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SIAM Journal on Applied Mathematics, Volume 18, Issue 3 (May, 1970), 658-674.

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SIAM J. APPL. MATH.
Vol. 18, No. 3, May 1970

LEIBNIZ RULE FOR FRACTIONAL DERIVATIVES GENERALIZED AND AN APPLICATION TO INFINITE SERIES*

THOMAS J. OSLER†

1. Introduction. In this paper certain generalizations of the Leibniz rule for the derivative of the product of two functions are examined and used to generate several infinite series expansions relating special functions. We first review various definitions which have been proposed to generalize the order of the differential operator $D_z^n(=d^n/dz^n)$, considering finally a derivative of arbitrary order α with respect to $g(z)$ of $f(z)$ which we denote by the symbol $D_{g(z)}^{\alpha}f(z)$. The latter reduces to the usual differential operator when $\alpha = 0, 1, 2, \cdots$ and $g(z) = z$. A new proof for the formula

$$D_{g(z)}^{\alpha}u(z)v(z) = \sum_{n=-\infty}^{\infty} \binom{\alpha}{\gamma+n} D_{g(z)}^{\alpha-\gamma-n}u(z)D_{g(z)}^{\gamma+n}v(z), \tag{1.1}$$

where

$$\binom{\alpha}{\gamma+n} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\gamma-n+1)\Gamma(\gamma+n+1)},$$

for the fractional derivative of the product uv is given which reveals for the first time the precise region of convergence of the series in the z-plane. The special case of (1.1) in which $\gamma = 0$ and $\alpha = 0, 1, 2, \cdots$ is the Leibniz rule from the elementary calculus. The formula

$$D_{g(z)}^{\alpha}f(z,z) = \sum_{n=-\infty}^{\infty} \binom{\alpha}{\gamma+n} D_{g(z),g(w)}^{\alpha-\gamma-n,\gamma+n}f(z,w)\Big|_{w=z} \tag{1.2}$$

is shown to be a natural outgrowth of the generalized Leibniz rule (1.1). Formula (1.1) is the special case of (1.2) in which $f(z, w) = u(z)v(w)$. Formula (1.2) and several of the series expansions obtained from it appear to be new.

Studies of a Leibniz rule for derivatives of arbitrary order date back to 1832 when Liouville [10, p. 117] gave the special case of (1.1) in which $\gamma = 0$,

$$D_z^{\alpha}u(z)v(z) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+1)D_z^{\alpha-n}u(z)D_z^n v(z)}{\Gamma(\alpha-n+1)n!}. \tag{1.3}$$

Liouville invented a fractional derivative by observing that the simple relation $D^n e^{az} = a^n e^{az}, n = 0, 1, 2, \cdots$, could be generalized for arbitrary α by $D^{\alpha}e^{az} = a^{\alpha}e^{az}$. Liouville then expanded a general function in a Fourier series and obtained its fractional derivative by differentiating term by term.

In 1867 and 1868 A. K. Grunwald [7, pp. 406–468] and A. V. Letnikov [9] found (1.3) by starting with a fractional derivative based on the so called Riemann–Liouville integral

$$D_z^{\alpha}f(z) = \frac{1}{\Gamma(-\alpha)}\int_0^z \frac{f(\zeta)\,d\zeta}{(z-\zeta)^{\alpha+1}}.$$

* Received by the editors January 14, 1969, and in expanded and revised form August 12, 1969.
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A further extension of (1.3) was given by E. Post [12, p. 755] in 1930 to functions of D more general than D^α. In 1961 M. A. Bassam [1] presented another derivation of (1.3), and recently M. Gaer and L. A. Rubel [6] have found a rule for the fractional derivative of a product which does not reduce to the Leibniz rule.

In 1931 Y. Watanabe [14] studied (1.3) and (1.1) by expanding $u(z)$ and $v(z)$ in power series in z. His method does not yield the precise region of convergence in the z-plane. On the following pages a simpler proof of (1.1) is given using a Cauchy integral formula for fractional derivatives. The use of this powerful tool permits us to determine the precise region of convergence.

Although the Cauchy integral formula for fractional derivatives appeared as early as 1888 [11], it seems appropriate to include a brief section motivating this concept to make the paper self-contained.

The concept of fractional derivative with respect to an arbitrary function has been used by A. Erdelyi [4], [5]. A natural notation is introduced for this concept which is useful and suggestive in applications, yet seems to be new. A very short table of special functions represented by fractional derivatives incorporating this notation is included.

Finally (1.2) is used to generate certain infinite series expansions relating special functions of mathematical physics by assigning specific values to the functions f and g, and to the parameters α and γ. Some of the expansions thus generated are known, while others appear to be new. It is remarkable that the proof of (1.2) is so simple, that it is easier than the usual derivations of known expansions obtainable from it. An example is Dougall's formula [2, vol. 1, p. 7]

$$\frac{\pi^2\Gamma(c+d-a-b-1)\csc \pi a \csc \pi b}{\Gamma(c-a)\Gamma(c-b)\Gamma(d-a)\Gamma(d-b)} = \sum_{n=-\infty}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)\Gamma(d+n)}$$

which can be obtained from (1.1) immediately by taking $u = z^{d-a-1}$, $v = z^{c-b-1}$, $g = z$, $\alpha = c-a-1$, and $\gamma = c-1$.

2. Motivation of fractional derivatives. When motivating the concept of a derivative whose order is not to be the usual integer value, it is perhaps simplest to begin with the function z^p. The elementary formula, for n and p as natural numbers, $D_z^n z^p = p!z^{p-n}/(p-n)!$, generalizes at once to

$$D_z^\alpha z^\beta = \frac{\Gamma(\beta+1)z^{\beta-\alpha}}{\Gamma(\beta-\alpha+1)},$$

where the only restriction is that $\beta \neq -1, -2, \cdots$. We can now define the fractional derivative of $z^\beta f(z)$, where $f(z)$ is analytic at $z = 0$, by differentiating the power series for $z^\beta f(z)$ term by term. We get

$$D_z^\alpha z^\beta f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)\Gamma(\beta+n+1)z^{\beta-\alpha+n}}{n!\Gamma(\beta-\alpha+n+1)}.$$

This series has the same circle of convergence as the power series for $f(z)$ about $z = 0$.

The usual starting point for a definition of fractional derivative taken in recent papers [1], [4], [5], [8], is the Riemann–Liouville fractional integral

$$D_z^\alpha f(z) = \frac{1}{\Gamma(-\alpha)} \int_0^z \frac{f(\zeta)\, d\zeta}{(z-\zeta)^{\alpha+1}}. \tag{2.1}$$

Here the path of integration is along a line from 0 to z in the complex ζ-plane, and $\operatorname{Re}(\alpha) < 0$. This integral can be motivated from the Cauchy formula for a repeated integral

$$\int_0^z \cdots \int_0^{t_3} \int_0^{t_2} f(t_1)\, dt_1\, dt_2 \cdots dt_n = \frac{1}{(n-1)!} \int_0^z \frac{f(t)\, dt}{(z-t)^{-n+1}}.$$

Setting $f(t_1) = t_1^\beta$ in this formula we see that the left-hand side becomes

$$\frac{z^{\beta+n}}{(\beta+1)(\beta+2)\cdots(\beta+n)} = \frac{\Gamma(\beta+1) z^{\beta+n}}{\Gamma(\beta+n+1)} = D_z^{-n} z^\beta.$$

Formally replacing $-n$ by α we get

$$D_z^\alpha z^\beta = \frac{1}{\Gamma(-\alpha)} \int_0^z \frac{t^\beta\, dt}{(z-t)^{\alpha+1}}.$$

Now the Riemann–Liouville integral seems somewhat reasonable as a definition for fractional derivative. In fact, the reader can quickly convince himself that this integral and the previous power series definition are equivalent using the elementary properties of the gamma and beta functions, provided $\operatorname{Re}(\alpha) < 0$.

A third method for motivating the concept of a derivative of arbitrary order is to examine Cauchy's integral formula from complex variables

$$D_z^n f(z) = \frac{n!}{2\pi i} \oint_C \frac{f(\zeta)\, d\zeta}{(\zeta - z)^{n+1}}.$$

If we replace n by noninteger α in this formula, $(\zeta - z)^{-\alpha-1}$ no longer has a pole at $\zeta = z$, but a branch point. We are no longer free to deform the contour C at will, for the value of the integral is now a function of the point where C crosses the branch line for $(\zeta - z)^{-\alpha-1}$. Take this point as zero as shown in Fig. 1.

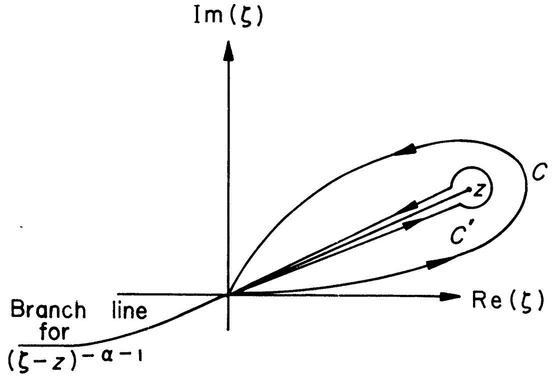

FIG. 1

If we deform the contour C to C' we see that

$$\frac{\Gamma(\alpha+1)}{2\pi i}\int_C \frac{f(\zeta)\,d\zeta}{(\zeta-z)^{\alpha+1}} = \frac{\Gamma(\alpha+1)}{2\pi i}[1-e^{-2\pi i(\alpha+1)}]\int_0^z \frac{f(z)\,d\zeta}{(\zeta-z)^{\alpha+1}}$$

$$= \frac{1}{\Gamma(-\alpha)}\int_0^z \frac{f(\zeta)\,d\zeta}{(z-\zeta)^{\alpha+1}}.$$

Since this is the Riemann–Liouville integral we have the following generalized Cauchy integral formula:

(2.2) $$D_z^\alpha f(z) = \frac{\Gamma(\alpha+1)}{2\pi i}\int_C \frac{f(\zeta)\,d\zeta}{(\zeta-z)^{\alpha+1}}.$$

The significance of the fact that C must start and end at $\zeta = 0$ will appear in the following section. This formula was given as far back as 1888 by Nekrassov [11].

It is important to note that each of the three methods for defining a fractional derivative feature certain peculiar advantages and disadvantages. Using the power series method we can differentiate z^p so long as p is not a negative integer. However, for the Riemann–Liouville integral, or the generalized Cauchy integral, we require $\operatorname{Re}(p) > -1$ so that the integrals are defined. The power series method fails to differentiate functions whose singularities at $z = 0$ are not of the type z^β. Such a function is $\log z$ which can be handled by the other two methods.

The Riemann–Liouville integral requires $\operatorname{Re}(\alpha) < 0$, whereas the power series method and the Cauchy formula have no restriction on α. The generalized Cauchy integral formula requires that the function $f(z)$ being differentiated be analytic in some finite sector of the z-plane with vertex at $z = 0$. We also need $\oint f(z)\,dz = 0$ along any closed path through $z = 0$, so that the angle at which C approaches $z = 0$ is arbitrary.

3. Derivatives with respect to any function and partial derivatives. We next assign a meaning to the derivative of order α with respect to an arbitrary function $g(z)$ of $f(z)$, and denote it by the symbol $D_{g(z)}^\alpha f(z)$. To generate a useful definition, consider the Riemann–Liouville integral

$$D_\omega^\alpha F(\omega) = \frac{1}{\Gamma(-\alpha)}\int_0^\omega \frac{F(\zeta)\,d\zeta}{(\omega-\zeta)^{\alpha+1}},$$

and formally set $\omega = g(z)$. We then require $F(g(z)) = f(z)$ and $\zeta = g(t)$. This is a simple change of variables, and is all that is needed for our purpose. We obtain

(3.1) $$D_{g(z)}^\alpha f(z) = \frac{1}{\Gamma(-\alpha)}\int_{g^{-1}(0)}^z \frac{f(t)g'(t)\,dt}{(g(z)-g(t))^{\alpha+1}}.$$

If we let $g(t) = ug(z)$ we get an equivalent form for the definition:

(3.2) $$D_{g(z)}^\alpha f(z) = \frac{g(z)^{-\alpha}}{\Gamma(-\alpha)}\int_0^1 \frac{f(g^{-1}(ug(z)))\,du}{(1-u)^{\alpha+1}}.$$

It is of particular interest to set $g(z) = z - a$. In this case $g^{-1}(0) = a$ and

(3.1) becomes

$$D^{\alpha}_{z-a}f(z) = \frac{1}{\Gamma(-\alpha)}\int_a^z \frac{f(t)\,dt}{(z-t)^{\alpha+1}}.$$

Other notations have been used by authors of recent papers to denote this last integral, but the above considerations appear to make this notation, presented here for apparently the first time, the most natural.

The reason why the contour in (2.2), defining the generalized Cauchy integral formula, passes through zero is now clear. It must be that

$$\frac{\Gamma(\alpha+1)}{2\pi i}\int_a^{(z+)} \frac{f(\zeta)\,d\zeta}{(\zeta - z)^{\alpha+1}} = D^{\alpha}_{z-a}f(z).$$

(The notation on this integral implies that the contour of integration starts at a, encircles z in the positive sense, and returns to a without enclosing singularities of f.)

We also require fractional partial derivatives. These have been introduced by M. Riesz [13], and M. A. Bassam [1]. The notation $D^{\alpha,\beta}_{g(z),h(w)}f(z, w)$ means the fractional derivative of $f(z, w)$ of order β with respect to $h(w)$ holding z fixed, followed by the derivative of order α with respect to $g(z)$ holding w fixed. This is given by

$$\begin{aligned} &D^{\alpha,\beta}_{g(z),h(w)}f(z, w) \\ (3.3)\qquad &= \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{-4\pi^2}\int_{g^{-1}(0)}^{(z+)} \frac{g'(\xi)}{(g(\xi)-g(z))^{\alpha+1}}\int_{h^{-1}(0)}^{(w+)} \frac{f(\xi,\zeta)h'(\zeta)\,d\zeta\,d\xi}{(h(\zeta)-h(w))^{\beta+1}}, \end{aligned}$$

where $f(z, w)$, $g(z)$ and $h(w)$ are assumed to possess sufficient regularity to give the definition meaning.

It is clear from the definition (3.3) that when $f(z, w) = u(z)v(z)$,

$$(3.4)\qquad D^{\alpha,\beta}_{g(z),g(w)}u(z)v(w)\big|_{w=z} = D^{\alpha}_{g(z)}u(z)D^{\beta}_{g(z)}v(z).$$

Thus it is easy to see that (1.2) is a generalization of the Leibniz rule (1.1).

4. The product rule. We next consider extending the elementary Leibniz rule for the derivative of the product of two functions, to fractional derivatives. The formula

$$D^N_z u(z)v(z) = \sum_{n=0}^{N} \frac{N!\,D^{N-n}_z u(z) D^n_z v(z)}{(N-n)!\,n!}$$

appears at once to generalize as

$$(4.1)\qquad D^{\alpha}_z u(z)v(z) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+1) D^{\alpha-n}_z u(z) D^n_z v(z)}{\Gamma(\alpha-n+1)\,n!}$$

for arbitrary α. This formula can be derived from the Riemann–Liouville definition of fractional derivative by integrating by parts. It can also be derived using the power series method if the functions $u(z)$ and $v(z)$ permit.

There is a disturbing feature of (4.1). It is obvious that $D^{\alpha}_z uv = D^{\alpha}_z vu$, but this fact is not clear on the right-hand side since u is differentiated fractionally while

v is differentiated in the usual elementary sense. Could it be that (4.1) is a special case of a more general formula in which the interchangeability of the functions u and v is obvious? To see that this is the case, formally differentiate

$$D_z^\gamma vu = \sum_{r=0}^{\infty} \frac{\Gamma(\gamma + 1) D_z^{\gamma - r} v D_z^r u}{\Gamma(\gamma - r + 1) r!}$$

with the operator $D_z^{\alpha - \gamma}$ and obtain

$$D_z^{\alpha-\gamma}(D_z^\gamma vu) = D_z^\alpha vu = \sum_{r=0}^{\infty} \frac{\Gamma(\gamma + 1) D_z^{\alpha-\gamma}(D_z^r u D_z^{\gamma - r} v)}{\Gamma(\gamma - r + 1) r!}.$$

Using (4.1) again we obtain

$$D_z^\alpha uv = \sum_{r=0}^{\infty} \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma - r + 1) r!} \sum_{k=0}^{\infty} \frac{\Gamma(\alpha - \gamma + 1) D_z^{\alpha - \gamma + r - k} u D_z^{\gamma - r + k} v}{\Gamma(\alpha - \gamma - k + 1) k!}.$$

Interchanging the order of summation, and summing diagonally by setting $n = k - r$ we obtain

$$D_z^\alpha uv = \sum_{n=-\infty}^{\infty} \sum_{r=0}^{\infty} \frac{\Gamma(\gamma + 1)\Gamma(\alpha - \gamma + 1) D_z^{\alpha - \gamma - n} u D_z^{\gamma + n} v}{\Gamma(\gamma - r + 1)\Gamma(\alpha - \gamma - n - r + 1)(n + r)! r!}.$$

Using elementary properties of the gamma function and the value of the hypergeometric function of unit argument we can sum over r to obtain the following generalization of (4.1):

$$D_z^\alpha uv = \sum_{n=-\infty}^{\infty} \frac{\Gamma(\alpha + 1) D_z^{\alpha - \gamma - n} u D_z^{\gamma + n} v}{\Gamma(\alpha - \gamma - n + 1)\Gamma(\gamma + n + 1)}. \tag{4.2}$$

It is now clear that (4.1) is the special case of (4.2) obtained by setting $\gamma = 0$. It is also clear that u and v can be interchanged in (4.2), and thus the question raised above concerning the existence of a general formula of which (4.1) is a special case has been answered affirmatively. The formula (4.2) can be derived rigorously by expanding u and v in power series and following the steps outlined above. However the proof is lengthy and tedious due to the need to generate inequalities for terms like $D_z^\alpha f$ to prove the convergence of the series encountered. A much simpler proof is given below using the generalized Cauchy integral formula and contour integration.

After discovering (4.2) the author was informed by a referee that Y. Watanabe [14] had published it in 1931. His proof is in two parts. First he derives (4.1) by expanding $u(z)$ and $v(z)$ in power series in z. Since power series by nature converge only in circles, this method demonstrates that (4.1) converges in the largest circle centered at the origin and interior to the full region of convergence [14, p. 12]. He then mistakenly concludes that the series (4.1) converges wherever $u(z)$ and $v(z)$ are analytic, which is not true as is shown below by an example. Finally Watanabe uses (4.1) to derive the general Leibniz rule (4.2) by a method somewhat like that outlined above.

The proof given below using the Cauchy integral formula for fractional derivatives is new. It yields the general Leibniz rule (4.2) as well as the precise region of convergence in one stroke.

THEOREM 1 (product rule). *Let $u(z)$ and $v(z)$ be analytic functions of z on the simply connected region $\mathscr{R}$. Suppose also that 0 is an interior or boundary point of $\mathscr{R}$ and that the integral along any simple closed path in $\mathscr{R}$ through 0 of u, v, and uv is zero. Call $\mathscr{S}$ the set of all z such that the closed disk $|\zeta - z| \leqq |z|$ contains only points ζ in $\mathscr{R} \cup \{0\}$. Then*

$$D_z^\alpha u(z)v(z) = \sum_{n=-\infty}^{\infty} \frac{\Gamma(\alpha+1) D_z^{\alpha-\gamma-n} u(z) D_z^{\gamma+n} v(z)}{\Gamma(\alpha-\gamma-n+1)\Gamma(\gamma+n+1)}$$

for z in $\mathscr{S}$ and all complex α and γ, except negative integral α.

Proof. Using the contours shown in Fig. 2 we know Cauchy's integral formula for fractional derivatives states that

$$\begin{aligned} D_z^\alpha u(z)v(z) &= \frac{\Gamma(\alpha+1)}{2\pi i} \int_{C_2} \frac{u(\xi)v(\xi)\,d\xi}{(\xi - z)^{\alpha+1}} \\ &= \frac{\Gamma(\alpha+1)}{2\pi i} \int_{C_2} \frac{u(\xi)}{(\xi - z)^{\alpha-\gamma+1}} \frac{v(\xi)}{(\xi - z)^{\gamma}}\,d\xi. \end{aligned}$$

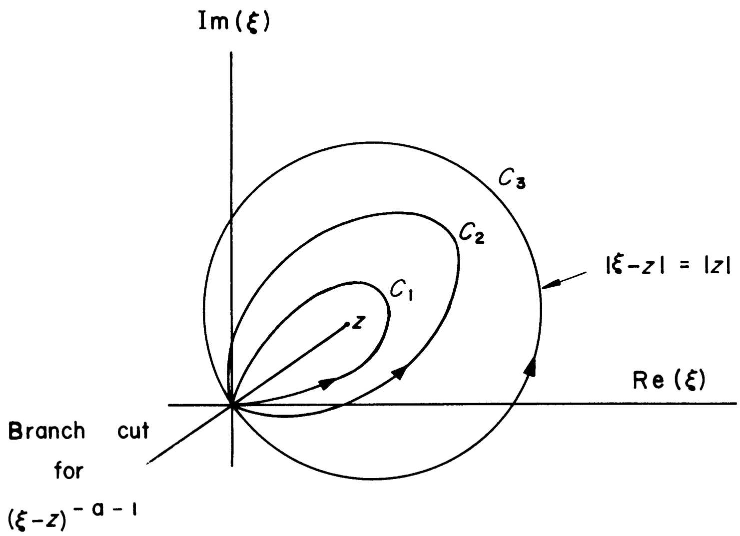

FIG. 2

Using the elementary Cauchy integral formula we can reduce this to

$$\begin{aligned} D_z^\alpha uv &= \frac{\Gamma(\alpha+1)}{-4\pi^2} \int_{C_2} \frac{u(\xi)}{(\xi - z)^{\alpha-\gamma+1}} \int_{C_3 - C_1} \frac{v(\zeta)\,d\zeta\,d\xi}{(\zeta - z)^{\gamma}(\zeta - \xi)} \\ &= \frac{\Gamma(\alpha+1)}{-4\pi^2} \left\{ \int_{C_2} \frac{u(\xi)}{(\xi - z)^{\alpha-\gamma+1}} \int_{C_3} \frac{v(\zeta)\,d\zeta\,d\xi}{(\zeta - z)^{\gamma}(\zeta - \xi)} \right. \\ &\quad \left. + \int_{C_2} \frac{u(\xi)}{(\xi - z)^{\alpha-\gamma+1}} \int_{C_1} \frac{v(\zeta)\,d\zeta\,d\xi}{(\zeta - z)^{\gamma}(\xi - \zeta)} \right\}. \end{aligned}$$

In the first term of this last expression, C_2 can be replaced by C_1 and in the second term, C_2 can be replaced by C_3. A little straightforward manipulation then yields

$$D_z^\alpha uv = \frac{\Gamma(\alpha+1)}{-4\pi^2}\left\{\int_{C_1}\frac{u(\xi)}{(\xi-z)^{\alpha-\gamma+1}}\int_{C_3}\frac{v(\zeta)\,d\zeta\,d\xi}{(\zeta-z)^{\gamma+1}(1-(\xi-z)/(\zeta-z))}\right.$$
$$\left.+\int_{C_3}\frac{u(\xi)}{(\xi-z)^{\alpha-\gamma+1}}\int_{C_1}\frac{v(\zeta)((\zeta-z)/(\xi-z))\,d\zeta\,d\xi}{(\zeta-z)^{\gamma+1}(1-(\zeta-z)/(\xi-z))}\right\}.$$

Expanding in power series we have

$$D_z^\alpha uv = \frac{\Gamma(\alpha+1)}{-4\pi^2}\left\{\int_{C_1}\frac{u(\xi)}{(\xi-z)^{\alpha-\gamma+1}}\int_{C_3}\frac{v(\zeta)}{(\zeta-z)^{\gamma+1}}\left[\sum_{n=0}^{N}\left(\frac{\xi-z}{\zeta-z}\right)^n\right.\right.$$
$$\left.+\frac{((\xi-z)/(\zeta-z))^{N+1}}{1-(\xi-z)/(\zeta-z)}\right]d\zeta\,d\xi$$
$$+\int_{C_3}\frac{u(\xi)}{(\xi-z)^{\alpha-\gamma+1}}\int_{C_1}\frac{v(\zeta)}{(\zeta-z)^{\gamma+1}}\left[\sum_{n=1}^{N}\left(\frac{\zeta-z}{\xi-z}\right)^n\right.$$
$$\left.\left.+\frac{((\zeta-z)/(\xi-z))^{N+1}}{1-(\zeta-z)/(\xi-z)}\right]d\zeta\,d\xi\right\}$$
$$=\sum_{n=-N}^{N}\frac{\Gamma(\alpha+1)}{-4\pi^2}\int_{C_2}\frac{u(\xi)\,d\xi}{(\xi-z)^{\alpha-\gamma-n+1}}\int_{C_2}\frac{v(\zeta)\,d\zeta}{(\zeta-z)^{\gamma+n+1}}$$
$$-\frac{\Gamma(\alpha+1)}{4\pi^2}\int_{C_1}\frac{u(\xi)}{(\xi-z)^{\alpha-\gamma}}\int_{C_3}\frac{v(\zeta)}{(\zeta-z)^{\gamma+1}}\frac{((\xi-z)/(\zeta-z))^N}{\zeta-\xi}\,d\zeta\,d\xi$$
$$-\frac{\Gamma(\alpha+1)}{4\pi^2}\int_{C_3}\frac{u(\xi)}{(\xi-z)^{\alpha-\gamma+1}}\int_{C_1}\frac{v(\zeta)}{(\zeta-z)^{\gamma}}\frac{((\zeta-z)/(\xi-z))^N}{\xi-\zeta}\,d\zeta\,d\xi.$$

We see at once from the generalized Cauchy integral formula that $\sum_{-N}^{N}$ in this last expression is

$$\sum_{n=-N}^{N}\frac{\Gamma(\alpha+1)D_z^{\alpha-\gamma-n}u(z)D_z^{\gamma+n}v(z)}{\Gamma(\alpha-\gamma-n+1)\Gamma(\gamma+n+1)}.$$

It is an elementary exercise to see that the remaining two terms approach zero as N grows large. This is because

$$\left|\frac{\xi-z}{\zeta-z}\right|=\left|\frac{\xi-z}{z}\right|<1$$

for ζ and ξ not zero in the first remainder, since C_3 is a circle. This shows the reason why $z\in\mathscr{S}$. A similar statement holds for the second remainder. Thus the theorem is proved.

If the region $\mathscr{R}$ of analyticity of u and v is the z-plane with only isolated points removed, then the region of convergence $\mathscr{S}$ of the generalized Leibniz rule consists of the interior of a polygon whose sides are the perpendicular bisectors of the line segments joining the singularities to the origin. As an example, consider the case where $\gamma = 0$, $u(z) \equiv 1$, and $v(z) = \sum_{r=0}^{R}(a_r - z)^{-1}$.

A little computation reveals that

$$D_z^\alpha \sum_{r=0}^{R} (a_r - z)^{-1} = \frac{\Gamma(\alpha + 1)}{\pi \csc \pi\alpha} \sum_{r=0}^{R} \frac{z^{-\alpha}}{a_r - z} \sum_{n=0}^{\infty} \frac{z^n}{(\alpha - n)(z - a_r)^n}.$$

It is easy to see that this series converges for $|z|/|a_r - z| < 1$, $r = 0, 1, 2, \cdots, R$. This is the polygon just described. (See Fig. 3.)

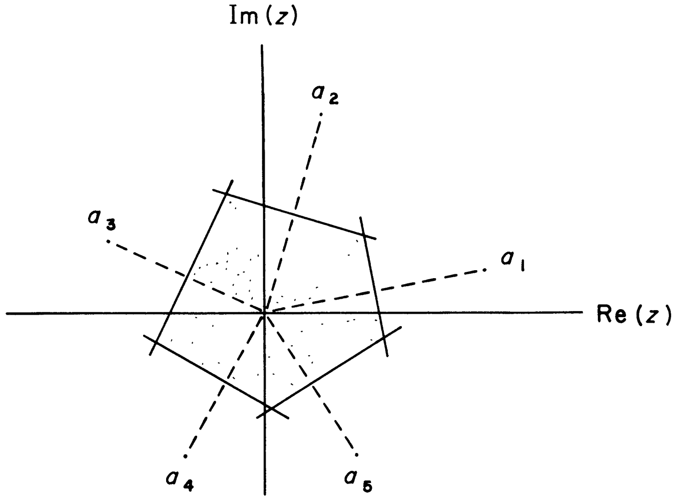

FIG. 3

In this example (4.2) diverges outside the closure of the region $\mathscr{S}$ described in the theorem. Thus with the exception of the boundary of $\mathscr{S}$ the precise region of convergence for general functions $u(z)$ and $v(z)$ has been determined. (Watanabe appears to state that (4.1) should converge wherever $u(z)$ and $v(z)$ are analytic [14, p. 15, Remark 2]. The above example shows that this is not correct.)

COROLLARY. *With the hypothesis of the previous theorem and the additional conditions*

(a) *$g(w)$ is analytic for w in $g^{-1}(\mathscr{R})$,*
(b) *$U(w) = u(g(w))$,*
(c) *$V(w) = v(g(w))$,*

then

$$D_{g(w)}^\alpha U(w)V(w) = \sum_{n=-\infty}^{\infty} \frac{\Gamma(\alpha + 1) D_{g(w)}^{\alpha-\gamma-n} U(w) D_{g(w)}^{\gamma+n} V(w)}{\Gamma(\alpha - \gamma - n + 1)\Gamma(\gamma + n + 1)} \tag{4.3}$$

for w in $g^{-1}(\mathscr{S})$ and all complex α and γ, except negative integral α.

The proof of this corollary follows immediately from the previous theorem by replacing z by $g(w)$. As an example, consider the case in which $\gamma = 0$, $u(z) \equiv 1$, $v(z) = (1 - z)^{-1}$, and $g(w) = w^2$. The region $\mathscr{S}$ of convergence of the generalized Leibniz rule (4.3) is $\operatorname{Re}(z) < \frac{1}{2}$. In the w-plane we have $w = \sqrt{z}$ which is the region bound on the right and left by the two branches of the hyperbola $2\mu^2 - 2\nu^2 = 1$, where $w = \mu + iv$. Therefore (4.3) reduces to

$$D_{w^2}^{\alpha}(1 - w^2)^{-1} = \frac{1}{\Gamma(-\alpha)} \sum_{n=0}^{\infty} \frac{w^{2(n-\alpha)}}{(\alpha - n)(w^2 - 1)^{n+1}}$$

for w in the region specified.

5. A further generalization of Leibniz rule. To see intuitively how the formula

$$D_z^{\alpha} f(z, z) = \sum_{n=-\infty}^{\infty} \binom{\alpha}{\gamma + n} D_{z,w}^{\alpha-\gamma-n, \gamma+n} f(z, w)|_{w=z} \tag{5.1}$$

follows from the Leibniz rule

$$D_z^{\alpha} u(z)v(z) = \sum_{n=-\infty}^{\infty} \binom{\alpha}{\gamma + n} D_z^{\alpha-\gamma-n} u(z) D_z^{\gamma+n} v(z), \tag{5.2}$$

we expand $f(z, w)$ in a power series in z and w, $f(z, w) = \sum a_{r,s} z^r w^s$. Operating with D_z^{α} and using (5.2) we get

$$\begin{aligned} D_z^{\alpha} f(z, z) &= \sum a_{r,s} D_z^{\alpha} z^r \cdot z^s \\ &= \sum a_{r,s} \sum_n \binom{\alpha}{\gamma + n} D_z^{\alpha-\gamma-n} z^r D_z^{\gamma+n} z^s. \end{aligned}$$

Interchanging the order of summation and using (3.4) we get

$$\begin{aligned} D_z^{\alpha} f(z, z) &= \sum_n \binom{\alpha}{\gamma + n} \sum a_{r,s} D_z^{\alpha-\gamma-n} z^r D_z^{\gamma+n} z^s \\ &= \sum_n \binom{\alpha}{\gamma + n} D_{z,w}^{\alpha-\gamma-n, \gamma+n} \sum a_{r,s} z^r w^s \Big|_{w=z}. \end{aligned}$$

This last equation is the desired generalization (5.1).

The steps outlined above could be made into a rigorous proof. However a far easier proof is obtained by using the Cauchy integral formula for fractional derivatives as demonstrated previously in the proof of the Leibniz rule in Theorem 1. In fact, if the reader replaces $u(\xi)v(\zeta)$ by $f(\xi, \zeta)$ in that proof he will find that (5.1) is obtained without any further modification. We state the precise conclusion as a theorem.

THEOREM 2. *Let $\mathscr{R}$ be a simply connected region in the complex plane. Let the origin be an interior or boundary point of $\mathscr{R}$. Let $f(z, w)$ be an analytic function for z and w in $\mathscr{R}$. Assume also that $\oint f(z, z)\, dz$, $\oint f(z, w)\, dz$, and $\oint f(z, w)\, dw$ vanish over any simple closed path in $\mathscr{R}$ through the origin. Call $\mathscr{S}$ the set of all z such that the closed disk $|\zeta - z| \leqq |z|$ contains only points ζ in $\mathscr{R} \cup \{0\}$. Then*

$$D_z^{\alpha} f(z, z) = \sum_{n=-\infty}^{\infty} \binom{\alpha}{\gamma + n} D_{z,w}^{\alpha-\gamma-n, \gamma+n} f(z, w) \Big|_{w=z}$$

for z in $\mathscr{S}$ and all complex α and γ, except negative integral α.

The more general result in which we differentiate with respect to $g(z)$, (1.2), is obtained immediately from the above theorem by replacing z by $g(z)$ as in the corollary to Theorem 1.

6. Series expansions. We conclude our examination of the generalization of the Leibniz rule by assigning specific values to f, g, α, and γ in the formula

$$D^{\alpha}_{g(z)} f(z, z) = \sum_{n=-\infty}^{\infty} \binom{\alpha}{\gamma + n} D^{\alpha-\gamma-n, \gamma+n}_{g(z), g(w)} f(z, w)\Big|_{w=z} \tag{6.1}$$

and simplifying the result. Table 3 shows the fruit of this procedure. It is not surprising that a wide variety of infinite series expansions is obtained since the generalized Leibniz rule (6.1) is a glorified integration by parts and many important series are obtained by this technique. For example, series 2 in Table 3 is simply the value of the hypergeometric function $F(a, b : c : z)$ at $z = 1$. Series 10 is known as Dougall's formula [2, vol. 1, p. 7].

TABLE 1
Special functions expressed as fractional derivatives

Name	Derivative Representation
Hypergeometric function	$F(\alpha, \beta; \gamma; x) = \dfrac{\Gamma(\gamma)x^{1-\gamma}}{\Gamma(\beta)} D_x^{\beta-\gamma} \dfrac{x^{\beta-1}}{(1-x)^{\alpha}}$
Degenerate hypergeometric function	${}_1F_1(\alpha; \gamma; x) = \dfrac{\Gamma(\gamma)x^{1-\gamma}}{\Gamma(\alpha)} D_x^{\alpha-\gamma} e^x x^{\alpha-1}$
Bessel function	$J_\nu(x) = \dfrac{x^{-\nu}}{2^\nu\sqrt{\pi}} D_{x^2}^{-\nu-1/2} \dfrac{\cos x}{x}$
Modified Bessel function	$I_\nu(x) = \dfrac{x^{-\nu}}{2^\nu\sqrt{\pi}} D_{x^2}^{-\nu-1/2} \dfrac{\cosh x}{x}$
Struve function	$\mathbf{H}_\nu(x) = \dfrac{x^{-\nu}}{2^\nu\sqrt{\pi}} D_{x^2}^{-\nu-1/2} \dfrac{\sin x}{x}$
Modified Struve function	$\mathbf{L}_\nu(x) = \dfrac{x^{-\nu}}{2^\nu\sqrt{\pi}} D_{x^2}^{-\nu-1/2} \dfrac{\sinh x}{x}$
Legendre function of the first kind	$P_\nu(x) = \dfrac{1}{\Gamma(\nu+1)2^\nu} D^\nu_{1-x}(1-x^2)^\nu$
Associated Legendre function of the first kind	$P^\mu_\nu(x) = \dfrac{(1-x^2)^{\mu/2}}{\Gamma(\nu+1)2^\nu} D^{\nu+\mu}_{1-x}(1-x^2)^\nu$
Laguerre function	$L^{(\alpha)}_\nu(x) = \dfrac{\Gamma(\alpha+\nu+1)x^{-\alpha}}{\Gamma(1+\nu)\Gamma(-\nu)} D_x^{-\alpha-\nu-1} e^x x^{-\nu-1}$
Incomplete gamma function	$\gamma(\alpha, x) = \Gamma(\alpha)e^{-x} D_x^{-\alpha} e^x$

TABLE 2
Choices for functions and parameters in the generalized Leibniz rule (1.2) *from which the series in Table 3 are derived*

Series Number	α	γ	$f(z, w)$	$g(z)$
1	$b - c$	0	$w^{b-1}(1 - w)^{-a}$	z
2	$-a$	0	$z^{c-a-1}w^{-b}$	z
3	$a - c$	0	$e^w w^{a-1}$	z
4	$-v - \frac{1}{2}$	0	$\frac{\cos w}{w}$	z^2
5	$-v - \frac{1}{2}$	0	$\frac{\cosh w}{w}$	z^2
6	$-v - \frac{1}{2}$	0	$\frac{\sin w}{w}$	z^2
7	$-v - \frac{1}{2}$	0	$\frac{\sinh w}{w}$	z^2
8	v	0	$(1 - w^2)^v$	$1 - z$
9	$b + B - d - D$	$B - D$	$z^{b-1}(1 - z)^{-a}w^{B-1}(1 - w)^{-A}$	z
10	$c - a - 1$	$c - 1$	$z^{d-a-1}w^{c-b-1}$	z
11	$a - b$	γ	$e^w w^{a-1}$	z
12	$-v - \frac{1}{2}$	$-\frac{1}{2} - b$	$\frac{\cos w}{w}$	z^2
13	$-v - \frac{1}{2}$	$-\frac{1}{2} - b$	$\frac{\cosh w}{w}$	z^2
14	$-v - \frac{1}{2}$	$-\frac{1}{2} - b$	$\frac{\sin w}{w}$	z^2
15	$-v - \frac{1}{2}$	$-\frac{1}{2} - b$	$\frac{\sinh w}{w}$	z^2
16	$v + \mu$	γ	$(1 - w^2)^v$	$1 - z$
17	$-a - v - 1$	γ	$e^w w^{-v-1}$	z
18	$-a$	c	e^w	z
19	$-a$	0	$z^{c-a-1}(z - w/x)^{-b}$	z
20	$-a$	0	$z^{b-1}\exp(wz^k)$	z
21	$c - a - 1$	$c - 1$	$z^{d-a-e-1}w^{c-b-1}(z^{-1} + w)^{-e}$	z

TABLE 3

Series expansions of special functions derived from the generalized Leibniz rule

Series Number	Series Expansion	Restrictions
1	$F(a, b; c; z) = \frac{\Gamma(c)\Gamma(b-c+1)\sin\pi(b-c)}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n F(a, b; b-n; z)}{\Gamma(b-n)n!(b-c-n)}$	$\mathrm{Re}(b) > 0$ $\mathrm{Re}(z) < \frac{1}{2}$
2	$\frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)} = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!}$	$\mathrm{Re}(c-a-b) > 0$
3	${}_1F_1(a; c; z) = \frac{\Gamma(c)\Gamma(a-c+1)\sin(a-c)\pi}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n {}_1F_1(a; a-n; z)}{\Gamma(a-n)n!(a-c-n)}$	$\mathrm{Re}(a) > 0$
4 through 7	$\mathscr{F}_\nu(z) = \frac{\Gamma(\frac{1}{2}-\nu)\cos\pi\nu}{\pi 2^{\nu+1/2}} \sum_{n=0}^{\infty} \frac{(-1)^n z^{n+\nu+1/2} \mathscr{F}_{-n-1/2}(z)}{2^n(\nu+n+\frac{1}{2})n!}$, where $\mathscr{F}_\nu = J_\nu,\ I_\nu,\ \mathbf{H}_\nu$ and $\mathbf{L}_\nu$, respectively, for series 4, 5, 6, and 7.	
8	$P_\nu(z) = \frac{\sin\pi\nu}{\pi 2^\nu} \sum_{n=0}^{\infty} \frac{(-2)^n(1-z)^{n-\nu}P_n(z)}{\nu-n}$	$\mathrm{Re}(\nu) > -1$ $\mathrm{Re}(z) > 0$
9	$F(a+A, b+B-1; d+D-1; z) = \frac{\Gamma(b+B-d-D+1)\Gamma(d+D-1)\Gamma(b)\Gamma(B)}{\Gamma(b+B-1)}$ $\sum_{n=-\infty}^{\infty} \frac{F(a, b; d+n; z)F(A, B; D-n; z)}{\Gamma(b-d-n+1)\Gamma(B-D+n+1)\Gamma(d+n)\Gamma(D-n)}$	$\mathrm{Re}(b+B) > 1$ $\mathrm{Re}(b) > 0$ $\mathrm{Re}(B) > 0$ $\mathrm{Re}(z) < \frac{1}{2}$
10	$\frac{\pi^2 \csc\pi a \csc\pi b\,\Gamma(c+d-a-b-1)}{\Gamma(d-b)\Gamma(c-a)\Gamma(d-a)\Gamma(c-b)} = \sum_{n=-\infty}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)\Gamma(d+n)}$	$\mathrm{Re}(c+d-a-b) > +1$

TABLE 3—*Continued*

Series Number	Series Expansion	Restrictions
11	${}_1F_1(a;b;z) = \frac{\Gamma(a-b+1)\Gamma(b)\sin(a-b-\gamma)\pi}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n \, {}_1F_1(a;a-\gamma-n;z)}{(a-b-\gamma-n)\Gamma(\gamma+n+1)\Gamma(a-\gamma-n)}$	$\mathrm{Re}\,(a) > 0$
12 through 15	$\mathscr{F}_\nu(z) = \frac{\Gamma(\frac{1}{2}-\nu)\, z^{\nu-b}\sin(b-\nu)\pi}{\pi 2^{\nu-b}} \sum_{n=-\infty}^{\infty} \frac{(-z)^n \mathscr{F}_{b-n}(z)}{2^n(b-\nu-n)\Gamma(\frac{1}{2}-b+n)}$, where $\mathscr{F}_\nu(z) = J_\nu(z)$, $I_\nu(z)$, $\mathbf{H}_\nu(z)$, and $\mathbf{L}_\nu(z)$ for series 12, 13, 14, and 15 respectively.	
16	$P_\nu^\mu(z) = \frac{\Gamma(\nu+\mu+1)\sin(\nu+\mu-\gamma)\pi}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n P_z^{\gamma-\nu+n}(z)}{\Gamma(\gamma+n+1)(\nu+\mu-\gamma-n)} \left(\frac{1-z}{1+z}\right)^{(n+\gamma-\mu-\nu)/2}$	$\mathrm{Re}\,(\nu) > -1$ $\mathrm{Re}\,(z) > 0$
17	$L_\nu^{(a)}(z) = -\frac{\sin(a+\nu+\gamma)\pi\sin(\gamma\pi)}{\pi\sin(a+\nu)\pi} \sum_{n=-\infty}^{\infty} \frac{L_\nu^{(-n-\nu-\gamma)}(z)}{n+a+\nu+\gamma}$	$\mathrm{Re}\,(\nu) < 0$
18	$\gamma(a,z) = -\frac{\sin(a+c)\pi\sin c\pi}{\pi\sin a\pi} \sum_{n=-\infty}^{\infty} \frac{z^{n+c+a}\gamma(-c-n,z)}{a+c+n}$	
19	$\frac{\Gamma(c)\Gamma(c-a-b)(1-x)^{-b}}{\Gamma(c-a)\Gamma(c-b)} = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} F(b+n, c-a; c+n; x)$	$x \neq -1$ $\mathrm{Re}\,(c-a-b) > 0$ $\mathrm{Re}\,(c-a) > 0$ $\mathrm{Re}\,(1-b) > 0$

20	$\dfrac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\,{}_{k+1}F_{k+1}\left(\begin{matrix}\dfrac{b}{k+1},\dfrac{b+1}{k+1},\ldots,\dfrac{b+k}{k+1};z^{k+1}\\ \dfrac{a+b}{k+1},\dfrac{a+b+1}{k+1},\ldots,\dfrac{a+b+k}{k+1}\end{matrix}\right)=\sum_{n=0}^{\infty}\dfrac{\Gamma(b+nk)\Gamma(a+n)(-z^{k+1})^n}{\Gamma(a+b+kn+n)n!}\cdot{}_kF_k\left(\begin{matrix}\dfrac{b+nk}{k},\dfrac{b+nk+1}{k},\ldots,\dfrac{b+nk+k-1}{k};z^{k+1}\\ \dfrac{a+b+nk+n}{k},\dfrac{a+b+nk+n+1}{k},\ldots,\dfrac{a+b+nk+n+k-1}{k}\end{matrix}\right)$	$\operatorname{Re}(b)>0$
21	$\dfrac{\pi^2\Gamma(c+d-a-b-1)\csc\pi a\csc\pi b}{\Gamma(c-a)\Gamma(c-b)\Gamma(d-a)\Gamma(d-b)}\,{}_3F_2\left(e,\dfrac{c+d-a-b-1}{2},\dfrac{c+d-a-b}{2};\dfrac{d-b}{2},\dfrac{d-b+1}{2};-z^2\right)=\sum_{n=-\infty}^{\infty}\dfrac{\Gamma(a+n)\Gamma(b+n)\,{}_3F_2(e,d-a,c-b;d+n,1-b-n;-z^2)}{\Gamma(c+n)\Gamma(d+n)}$	$\lvert\operatorname{Im}(z)\rvert<\frac{1}{2}$ $\operatorname{Re}(d-a-e)>0$ $\operatorname{Re}(c-b)>0$ $\operatorname{Re}(1-e)>0$ $\operatorname{Re}(c+d-a-b-1)>0$

Table 2 exhibits the special choices of f, g, α and γ used to generate the corresponding series in Table 3. After assigning these choices to (6.1), it is useful to simplify by expressing the resulting fractional derivatives in terms of well-known special functions of mathematical physics. For this purpose it is convenient to have a table of special functions expressed as fractional derivatives. A brief list of this type is included in Table 1, where the notation $D^{\alpha}_{g(z)}$ introduced in § 3 is used to advantage. A more extensive table for this purpose is found in [3, vol. 2, pp. 185–212]. The notation for the special functions used is that of Erdélyi, et al. [2], [3]. Higgins [8] has also compiled an extensive table of special functions represented by fractional derivatives incorporating a different notation.

Listed below are four special forms of the generalized product rule which are obtained from (1.1) by taking specific values for α, γ and v. These formulas are of interest, for when combined judiciously with a table of fractional integrals or derivatives, a host of series expansions relating special functions is envisioned.

Case 1 ($\gamma = 0$).

$$D^{\alpha}_{g(z)}u(z)v(z) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha + 1)D^{\alpha-n}_{g(z)}u(z)D^{n}_{g(z)}v(z)}{\Gamma(\alpha - n + 1)n!}.$$

Case 2 ($v \equiv 1$).

$$D^{\alpha}_{g(z)}u(z) = \frac{\Gamma(\alpha + 1)\sin(\alpha - \gamma)\pi}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n g(z)^{n+\gamma-\alpha} D^{\gamma+n}_{g(z)}u(z)}{(\alpha - \gamma - n)\Gamma(\gamma + n + 1)}.$$

Case 3 ($\alpha = 0$).

$$u(z)v(z) = \frac{\sin \gamma\pi}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n D^{-\gamma-n}_{g(z)}u(z)D^{\gamma+n}_{g(z)}v(z)}{\gamma + n}.$$

Case 4 ($\alpha = 0$, $v \equiv 1$).

$$u(z) = \frac{\sin^2 \gamma\pi}{\pi^2} \sum_{n=-\infty}^{\infty} \frac{\Gamma(\gamma + n)g(z)^{-\gamma-n}D^{-\gamma-n}_{g(z)}u(z)}{\gamma + n}.$$

7. Acknowledgment. The author wishes to express his indebtedness to Professor S. N. Karp of the Courant Institute of Mathematical Sciences at N.Y.U. for his guidance and encouragement, and to T. P. Higgins, L. A. Rubel and the referees for valuable references.

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