to \infty} u_n = 0 .$$

4. Weierstrass' factor theorem

Suppose $f(z)$ is a given entire function having zeros at r_n of multiplicity m_n, where $n = 1, 2, 3, \ldots$. Then $f(z)$ can be expressed in the form of the convergent product

$$f(z) = e^{h(z)} z^{m_o} \prod_{n=1}^{\infty} \left[\left(1 - \frac{z}{r_n}\right) \exp\left(\frac{z}{r_m} + \frac{1}{2}\left(\frac{z}{r_n}\right)^2 + \ldots \frac{1}{k_n}\left(\frac{z}{r_n}\right)^{k_n} \right) \right]^{m_n}$$

where $h(z)$ is some entire function. The constants k_n, $n = 1, 2, 3, \ldots$, must be determined so as to make the infinite product converge. The factor z^{m_o} reveals a zero of multiplicity m_o at the origin and is to be suppressed in case the origin is not a zero of the given function $f(z)$.

5. Mittag - Leffler's partial fractions theorem

Suppose $f(z)$ is a given meromorphic function having poles at the points $z = r_n$ for positive integral n, and suppose that the principal part of $f(z)$ at r_n is given as

$$P_n(z) = \sum_{k=1}^{m_n} a_{n,k}(z - r_n)^{-k} .$$

Then $f(z)$ can be expressed in the form

$$f(z) = g(z) + \sum_{n=1}^{\infty} [\, P_n(z) + h_n(z) \,]$$

where $g(z)$ is some entire function and the functions $h_n(z)$ are polynomials selected so as to make the infinite series converge for all $z \neq r_n$.