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LEIBNIZ RULE FOR FRACTIONAL DERIVATIVES
USED TO GENERALIZE FORMULAS
OF WALKER AND CAUCHY

BY

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1. Introduction

There are two extensions of the Leibniz rule for the N th derivative of the product of two functions

$$D^N(uv) = \sum_{n=0}^N \binom{N}{n} D^{N-n} u D^n v,$$

one due to J. J. Walker in 1800

$$(1) \quad D^N[f^N uv] = \sum_{n=0}^N W(N, n),$$

where

$$W(\alpha, \omega) = \binom{\alpha}{\omega} D^{\alpha-\omega} [f^{\alpha-\omega} v] D^{\omega-1} [f^{\omega} u'],$$

and the other due to A. L. Cauchy in 1826

$$(2) \quad D^{N-1}[f^N D(uv)] = \sum_{n=0}^N C(N, n)$$

where

$$C(\alpha, \omega) = \binom{\alpha}{\omega} D^{\alpha-\omega-1} [f^{\alpha-\omega} v'] D^{\omega-1} [f^{\omega} u'].$$

In a recent paper, H. N. Gould [2] discussed the history of (1) and (2), showed several methods of proof, and mentioned their relation to combinatorial identities.

In this paper we extend (1) and (2) to the case where the non-negative integer N is replaced by arbitrary (rational, irrational or complex) α . The resulting derivatives of the form $D_z^\alpha g(z)$ are called fractional derivatives and can be defined by

$$(3) \quad D_z^\alpha g(z) = \frac{\Gamma(\alpha + 1)}{2\pi i} \int_0^{(z^+)} \frac{g(t) dt}{(t - z)^{\alpha+1}}.$$

The definition (3) is explained in detail in [3].

In the next section, we prove that the extension of Walker's formula (1) to fractional derivatives has a series and an integral form

$$(4) \quad D_z^\alpha [f^\alpha uv] = \sum_{n=-\infty}^{\infty} W(\alpha, an + \gamma)a = \int_{-\infty}^{\infty} W(\alpha, \omega + \gamma)d\omega,$$

and that the extension of Cauchy's formula (2) also assumes a series and an integral form

$$(5) \quad D_z^{\alpha-1} [f^\alpha D(uv)] = \sum_{n=-\infty}^{\infty} C(\alpha, an + \gamma)a = \int_{-\infty}^{\infty} C(\alpha, \omega + \gamma) d\omega.$$

Here $0 < a \leq 1$ and γ is an arbitrary complex number.

In Section 3 we examine special cases of (4) and (5) in which specific functions are selected for f , u and v .

2. Proofs of the Generalized Formulas

Theorem. (i) Let R be a simply connected open set in the complex plane having the origin as an interior or boundary point.

(ii) Let $u(z) = z^p g(z)$ and $v(z) = z^q h(z)$ where $g(z)$ and $h(z)$ are analytic on $R \cup \{0\}$.

(iii) Let $f(z)$ be analytic and never zero on $R \cup \{0\}$.

(iv) Assume that the curves $C(z) = \{t \mid (t-z)/f(t) = |-z/f(0)|\}$ are simple and closed for each z such that $C(z) \subset R \cup \{0\}$. Assume also that each curve defined by $\{t \mid |(t-z)/f(t)| = \text{constant}\}$ interior to $C(z)$ is simple and closed.

(v) Call $S = \{z \mid C(z) \subset R \cup \{0\}\}$.

Then for $z \in S$, $0 < a \leq 1$, and all α and γ such that $\left(an + \frac{\alpha}{\gamma}\right)$ defined, the generalized formula of Walker (4) is true for $\text{Re}(P) > 0$, $\text{Re}(Q) > -1$, and the generalized formula of Cauchy (5) is true for $\text{Re}(P) > 0$ and $\text{Re}(Q) > 0$.

Proof. In [3, p. 3, (1.4 a)] the author proved that

$$D_z^\alpha UV = \sum_{n=-\infty}^{\infty} a \left(an + \frac{\alpha}{\gamma} \right) D^{\alpha-an-\gamma} [Uq^{an+\gamma}] D_z^{an+\gamma-1} [V'q^{-an-\gamma}],$$

where $V(0) = 0$. Replacing U by $f^\alpha v$, V by u and q by $1/f$ we get the series form of Walker's formula (4) at once. In [4] the author showed that

$\sum_{n=-\infty}^{\infty} \dots a$, could be replaced by $\int_{-\infty}^{\infty} \dots d\omega$ and an by ω in the series form

of the Leibniz rule to get a valid integral form. Thus we have shown that the generalized formulas of Walker are simply formulas from [3] and [4] with a change in notation.

To obtain the generalized formulas of Cauchy we start with (4) and „factor a D_z “ from both sides in the form

$$(6) \quad D_z^{\alpha-1} D_z [f^\alpha uv] = \sum_{n=-\infty}^{\infty} \dots D_z^{\alpha-an-\gamma-1} D_z [f^{\alpha-an-\gamma} v].$$

(One can say that $D_z^{\alpha-1} D_z G(z) = D_z^\alpha G(z)$ provided $G(0) = 0$. See [3, p. 9]). Now (6) becomes

$$(7) \quad D_z^{\alpha-1} [f^\alpha D_z (uv)] + \alpha D^{\alpha-1} [f^{\alpha-1} uvf'] = \sum \dots D_z^{\alpha-an-\gamma-1} [f^{\alpha-an-\gamma} v'] + \\ + \sum \dots (\alpha - an - \gamma) D_z^{\alpha-an-\gamma-1} [f^{\alpha-an-\gamma-1} v'f'].$$

If we replace α by $\alpha - 1$ and v by vf' in the generalized Walker's formula (4), we see that the second term on the left hand side of (7) equals the second term on the right hand side of (7). Canceling these terms we have the series form of the generalized Cauchy's formula (5).

The integral form of (5) follows at once from [4]. It is a simple matter to check that the hypothesis of this theorem is sufficient to assure that the above formulas and manipulations are valid by comparison with [3, theorem 4.1 and Corollary 4.1].

3. Examples

In this section we select specific functions for $u(z)$, $v(z)$, $f(z)$ and we examine the resulting generalized formulas of Cauchy (5). We do not give any examples of the generalized formulas of Walker (4), since the above proof demonstrated that (4) is simply a result given previously in [3] and [4] with a change in notation.

We give only the series form of (5). In each example $\sum_{-\infty}^{\infty} \dots a$ can be replaced by $\int_{-\infty}^{\infty} \dots d\omega$ and an can be replaced by ω to get the corresponding integral form.

In all the examples given below, we have set $u(z) = z^P$ and $v(z) = z^Q$ with $\text{Re}(P)$ and $\text{Re}(Q)$ positive in (5). The resulting fractional derivatives were found in [1, pp. 186-7].

Example 1. Let $f(z) = e^{Az}$ in (5) and get

$$\frac{\Gamma(P+a+1) {}_1F_1 \left(\begin{matrix} P+Q \\ P+Q-\alpha+1 \end{matrix} \middle| \alpha Az \right)}{\Gamma(P+Q-\alpha+1) \Gamma(\alpha+1) \Gamma(P+1) \Gamma(Q+1)} = \\ = \sum_{n=-\infty}^{\infty} \frac{{}_1F_1 \left(\begin{matrix} Q \\ Q-\alpha+an+\gamma+1 \end{matrix} \middle| (\alpha-an-\gamma)Az \right) {}_1F_1 \left(\begin{matrix} P \\ P-an-\gamma+1 \end{matrix} \middle| (an+\gamma)Az \right) a}{\Gamma(\alpha-an-\gamma+1) \Gamma(Q-\alpha+an+\gamma+1) \Gamma(an+\gamma+1) \Gamma(P-an-\gamma+1)}.$$

Example 2. Let $f(z) = (z + A)^B$ in (5) and get

$$\frac{\Gamma(P + Q + 1) {}_2F_1\left(-\alpha B, P + Q \mid -\frac{z}{A}\right)}{\Gamma(P + Q - \alpha + 1)\Gamma(\alpha + 1)\Gamma(P + 1)\Gamma(Q + 1)} =$$

$$= \sum_{n=-\infty}^{\infty} \frac{{}_2F_1\left(B(an + \gamma - \alpha), Q \mid -\frac{Z}{A}\right) {}_2F_1\left(-B(an + \gamma), P \mid -\frac{Z}{A}\right) a}{\Gamma(\alpha - an - \gamma + 1)\Gamma(Q - \alpha + an + \gamma + 1)\Gamma(an + \gamma + 1)\Gamma(P - an - \gamma + 1)}$$

Example 3. In this example we generalize Example 1 by taking $f(z) = e^{Az^k}$, $k = 1, 2, 3, \dots$, and get

$$\frac{\Gamma(P + Q + 1) {}_kF_k\left(\{P + Q\}_k \mid \alpha A z^k\right)}{\Gamma(P + Q - \alpha + 1)\Gamma(\alpha + 1)\Gamma(P + 1)\Gamma(Q + 1)} =$$

$$= \sum_{n=-\infty}^{\infty} \frac{{}_kF_k\left(\{Q\}_k \mid (\alpha - an - \gamma) A z^k\right) {}_kF_k\left(\{P\}_k \mid (an + \gamma) A z^k\right) a}{\Gamma(\alpha - an - \gamma + 1)\Gamma(Q - \alpha + an + \gamma + 1)\Gamma(an + \gamma + 1)\Gamma(P - an - \gamma + 1)}$$

where the notation $\{P\}_k$ stands for the set of k parameters $\frac{P}{k}, \frac{P+1}{k}, \dots, \frac{P+k-1}{k}$.

Example 4. Finally, we extend Example 2 by selecting $f(z) = (z^k + A^k)^B$ and we get

$$\frac{\Gamma(P + Q + 1) {}_{k+1}F_k\left(-\alpha B, \{P + Q\}_k \mid -\frac{z^k}{A^k}\right)}{\Gamma(P + Q - \alpha + 1)\Gamma(\alpha + 1)\Gamma(P + 1)\Gamma(Q + 1)} =$$

$$= \sum_{n=-\infty}^{\infty} \frac{{}_{k+1}F_k\left(B(an + \gamma - \alpha), \{Q\}_k \mid -\frac{z^k}{A^k}\right) {}_{k+1}F_k\left(-B(an + \gamma), \{P\}_k \mid -\frac{z^k}{A^k}\right) a}{\Gamma(\alpha - an - \gamma + 1)\Gamma(Q - \alpha + an + \gamma + 1)\Gamma(an + \gamma + 1)\Gamma(P - an - \gamma + 1)}$$

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REGULA LUI LEIBNIZ PENTRU DERIVATE FRAȚIONARE UTILIZATE PENTRU GENERALIZAREA FORMULELOR LUI WALKER ȘI CAUCHY

(Rezumat)

Se extind formulele (1) și (2) pentru cazul când întregul nenegativ N este înlocuit printr-un număr arbitrar α (rațional, irațional sau complex).