APPENDIX I

SOLUTIONS TO PROBLEMS

Problems from Chapter 4

1/ Replace z by z^2 in (1) and get the result at once.

2/ $$\frac{1}{a-z} = \frac{1}{a}\left(\frac{1}{1-\frac{z}{a}}\right) = \frac{1}{a}\sum_{n=0}^{\infty}\left(\frac{z}{a}\right)^n$$

$$= \sum_{n=0}^{\infty}\frac{z^n}{a^{n+1}}.$$

3/ $$\frac{1}{2-z} = \frac{1}{1+1-z} = \frac{1}{1-(z-1)} = \sum_{n=0}^{\infty}(z-1)^n.$$

4/ $$\frac{1}{4+z} = \frac{1}{3+(z+1)} = \frac{1}{3}\left(\frac{1}{1+\frac{z+1}{3}}\right) = \frac{1}{3}\left(\frac{1}{1-\left(-\frac{z+1}{3}\right)}\right)$$

$$= \frac{1}{3}\sum_{n=0}^{\infty}\left(-\frac{z+1}{3}\right)^n = \sum_{n=0}\frac{(-1)^n(z+1)^n}{3^{n+1}}$$

5/ $$\frac{1}{a-z} = \frac{1}{a-z_0-z+z_0} = \frac{1}{(a-z_0)-(z-z_0)} =$$

$$\frac{1}{a-z_0}\left(\frac{1}{1-\frac{z-z_0}{a-z_0}}\right) = \frac{1}{a-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{a-z_0}\right)^n =$$

$$= \sum_{n=0}^{\infty}\frac{(z-z_0)^n}{(a-z_0)^{n+1}}$$

6/ Forming $\sin^2 z$ we have

$$\begin{array}{l}
z - \frac{z^3}{6} + \frac{z^5}{120} - \cdots \\
z - \frac{z^3}{6} + \frac{z^5}{120} - \cdots \\
\hline
z^2 - \frac{z^4}{6} + \frac{z^6}{120} - \cdots \\
\qquad - \frac{z^4}{6} + \frac{z^6}{36} - \cdots \\
\qquad\qquad + \frac{z^6}{120} - \cdots \\
\hline
z^2 - \frac{z^4}{3} + \frac{2z^6}{45} - \cdots
\end{array}$$

Forming $\cos^2 z$ we have

$$\begin{array}{l}
1 - \frac{z^2}{2} + \frac{z^4}{24} - \frac{z^6}{720} + \cdots \\
1 - \frac{z^2}{2} + \frac{z^4}{24} - \frac{z^6}{720} + \cdots \\
\hline
1 - \frac{z^2}{2} + \frac{z^4}{24} - \frac{z^6}{720} + \cdots \\
\quad - \frac{z^2}{2} + \frac{z^4}{4} - \frac{z^6}{48} + \cdots \\
\qquad\qquad + \frac{z^4}{24} - \frac{z^6}{48} + \cdots \\
\qquad\qquad\qquad - \frac{z^6}{720} + \cdots \\
\hline
1 - z^2 + \frac{z^4}{3} - \frac{2z^6}{45} + \cdots
\end{array}$$

Adding these series for $\sin^2 z$ and $\cos^2 z$ we get

$$1 + 0 + 0 + 0 + \cdots = 1,$$

7/ $$e^{iz} = \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} = 1 + \frac{iz}{1} + \frac{(iz)^2}{2} + \frac{(iz)^3}{3!} + \frac{(iz)^4}{4!} + \cdots$$

$$e^{iz} = 1 + iz - \frac{z^2}{2} - i\frac{z^3}{3!} + \frac{z^4}{4!} + i\frac{z^5}{5!} + \cdots$$

$$e^{-iz} = 1 + (-iz) + \frac{(-iz)^2}{2} + \frac{(-iz)^3}{3!} + \frac{(-iz)^4}{4!} + \frac{(-iz)^5}{5!} + \cdots$$

$$e^{-iz} = 1 - iz - \frac{z^2}{2} + i\frac{z^3}{3!} + \frac{z^4}{4!} - i\frac{z^5}{5!} + \cdots$$

adding we get

$$e^{iz} + e^{-iz} = 2 - 2\frac{z^2}{2} + 2\frac{z^4}{4!} + \cdots$$

$$= 2\left(1 - \frac{z^2}{2} + \frac{z^4}{4!} + \cdots\right) = 2\cos z,$$

8/ Subtracting the series for e^{iz} and e^{-iz} obtained in problem 7 we get

$$e^{iz} - e^{-iz} = 2iz - 2i\frac{z^3}{3!} + 2i\frac{z^5}{5!} + \cdots$$

$$= 2i\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right)$$

$$= 2i \sin z,$$

9/ $\cosh z = \frac{1}{2}(e^z + e^{-z})$

$$= \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{z^n}{n!} + \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{n!}\right)$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z^n}{n!} + \frac{(-1)^n z^n}{n!}\right)$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}(1 + (-1)^n)\frac{z^n}{n!}$$

$$= \frac{1}{2}\left(2 + 2\frac{z^2}{2!} + 2\frac{z^4}{4!} + 2\frac{z^6}{6!} + \cdots\right)$$

$$= 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots$$

$$= \sum_{n=0}^{\infty}\frac{z^{2n}}{(2n)!}.$$

10/ $\tan z = \dfrac{\sin z}{\cos z}$, Dividing the series for cos z into the series for sin z we get

$$
\begin{array}{rl}
 & z + \frac{z^3}{3} + \frac{2z^5}{15} \\
\cline{2-2}
1 - \frac{z^2}{2} + \frac{z^4}{24} - \cdots \Big) & z - \frac{z^3}{6} + \frac{z^5}{120} - \cdots \\
 & z - \frac{z^3}{2} + \frac{z^5}{24} - \cdots \\
\cline{2-2}
 & \frac{z^3}{3} - \frac{z^5}{30} + \cdots \\
 & \frac{z^3}{3} - \frac{z^5}{6} + \cdots \\
\cline{2-2}
 & \frac{2z^5}{15} + \cdots \\
 & \frac{2z^5}{15} + \cdots \\
\cline{2-2}
 & \cdots
\end{array}
$$

Thus $\tan z = z + \dfrac{z^3}{3} + \dfrac{2z^5}{15} + \cdots$,

11/ $\cosh z = 1 + \dfrac{z^2}{2!} + \dfrac{z^4}{4!} + \dfrac{z^6}{6!} + \cdots$

$$\frac{d \cosh z}{dz} = \frac{2z}{2!} + \frac{4z^3}{4!} + \frac{6z^5}{6!} + \cdots$$

$$\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$$

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} ,$$

12/

$$\frac{1}{1+t} = \frac{1}{1-(-t)} = \sum_{n=0}^{\infty} (-t)^n = \sum_{n=0}^{\infty} (-1)^n t^n$$

$$\int_0^z \frac{1}{1+t}\, dt = \sum_{n=0}^{\infty} (-1)^n \int_0^z t^n\, dt$$

$$\mathrm{Log}(z+1) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{n+1}}{n+1},$$

13/ Replacing z by $z-1$ in the previous result we get

$$\mathrm{Log}\left((z-1)+1\right) = \sum_{n=0}^{\infty} (-1)^n \frac{(z-1)^{n+1}}{n+1}$$

$$\mathrm{Log}\, z = \quad '' \quad ,$$

14/

$$\frac{d}{dz}(1-z)^{-1} = \sum_{n=0}^{\infty} \frac{d}{dz} z^n$$

$$(1-z)^{-2} = \sum_{n=1}^{\infty} n z^{n-1} = \sum_{n=0}^{\infty} (n+1) z^n$$

15/

$$\frac{d}{dz}(1-z)^{-2} = \sum_{n=0}^{\infty} (n+1) \frac{d}{dz} z^n$$

$$2(1-z)^{-3} = \sum_{n=1}^{\infty} (n+1) n\, z^{n-1} = \sum_{n=0}^{\infty} (n+2)(n+1) z^n$$

$$(1-z)^{-3} = \sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{2} z^n$$

16/ $D^n e^z\Big|_{z=0} = e^z\Big|_{z=0} = e^0 = 1.$

THUS $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$.

17/ FIRST; $e^z = e^{z_0} e^{z-z_0} = e^{z_0} \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{n!} = \sum_{n=0}^{\infty} \frac{e^{z_0}}{n!} (z-z_0)^n,$

SECOND; $D^n e^z\Big|_{z=z_0} = e^{z_0},$ THUS $e^z = \sum_{n=0}^{\infty} \frac{e^{z_0}}{n!}(z-z_0)^n,$

18/ $D^0 (a-z)^{-1}\Big|_{z_0} = (a-z)^{-1}\Big|_{z_0} = (a-z_0)^{-1}$

$D^1 (a-z)^{-1}\Big|_{z_0} = (a-z)^{-2}\Big|_{z_0} = (a-z_0)^{-2}$

$D^2 (a-z)^{-1}\Big|_{z_0} = 2(a-z)^{-3}\Big|_{z_0} = 2(a-z_0)^{-3}$

$D^3 (a-z)^{-1}\Big|_{z_0} = 2\cdot 3(a-z)^{-4}\Big|_{z_0} = 3!(a-z_0)^{-4}$

$\vdots$

$D^n (a-z)^{-1}\Big|_{z_0} = \qquad = n!(a-z_0)^{-n-1}$

THUS $(a-z)^{-1} = \sum_{n=0}^{\infty} (a-z_0)^{-n-1}(z-z_0)^n,$

19/ $D^0 \operatorname{Log} z\Big|_1 = \operatorname{Log} z\Big|_1 = 0$

$D^1 \operatorname{Log} z\Big|_1 = z^{-1}\Big|_1 = 1$

$D^2 \operatorname{Log} z\Big|_1 = -z^{-2}\Big|_1 = -1$

$D^3 \operatorname{Log} z\Big|_1 = 2z^{-3}\Big|_1 = 2$

$\vdots$

$D^n \operatorname{Log} z\Big|_1 = (n-1)!(-1)^{n-1} z^{-n}\Big|_1 = (-1)^{n-1}(n-1)!$

THUS $\operatorname{Log} z = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(z-1)^n$.

20/ $\sin i = i - \frac{(i)^3}{3!} + \frac{(i)^5}{5!} - \frac{(i)^7}{7!} + \frac{(i)^9}{9!} - \cdots$

$$\cong i + \frac{i}{3!} + \frac{i}{5!} + \frac{i}{7!} + \frac{i}{9!}$$

$$\cong i\left(1 + \frac{1}{3!} + \frac{1}{5!} + \frac{1}{7!} + \frac{1}{9!}\right)$$

Using the table in Example 1 we have

$\sin i \cong (1{,}1752012)i$, The series converges,

21/ $\operatorname{Log} z = (z-1) - \frac{(z-1)^2}{2} + \frac{(z-1)^3}{3} - \frac{(z-1)^4}{4} + \frac{(z-1)^5}{5} - \cdots$

$$\operatorname{Log}(1{,}1) = 0{,}1 - \frac{0{,}1^2}{2} + \frac{0{,}1^3}{3} - \frac{0{,}1^4}{4} + \frac{0{,}1^5}{5} - \cdots$$

adding these first five terms we have

$$\begin{array}{r} 0{,}100000 \\ -0{,}005000 \\ 0{,}000333 \\ -0{,}000025 \\ 0{,}000002 \\ \hline \end{array}$$

$\operatorname{Log} 1{,}1 \approx 0{,}095300$ The series converges,

22/ $\operatorname{Log}(-9) = (-10) - \frac{(-10)^2}{2} + \frac{(-10)^3}{3} - \cdots$

$$= -\left[10 + \frac{100}{2} + \frac{1000}{3} + \frac{10000}{4} + \cdots\right]$$

This series makes no sense at all, and diverges, 23/ $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \frac{z^{10}}{10!} + \cdots$

SET $z = \frac{1+i}{\sqrt{2}} = e^{i\frac{\pi}{4}}$, THEN $z^2 = e^{i\frac{2\pi}{4}} = i$,

$z^4 = (z^2)^2 = (i)^2 = -1$, $z^6 = -i$, $z^8 = 1$, $z^{10} = i$.

$$\cos\left(\frac{1+i}{\sqrt{2}}\right) = 1 - \frac{i}{2!} + \frac{-1}{4!} - \frac{-i}{6!} + \frac{1}{8!} - \frac{i}{10!} + \cdots$$

$$= \left(1 - \frac{1}{4!} + \frac{1}{8!}\right) + i\left(-\frac{1}{2!} + \frac{1}{6!} - \frac{1}{10!}\right) + \cdots$$

$$\cong (1.0000000 - 0.0416667 + 0.0000248)$$

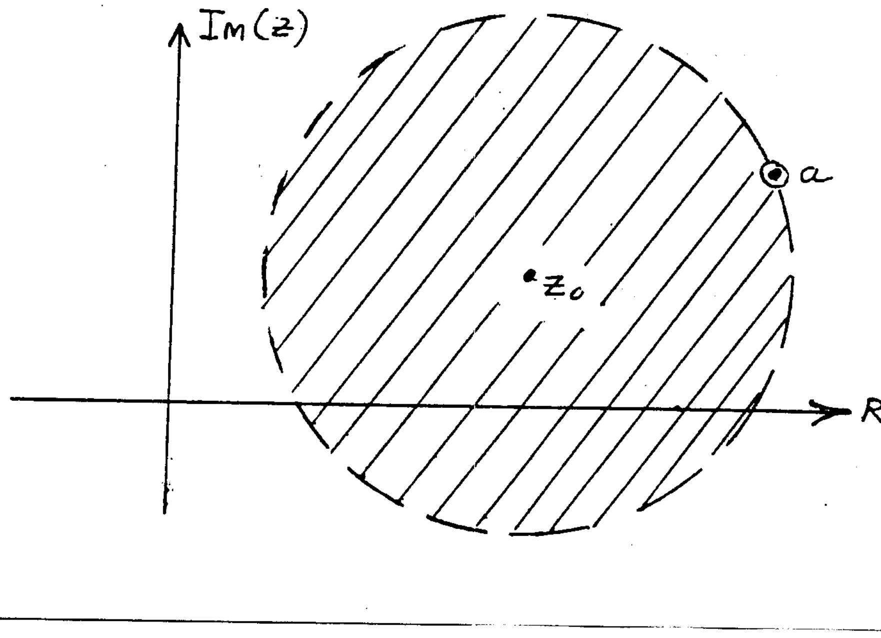
$$+ i(-0.5000000 + 0.0013889 - 0.0000003)$$

$$\cong 0.9583581 - 0.4986114\, i$$

The series converges. The above result is accurate to at least the first five decimal places.

24/ WE REQUIRE THAT $\left|\frac{z - z_0}{a - z_0}\right| < 1$, WHICH MEANS THAT

$|z - z_0| < |a - z_0|$.



25/ 1, $|z| < 1$

2, $|z| < |a|$

3, $|z-1| < 1$

4, $|z+1| < 3$

5, $|z - z_0| < |a - z_0|$

6,, 7, and 8, The series for e^z, $\sin z$ and $\cos z$ converge for all z,

9, Converges for all z,

10, $|z| < \frac{\pi}{2}$

11, Converges for all z,

12, $|z| < 1$

13, $|z-1| < 1$

14, $|z| < 1$

15, $|z| < 1$

26/ $$\lim_{n \to \infty} \left| \frac{z^{n+1}}{z^n} \right| = |z|$$

Thus the series converges for $|z| < 1$ and diverges for $|z| > 1$,

27/ $$\lim_{n \to \infty} \left| \frac{(a)_{n+1} (b)_{n+1} \; z^{n+1} \; (c)_n \; n!}{(c)_{n+1} \; (n+1)! \; (a)_n (b)_n \; z^n} \right|$$

$$= \lim_{n \to \infty} \left| \frac{(a+n)(b+n) z}{(c+n)(n+1)} \right| \quad \left(\text{since } \frac{(a)_{n+1}}{(a)_n} = (a+n)\right)$$

$$= \lim_{n\to\infty} \left| \frac{(1+\frac{a}{n})(1+\frac{b}{n})\ z}{(1+\frac{c}{n})(1+\frac{1}{n})} \right| = |z|$$

Thus the series converges for $|z|<1$ and diverges for $|z|>1$.

28/

$$\lim_{n\to\infty} \left| \frac{(a)_{n+1}\ z^{n+1}\ \ (b)_n\ \ n!}{(b)_{n+1}\ (n+1)!\ \ (a)_n\ \ z^n} \right| =$$

$$\lim_{n\to\infty} \left| \frac{(a+n)\ z}{(b+n)(n+1)} \right| = \lim_{n\to\infty} \left| \frac{(1+\frac{a}{n})\ z}{(1+\frac{b}{n})\ n} \right| = 0.$$

Thus the series converges for all z,

29/ $$(a+z)^{\frac{1}{2}} = ((a+z_0)+(z-z_0))^{1/2} = (a+z_0)^{1/2}\left(1+\frac{z-z_0}{a+z_0}\right)^{1/2}$$

$$= (a+z_0)^{1/2} \sum_{n=0}^{\infty} \binom{1/2}{n} \left\{ \frac{z-z_0}{a+z_0} \right\}^n$$

valid for $|z-z_0| < |a+z_0|$.

30/ $$(1-x^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \binom{-1/2}{n} (-x^2)^n = \sum_{n=0}^{\infty} \binom{-1/2}{n} (-1)^n x^{2n}$$

$$\int_0^x (1-x^2)^{-\frac{1}{2}} dx = \sum_{n=0}^{\infty} \binom{-1/2}{n} \frac{(-1)^n}{2n+1} x^{2n+1} = \arcsin x,$$

31/
$$D^0 (1+z)^p \Big|_{z=0} = (1+z)^p \Big|_{z=0} = 1$$

$$D^1 (1+z)^p \Big|_{z=0} = p(1+z)^{p-1} \Big|_{z=0} = p$$

$$D^2 (1+z)^p \Big|_{z=0} = p(p-1)(1+z)^{p-2} \Big|_{z=0} = p(p-1)$$

$$D^3 (1+z)^p \Big|_{z=0} = p(p-1)(p-2)(1+z)^{p-3} \Big|_{z=0} = p(p-1)(p-2)$$

$$\vdots$$

$$D^n (1+z)^p \Big|_{z=0} = \cdots = p(p-1)(p-2)\cdots(p-n+1)$$

THEREFORE

$$(1+z)^p = \sum_{n=0}^{\infty} \frac{D^n (1+z)^p \big|_{z=0}}{n!} z^n$$

$$= \sum_{n=0}^{\infty} \frac{p(p-1)(p-2)\cdots(p-n+1)}{n!} z^n$$

valid for $|z| < 1$.

32/ $$\frac{1}{a+z} = \frac{1}{(a+z_0)+(z-z_0)} = \frac{1}{z-z_0}\left[\frac{1}{1+\frac{a+z_0}{z-z_0}}\right]$$

$$= \frac{1}{(z-z_0)}\left[\frac{1}{1-\left\{-\frac{a+z_0}{z-z_0}\right\}}\right]$$

$$= \frac{1}{z-z_0}\sum_{n=0}^{\infty}\left\{-\frac{a+z_0}{z-z_0}\right\}^n$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n(a+z_0)^n}{(z-z_0)^{n+1}} \quad \text{for } \left|\frac{a+z_0}{z-z_0}\right| < 1$$

The region of convergence is $|a+z_0| < |z-z_0|$,

33/ Since $e^w = \sum_{n=0}^{\infty}\frac{w^n}{n!}$, we can replace w by $\frac{1}{z}$ and get

$$z^5 e^{\frac{1}{z}} = z^5\sum_{n=0}^{\infty}\frac{z^{-n}}{n!} = \sum_{n=0}^{\infty}\frac{z^{5-n}}{n!}$$

convergent for $0 < |z|$,

34/ $$\text{Log}\,\frac{z-1}{z} = \text{Log}\left(1-\frac{1}{z}\right)$$

Since $\text{Log}\, w = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(w-1)^n}{n}$ for $|w$

we can set $w = 1-\frac{1}{z}$ and get $w-1 = -\frac{1}{z}$ so that

$$\operatorname{Log}\frac{z-1}{z} = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{\left(-\frac{1}{z}\right)^n}{n} \quad \text{for} \quad \left|-\tfrac{1}{z}\right| < 1$$

$$= -\sum_{n=1}^{\infty} \frac{z^{-n}}{n} \quad \text{for} \quad 1 < |z| \,,$$

35/ $$\frac{1}{z^2+z-6} = \frac{1}{(z+3)(z-2)} = \frac{A}{z+3} + \frac{B}{z-2}$$

Multiply by $z+3$ and get

$$\frac{1}{z-2} = A + \frac{B(z+3)}{z-2}$$

Set $z=-3$ and get $A = -\frac{1}{5}$, Similarly, multiply by $z-2$ and get

$$\frac{1}{z+3} = \frac{A(z-2)}{z+3} + B \,.$$

Set $z=2$ and get $B = \frac{1}{5}$, Therefore

$$\frac{1}{z^2+z+6} = -\frac{1}{5}\left(\frac{1}{z+3}\right) + \frac{1}{5}\left(\frac{1}{z-2}\right).$$

(A) $$\frac{1}{z+3} = \frac{1}{3\left(1-\left(-\frac{z}{3}\right)\right)} = \frac{1}{3}\sum_{n=0}^{\infty}\left(-\frac{z}{3}\right)^n \quad \text{for} \quad |z| < 3$$

(B) $$\text{''} = \frac{1}{z\left(1-\left(-\frac{3}{z}\right)\right)} = \frac{1}{z}\sum_{n=0}^{\infty}\left(-\frac{3}{z}\right)^n \quad \text{for} \quad 3 < |z|$$

(C) $$\frac{1}{z-2} = \frac{1}{-2\left(1-\frac{z}{2}\right)} = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n \quad \text{for} \quad |z| < 2$$

(D) $$\text{''} = \frac{1}{z\left(1-\frac{2}{z}\right)} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n \quad \text{for} \quad 2 < |z| \,,$$

Combining A and C we get for $|z| < 2$

$$\frac{1}{z^2+z-6} = -\frac{1}{15}\sum_{n=0}^{\infty}\frac{(-1)^n z^n}{3^n} - \frac{1}{10}\sum_{n=0}^{\infty}\frac{z^n}{2^n}$$

$$= -\frac{1}{5}\sum_{n=0}^{\infty}\left[\frac{(-1)^n}{3^{n+1}} + \frac{1}{2^{n+1}}\right] z^n ,$$

Combining A and D we get for $2 < |z| < 3$

$$\frac{1}{z^2+z-6} = -\frac{1}{15}\sum_{n=0}^{\infty}\frac{(-1)^n z^n}{3^n} + \frac{1}{5}\sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}} ,$$

Combining B and D we get for $3 < |z|$

$$\frac{1}{z^2+z-6} = -\frac{1}{5}\sum_{n=0}^{\infty}\frac{(-1)^n 3^n}{z^{n+1}} + \frac{1}{5}\sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}}$$

$$= \frac{1}{5}\sum_{n=0}^{\infty}\left[2^n - (-1)^n 3^n\right] z^{-n-1} .$$

36/ Now we have

(A) $\frac{1}{z+3} = \frac{1}{4+(z-1)} = \sum_{n=0}^{\infty}\frac{(-1)^n (z-1)^n}{4^{n+1}}$ for $|z-1| < 4$

(B) " $= \frac{1}{(z-1)\left(1+\frac{4}{z-1}\right)} = \sum_{n=0}^{\infty}\frac{(-1)^n 4^n}{(z-1)^{n+1}}$ for $4 < |z-1|$

(C) $\frac{1}{z-2} = \frac{-1}{1-(z-1)} = -\sum_{n=0}^{\infty}(z-1)^n$ for $|z-1| < 1$

(D) " $= \frac{1}{(z-1)\left(1-\frac{1}{z-1}\right)} = \sum_{n=0}^{\infty}(z-1)^{-n-1}$ for $|z-1| > 1$

36/(Continued)

Combining A and C we get for $|z-1|<1$

$$\frac{1}{z^2+z-6} = -\frac{1}{5}\sum_{n=0}^{\infty}\left[1+\frac{(-1)^n}{4^{n+1}}\right](z-1)^n .$$

Combining A and D we get for $1<|z-1|<4$

$$\frac{1}{z^2+z-6} = \frac{1}{5}\sum_{n=0}^{\infty}(z-1)^{-n-1} - \frac{1}{5}\sum_{n=0}^{\infty}\frac{(-1)^n(z-1)^n}{4^{n+1}} ,$$

Combining B and D we get for $4<|z-1|$

$$\frac{1}{z^2+z-6} = \frac{1}{5}\sum_{n=0}^{\infty}\left[1-(-1)^n 4^n\right](z-1)^{-n-1} ,$$

37/

(a) $\frac{\sin z}{z} = \frac{1}{z}\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots\right) = 1-\frac{z^2}{3!}+\frac{z^4}{5!}-\cdots$

convergent for $|z|<\infty$. Thus there are no singularities in the finite z-plane. Set $z=\frac{1}{\zeta}$ in this last series and get $1-\frac{1}{3!\,\zeta^2}+\frac{1}{5!\,\zeta^4}-\cdots$. Since this is an essential singularity at $\zeta=0$, we say that $\frac{\sin z}{z}$ has an essential singularity at infinity.

(b) $\sqrt{z}\sinh\sqrt{z} = \sqrt{z}\left(\sqrt{z}+\frac{z\sqrt{z}}{3!}+\frac{z^2\sqrt{z}}{5!}+\cdots\right)$

$= z+\frac{z^2}{3!}+\frac{z^3}{5!}+\cdots$ for $|z|<\infty$.

Setting $z=\frac{1}{\zeta}$ we find that the only singularity is an essential singularity at infinity.

37/ (c) $$\frac{1-\cos z}{z^2} = \frac{1}{z^2}\left(1 - 1 + \frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} - \cdots\right)$$

$$= \frac{1}{2!} - \frac{z^2}{4!} + \frac{z^4}{6!} - \cdots \quad \text{for } |z| < \infty,$$

The only singularity is an essential singularity at infinity,

(d) $$\frac{e^z - 1}{z^2} = \frac{1}{z^2}\left(1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots \quad - 1\right)$$

$$= \frac{1}{z} + \frac{1}{2!} + \frac{z}{3!} + \frac{z^2}{4!} + \cdots \quad 0 < |z| < \infty$$

There is a simple pole at $z = 0$ and an essential singularity at infinity.

38/ We see that $f(z) = 3\,G(z-1) + 2$, where G is defined in the Example 1, Since $G^{(n)}(0) = 0$ for $n = 0, 1, 2, \cdots$, then $f(1) = 2$, $f^{(1)}(1) = 0$, $f^{(2)}(1) = 0, \cdots$. Therefore

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(z-1)^n = 2 + 0(z-1) + 0(z-1)^2 + \cdots$$

$$\equiv 2.$$

Since $f(z)$ is not identically the constant 2, The Taylor's series does not represent $f(z)$, Therefore $f(z)$ is not analytic at $z = 1$, 39/ Since $f(x) = x^2$ for $1 \le x < 2$, we know that $f(1) = 1$, $f^{(1)}(1) = 2$, $f^{(2)}(1) = 2$, $f^{(3)}(1) = 0$, $f^{(4)}(1) = 0, \ldots$, Therefore, if $f(z)$ is analytic, it can be expanded in a Taylor's series about $z = 1$ and we get

$$f(z) = 1 + 2(z-1) + (z-1)^2 + 0$$

$$= z^2 \qquad \text{for all } z,$$

Thus $f(0)$ cannot equal -1,

40/ YES, BECAUSE THE LARGEST POSSIBLE DOMAIN TO WHICH AN ANALYTIC FUNCTION CAN BE EXTENDED IS ALWAYS AN OPEN SET. SINCE $|z| \le 1$ IS A CLOSED SET; THERE EXISTS SOME OPEN SET $\mathcal{R}$ CONTAINING $|z| \le 1$ ON WHICH $f(z)$ IS ANALYTIC,

41/ SINCE $(1-z)^{-1} = \sum_{n=0}^{\infty} z^n$ FOR $|z| < 1$, WE SEE THAT TERM BY TERM DIFFERENTIATION YIELDS $(1-z)^{-2} = \sum_{n=0}^{\infty} n z^{n-1}$, THUS THE DESIRED ANALYTIC CONTINUATION IS GIVEN SIMPLY BY $(1-z)^{-2}$ FOR ALL z EXCEPT $z = 0$,

42/ DIFFERENTIATING THE SERIES FOR $(1-z)^{-2}$ IN THE PREVIOUS PROBLEM WE GET $2(1-z)^{-3} = \sum_{n=2}^{\infty} n(n-1) z^{n-2}$, THUS THE DESIRED CONTINUATION IS GIVEN BY $2(1-z)^{-3}$ FOR ALL z EXCEPT $z = 0$,

Solutions to Review Problems from Chapter 4

1/ (a) $e^z - 1 = 0$ when $z = 2\pi n i$, with $n = 0, \pm 1, \pm 2, \ldots$, Thus the zeros closest to the origin are $z = \pm 2\pi i$. Therefore the circle of convergence is $|z| < 2\pi$,

(b) $$e^z - 1 = \sum_{n=0}^{\infty} \frac{z^n}{n!} - 1 = \sum_{n=1}^{\infty} \frac{z^n}{n!}$$

$$\begin{array}{r|l}
 & 1 - \frac{z}{2} + \frac{z^2}{12} - \frac{z^4}{720} + \cdots \\
\hline
z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + \frac{z^5}{120} + \cdots & z \\
 & z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + \frac{z^5}{120} + \cdots \\
\cline{2-2}
 & -\frac{z^2}{2} - \frac{z^3}{6} - \frac{z^4}{24} - \frac{z^5}{120} - \cdots \\
 & -\frac{z^2}{2} - \frac{z^3}{4} - \frac{z^4}{12} - \frac{z^5}{48} \\
\cline{2-2}
 & \frac{z^3}{12} + \frac{z^4}{24} + \frac{z^5}{80} + \cdots \\
 & \frac{z^3}{12} + \frac{z^4}{24} + \frac{z^5}{72} + \cdots \\
\cline{2-2}
 & -\frac{z^5}{720} + \cdots
\end{array}$$

Since

$$\frac{z}{e^z - 1} = 1 - \frac{z}{2} + \frac{B_1 z^2}{2} - \frac{B_2 z^4}{24} + \cdots,$$

$B_1 = \frac{1}{6}$ and $B_2 = \frac{1}{30}$.

2/ $${}_2F_1(a, 1; 1; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} z^n$$

$$= \sum_{n=0}^{\infty} \frac{a(a+1)(a+2)\cdots(a+n-1)}{1 \cdot 2 \cdot 3 \cdots n} z^n =$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{(-a)}{1}\frac{(-a-1)}{2}\frac{(-a-2)}{3}\cdots\frac{(-a-n+1)}{n} z^n$$

$$= \sum_{n=0}^{\infty} \frac{(-a)}{1}\cdot\frac{(-a-1)}{2}\cdot\frac{(-a-2)}{3}\cdots\frac{(-a-n+1)}{n}(-z)^n$$

$$= \sum_{n=0}^{\infty} \binom{-a}{n}(-z)^n = (1-z)^{-a},$$

3/ $$\left(\frac{1}{1-z}\right)^2 = \left(\sum_{n=0}^{\infty} z^n\right)^2 = \sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n} 1\right\} z^n$$

$$= \sum_{n=0}^{\infty} (n+1) z^n \qquad \text{for } |z|<1.$$

This is the desired Maclaurin Series.

To get the Laurent series we write

$$\left(\frac{1}{1-z}\right)^2 = \left(-\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}\right)^2 = \left(\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}\right)^2.$$

$$\begin{array}{l}
\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\frac{1}{z^4}+\cdots \\
\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\frac{1}{z^4}+\cdots \\
\hline
\frac{1}{z^2}+\frac{1}{z^3}+\frac{1}{z^4}+\frac{1}{z^5}+\cdots \\
\qquad\ \frac{1}{z^3}+\frac{1}{z^4}+\frac{1}{z^5}+\cdots \\
\qquad\qquad +\frac{1}{z^4}+\frac{1}{z^5}+\cdots \\
\qquad\qquad\qquad\ \ \frac{1}{z^5}+\cdots \\
\qquad\qquad\qquad\quad \vdots \\
\hline
\frac{1}{z^2}+\frac{2}{z^3}+\frac{3}{z^4}+\frac{4}{z^5}+\cdots
\end{array} = \sum_{n=1}^{\infty}\frac{n}{z^{n+1}} \quad \text{for } |z|>1,$$

4/ Since $\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$ for all z,

$(\sin z)^3 = z^3 + a_1 z^5 + a_2 z^7 + \cdots$ where $a_1, a_2, \ldots$ can be found by multiplying the series for $\sin z$ times itself three times. Of necessity, the series for $(\sin z)^3$ converges over the same region as does the series for $\sin z$; $|z| < \infty$. Dividing our series for $(\sin z)^3$ by z^5 we get

$$(1) \quad \frac{(\sin z)^3}{z^5} = \frac{1}{z^2} + a_1 + a_2 z^2 + \cdots ,$$

THUS WE HAVE A POLE OF ORDER TWO AT $z=0$. The only other possible singularity is at infinity. We set $z = \frac{1}{\zeta}$ in (1) and get

$$\zeta^5 (\sin \tfrac{1}{\zeta})^3 = \zeta^2 + a_1 + \frac{a_2}{\zeta^2} + \cdots ,$$

Since this last series has INFINITELY MANY TERMS WITH NEGATIVE POWERS OF ζ, we have an essential singularity at $\zeta = 0$. This means that at $z = \infty$ we have an essential singularity.

APPENDIX II

ANSWERS TO CONJECTURES

Chapter 4

4.1 The circle of convergence of a Taylor's series

If $f(z)$ is analytic at the point $z=z_0$, then $f(z)$ can be expanded in a Taylor's series $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ which converges to $f(z)$ for all z inside the largest open circle with center at $z = z_0$ containing no singularities of $f(z)$.

Discussion:

Let z_1 be the singular point of $f(z)$ that is nearest to the point z_0. (There might be several points which have this distinction, and in this case, z_1 is any one of them.) Then if we call $|z_1 - z_0| = R$, our Taylor's series converges for $|z - z_0| < R$, diverges for $|z - z_0| > R$, and might converge at some points and diverge at others on the boundary $|z - z_0| = R$ of the circle itself.

4.2 Formal manipulations of addition subtraction and multiplication

Let

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n \quad \text{for } |z-z_0| < r, \text{ and}$$

$$g(z) = \sum_{n=0}^{\infty} b_n(z-z_0)^n \quad \text{for } |z-z_0| < R .$$

The formal addition, subtraction, and multiplication of these series is valid inside the smaller of the two circles of convergence.

4.3 The formal division of two Taylor's series

The formal division yielding

$$\frac{f(z)}{g(z)} = \frac{a_0 + a_1 z + a_2 z^2 + \dots}{b_0 + b_1 z + b_2 z^2 + \dots} = \frac{a_0}{b_0} + \frac{a_1 b_0 - b_1}{b_0} z + \dots$$

is valid for all z inside the largest circle centered at $z = 0$ containing no singularities of the function $f(z)/g(z)$. We must assume that $b_0 \neq 0$, since then $f(z)/g(z)$ would have a pole at $z = 0$ when a_0 is not zero. If both a_0 and b_0 are zero, them we can divide the series for f and for g by z and then begin again.

4.4 The composite function

If the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ converges for $|z| < R$, then we can replace z by $h(z)$ provided $|h(z)| < R$. Thus there is some region in the z-plane for which $|h(z)| < R$, and for these z we have $f(\,h(z)\,) = \sum_{n=0}^{\infty} a_n h(z)^n$.

4.5 Term by term differentiation and integration of series

We may differentiate a power series term by term at each point inside the circle of convergence of the original series. We may integrate a power series term by term provided that the path of integration is strictly inside the circle of convergence. Differentiating or integrating a power series does not alter the size of the circle of convergence.

4.6 Laurent series expansions

The boundaries of the regions of convergence are circles with centers at z_0 passing through the singular points z_1 , z_2 and z_3 .

In region I all the $b_n = 0$ for we have a Taylor's series expansion about the regular point z_0 .

In region II we have both positive and negative powers of $z-z_0$ in the Laurent series expansion. In fact, there must be infinitely many non-zero a_n , or otherwise the series in positive powers would be finite and thus would converge for all z, contradicting the finite circle of convergence which passes through the point z_2 . Similarly, there must be infinitely many non-zero b_n . If not, the series in negative powers of $z-z_0$ would be finite and thus would converge right up to the point z_0 itself, contradicting the boundary which passes through z_1 .

In region III no Laurent series is possible because of the discontinuity at the branch line.

In region IV it might happen that all the $a_n = 0$.